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ACM bundles, quintic threefolds and counting problems

Research Article

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Abstract: We review some facts about rank two arithmetically Cohen–Macaulay bundles on quintic threefolds. In particular, we separate them into seventeen natural classes, only fourteen of which can appear on a general quintic. We discuss some enumerative problems arising from these.

MSC: 14F05

Keywords: Vector bundles • Quintic threefolds • Arithmetically Cohen–Macaulay

1. Introduction

In their paper [5], Chiantini and Madonna prove that any rank two non-split ACM vector bundle (defined in the next sentence) on a general quintic threefold in $\mathbb{P}^4$ is rigid. A vector bundle E on a quintic hypersurface $X \subset \mathbb{P}^4$ is called arithmetically Cohen–Macaulay (ACM) if $H^i(E(v)) = 0$ for $v \in \mathbb{Z}$, $i = 1, 2$, and rigid if $H^1(\operatorname{End} E) = 0$. Their proof uses a restriction on the Chern classes of an ACM bundle E found in [14], if one assumes that $H^0(E) \neq 0$, $H^0(E(-1)) = 0$. Using this restriction, they classify all possible Chern classes and prove the rigidity for each case by analyzing curves arising from each possible Chern class pair. This finishes the proof of rigidity, with a slight inconclusiveness about the precise existence of each possible Chern class pair.

In [4], Chiantini and Faenzi look at rank two non-split ACM bundles on a general quintic surface in $\mathbb{P}^3$. They prove that there are 14 possible Chern class pairs, which are identical to the pairs discovered on quintic threefolds, and they prove

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that for each possible pair, there is an ACM rank two bundle on the general quintic surface. Unlike many prior works in this direction which use computer calculations, their proofs are qualitative.

In [15], the rigidity of rank two ACM bundles on a generic quintic threefold X was proved without a prior classification. Since $H^1(\operatorname{End} E) = H^2(\operatorname{End} E) = 0$ for such a bundle E, this opens up a problem in enumerative geometry: *viz.* count the number of ACM rank two bundles on a general quintic threefold (after identifying E and $E(v)$). The classification of possible Chern class pairs reduces this to perhaps simpler counting problems, many involving curves on X. We revisit the work of Chiantini–Madonna using Beauville's description of ACM bundles on X in [2] and obtain a more precise classification of various types of bundles. We also discuss existence problems on a general X, using computer calculations or otherwise, and collect answers to various counting problems in this context which have been obtained in enumerative geometry.

2. Admissible degrees for rank two ACM bundles on a quintic threefold

Let E be a non-split ACM rank two vector bundle on a hypersurface X of degree d in $\mathbb{P}^n$, with $\det E = \mathcal{O}_X(e)$ for some integer e. It has a minimal resolution by sums of line bundles on $\mathbb{P}^n$ of the form

$$0 \to L^\vee(t) \xrightarrow{\phi} L \to E \to 0,$$

where $t = e - d$ and the map ϕ is skew-symmetric [2]. Hence L has even rank $2s$. Choosing an ordering of the degrees $a_1 \geq a_2 \geq \ldots \geq a_{2s}$ of $L = \bigoplus_1^{2s} \mathcal{O}_{\mathbb{P}^n}(a_i)$, with $L^\vee(t) = \bigoplus_1^{2s} \mathcal{O}_{\mathbb{P}^n}(-a_i + t)$, we may assume that ϕ is given by a skew-symmetric matrix of homogeneous forms (minimal, in the sense that no entry is equal to a non-zero scalar). The defining polynomial f of X is given by the Pfaffian of ϕ. There are some basic restrictions on the values of a_i when $n \geq 4$.

Without loss of generality, we may twist E and assume that t has two possible values, -1 or 0.

Lemma 2.1 (restrictions on a_i).

(1) *$\sum a_i = st + d$, by taking first Chern classes of terms in the sequence, where E as a sheaf on $\mathbb{P}^n$ has first Chern class equal to $2d$.*

(2) *$a_1 \geq (st + d)/2s$ by the above.*

(3) *$a_1 \leq d + (t - 3)/2$ by [14, Theorem 1.1] (for both (2) and (3)) or by the proof of [16, Lemma 3.3].*

(4) *$a_i + a_j < e$ for $i \neq j$. This follows since the minimal section s_i of E corresponding to a_i yields an exact sequence $0 \to \mathcal{O}_X(a_i) \to E \to \mathcal{O}_X(e - a_i)$ of sheaves on X, and the section s_j maps to a nonzero (and non-identity) section of $\mathcal{O}_X(e - a_i - a_j)$.*

(5) *$a_i \geq t - a_1 + 1$ for $i > 1$. This follows since ϕ is an inclusion of sheaves. The entry $\phi_{i,j}$ of the matrix has degree $a_j - t + a_i$. The degrees decrease as the row or column increases. Since the last column of ϕ is non-zero, the topmost entry $\phi_{1,2s}$ must have positive degree. Thus $a_i \geq a_{2s} \geq t - a_1 + 1$.*

(6) *$a_i + a_{2s+3-i} > t$ for each value of $i = 3, \ldots, s + 1$, if X is a smooth hypersurface of dimension greater or equal to 3. This follows from the lemma below.*

Lemma 2.2.

Suppose X is a smooth hypersurface of dimension greater or equal to 3 with equation given by the Pfaffian of a skew-symmetric minimal matrix ϕ with invariants $2s$, t, $a_1 \geq a_2 \geq \ldots \geq a_{2s}$, as above. Then the entries $\phi_{3,2s}, \phi_{4,2s-1}, \ldots, \phi_{s+1,s+2}$ must all have positive degrees.

Proof. If $\phi_{i,2s+3-i}$ has nonpositive degree for some i between 3 and $s + 1$, by the minimality of the matrix and the ordering of the a_i's, ϕ has a block form

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix},$$

where B has size $(i-1)\times(i-2)$. Consider the ideal I generated by the $(i-2)\times(i-2)$ minors of B. Since the Pfaffian f of ϕ is expressible as an element in this ideal, with positive degree coefficients, and since f is irreducible, I is neither zero nor does it define a subscheme of $\mathbb{P}^n$ with a hypersurface component. Hence I defines a subscheme which is a divisor on X. Since it is not a complete intersection of X with a hypersurface of $\mathbb{P}^n$, we contradict the fact that $\operatorname{Pic} X = \mathbb{Z}$ when $\dim X \geq 3$. □

Proposition 2.3.

Let X be a smooth quintic threefold in $\mathbb{P}^4$. Let E be a non-split ACM vector bundle on X. Then the following are the only possibilities for the shape of the matrix ϕ resolving E, if t is normalized to be -1 or 0.

(1) $s=5$, $t=-1$, $[a_1,\dots,a_{10}]=[0,0,0,0,0,0,0,0,0,0]$,

(2) $s=4$, $t=-1$, $[a_1,\dots,a_8]=[1,0,0,0,0,0,0,0]$,

(3) $s=4$, $t=0$, $[a_1,\dots,a_8]=[1,1,1,1,1,0,0,0]$,

(4) $s=3$, $t=-1$, $[a_1,\dots,a_6]=[2,0,0,0,0,0]$,

(5) $s=3$, $t=-1$, $[a_1,\dots,a_6]=[1,1,1,1,-1,-1]$,

(6) $s=3$, $t=-1$, $[a_1,\dots,a_6]=[1,1,1,0,0,-1]$,

(7) $s=3$, $t=-1$, $[a_1,\dots,a_6]=[1,1,0,0,0,0]$,

(8) $s=3$, $t=0$, $[a_1,\dots,a_6]=[2,1,1,1,0,0]$,

(9) $s=3$, $t=0$, $[a_1,\dots,a_6]=[1,1,1,1,1,0]$,

(10) $s=2$, $t=-1$, $[a_1,\dots,a_4]=[3,0,0,0]$,

(11) $s=2$, $t=-1$, $[a_1,\dots,a_4]=[2,1,1,-1]$,

(12) $s=2$, $t=-1$, $[a_1,\dots,a_4]=[2,1,0,0]$,

(13) $s=2$, $t=-1$, $[a_1,\dots,a_4]=[1,1,1,0]$,

(14) $s=2$, $t=0$, $[a_1,\dots,a_4]=[3,1,1,0]$,

(15) $s=2$, $t=0$, $[a_1,\dots,a_4]=[2,2,2,-1]$,

(16) $s=2$, $t=0$, $[a_1,\dots,a_4]=[2,2,1,0]$,

(17) $s=2$, $t=0$, $[a_1,\dots,a_4]=[2,1,1,1]$.

Proof. This is a consequence of a mechanical application of the earlier restrictions on the a_i. Since $\operatorname{Pic} X = \mathbb{Z}$, $\det E = \mathcal{O}_X(e)$ for some e. First of all notice that since X is a quintic defined by the Pfaffian of a minimal matrix ϕ, the determinant of ϕ has degree 10, hence $2s \leq 10$. As an example, suppose that $s=3$, $t=-1$. Using the restrictions of Lemma 2.1, $1 \leq a_1 \leq 3$ from inequalities (1) and (3). We get $a_3, a_4 \geq 0$ by inequality (6) of the same lemma, since $a_3+a_6 \geq 0$, $a_4+a_5 \geq 0$.

Suppose $a_1 = 3$. Then $a_2 \leq 0$ by inequality (4) of Lemma 2.1, hence $a_2,\dots,a_6$ are all 0, violating the equality in (1) in the lemma.

Suppose $a_1 = 2$. Then $0 \leq a_2 \leq 1$ by inequality (4) of Lemma 2.1. So either $a_2 = 0$, giving the case $[2,0,0,0,0,0]$ or $a_2 = 1$. In the second case, $a_1+a_2+(a_3+a_6)+(a_4+a_5) \geq 3$, violating the equality (1) in the same lemma.

Suppose $a_1 = 1$. Then $0 \leq a_2 \leq 1$. If $a_2 = 0$, then $a_3,\dots,a_6$ are all zero, violating equality (1). Hence $a_2 = 1$ and by the equality (1), $a_3+a_6 = 0$, $a_4+a_5 = 0$, giving the three cases $[1,1,0,0,0,0]$, $[1,1,1,0,0,-1]$, $[1,1,1,1,-1,-1]$. □

For each shape admissible on a quintic hypersurface, we get a skew-symmetric minimal homomorphism ϕ from $L^\vee(t)$ to L. Let V be the vector space of all skew symmetric, minimal maps from $L^\vee(t)$ to L. There is an open subset U of V corresponding to maps ϕ, where $\operatorname{coker}\phi$ is a rank two vector bundle E on a smooth quintic hypersurface X. *A priori*, this open set U may be empty. We will visit this question later. In some cases, V is a proper vector subspace of the vector space of all skew symmetric maps (not necessarily minimal). See for example case (3) in the above proposition. Assuming that the open set U is nonempty, we wish to perform a parameter count of the number of non-isomorphic pairs (X, E) obtained for this admissible shape.

First of all, we discuss the endomorphisms of E. Let $\phi\colon L^\vee(t) \to L$ be in U, giving a rank two bundle E on a hypersurface X. In many cases, the vector bundle E on the quintic is not stable. Since when $t = 0$, E will have first Chern class equal to 5, this means that $a_1 \geq 3$. Likewise, when $t = -1$, $e = 4$, and this means that $a_1 \geq 2$. More generally, we have

Proposition 2.4.

Suppose E is a non-split ACM bundle on a hypersurface X in $\mathbb{P}^n$.

(1) *If E is a stable bundle, then* $\operatorname{Aut} E$ *equals* $\{\lambda \mathrm{Id}_E : \lambda \in k^*\}$.

(2) *If E is not stable, with a destabilizing section in $E(-a_1)$, then*

$$\operatorname{Aut} E = \big\{\lambda \mathrm{Id}_E + g\rho \,:\, \lambda \in k^*,\ g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2a_1 - d - t))\big\},$$

where $\rho\colon E \to E(d + t - 2a_1)$ is the endomorphism induced by the destabilizing section.

(3) *If E is not stable, the endomorphism ρ lifts to a commuting diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^\vee(t) & \xrightarrow{\ \phi\ } & L & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle S'} & & \downarrow{\scriptstyle S} & & \downarrow{\scriptstyle \rho} & & \\
0 & \longrightarrow & L^\vee(d+2t-2a_1) & \xrightarrow{\ \phi\ } & L(d+t-2a_1) & \longrightarrow & E(d+t-2a_1) & \longrightarrow & 0,
\end{array}$$

where $S = \begin{bmatrix} 0 & q_2 & q_3 & \dots & q_{2s} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$, $S' = -S^\vee$.

Proof. The first statement is the assertion that a stable bundle on X is simple. For the remaining statements, suppose that E is not stable. Then $e - 2a_1 \leq 0$ and the section t_1 of $E(-a_1)$ in lowest degree gives a codimension two subscheme Y in X, with the resolution $0 \to \mathcal{O}_X \xrightarrow{t_1} E(-a_1) \to \mathcal{I}_Y(e - 2a_1) \to 0$. The map ρ is given by $E(-a_1) \twoheadrightarrow \mathcal{I}_Y(e - 2a_1) \subseteq \mathcal{O}_X(e - 2a_1) \xrightarrow{t_1} E(e - 3a_1)$. Note that ρ is nilpotent since $\rho \circ \rho = 0$.

Let $t_2, t_3, \dots, t_{2s}$ be the other sections of E given by the generators of L. They give nonzero elements q_j in $H^0(\mathcal{I}_Y(e - a_1 - a_j))$. Then ρ carries t_1 to zero, and t_j to $q_j t_1$. This shows that S can be chosen as described. In fact, a precise description of $q_2, q_3, \dots, q_{2s}$ is as follows. Let $\operatorname{Pf}\phi(i, j)$ denote the (i, j)-Pfaffian of ϕ, and let ψ be the skew symmetric matrix where ψ_{ij} equals $(-1)^{i+j}\operatorname{Pf}\phi(i, j)$. Then $\psi\phi = \phi\psi = f \cdot I_{2s}$, where f is a defining polynomial for X, and the first column of the matrix ψ consists of $0, q_2, q_3, \dots, q_{2s}$. Hence $S\phi$ is a diagonal matrix with the first and only nonzero entry equal to $-f$. Thus $S' = -S^\vee$ makes the diagram commute.

Since $e - a_1 - a_j > 0$ and $e - 2a_1 \leq 0$, we get $a_j < a_1$ for $j > 1$. Any endomorphism $\alpha\colon E \to E$, viewed as an endomorphism of $E(-a_1)$, must hence carry the section t_1 to a multiple λ of itself. Therefore $\alpha - \lambda \cdot \mathrm{Id}_E$ carries t_1 to zero. Therefore if $\alpha - \lambda \cdot \mathrm{Id}_E \neq 0$, it factors through an inclusion $\mathcal{I}_Y(e - 2a_1) \hookrightarrow E(-a_1)$, or after taking double duals, through an inclusion $\mathcal{O}_X(e - 2a_1) \hookrightarrow E(-a_1)$. This section of $E(a_1 - e)$ is a multiple gt_1 of t_1 since $a_1 + a_2 < e$. So $\alpha = \lambda \cdot \mathrm{Id}_E + g\rho$. Observe that $2a_1 - e \geq 0$ and by inequality (3) among the restrictions, $2a_1 \leq 2d + t - 3$, hence $2a_1 - e = 2a_1 - t - d \leq d - 3$. Thus $H^0(X, \mathcal{O}_X(2a_1 - e)) = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2a_1 - e))$. Since ρ is nilpotent when λ belongs to k^*, then $\lambda \cdot \mathrm{Id}_E + g\rho$ is in $\operatorname{Aut} E$. □

Proposition 2.5.

Let U denote the open subset of all skew symmetric minimal maps $L^\vee(t) \xrightarrow{\phi} L$, where each point in U determines a rank two ACM bundle on a smooth hypersurface X of degree d. The group $\operatorname{Aut} L$ acts on U by the action $(\alpha, \phi) \mapsto \alpha \circ \phi \circ \alpha^\vee$. The map from $U/\operatorname{Aut} L$ to the set of isomorphism classes of pairs (X, E) is bijective.

Proof. Let ϕ and ξ be two elements in U, giving bundles E_ϕ, E_ξ which are isomorphic (on the same hypersurface X). The isomorphism $E_\phi \to E_\xi$ can be lifted to the presentations by ϕ and ξ as automorphisms γ, γ' of L, $L^\vee(t)$ respectively, such that $\gamma\phi = \xi\gamma'$. Then letting $\beta = \gamma'\gamma^\vee$, $\chi = \gamma\phi\gamma^\vee$, we get χ skew-symmetric, in the orbit of ϕ, and with $\chi = \xi\beta = \beta^\vee\xi$.

The equation $\xi\beta = \beta^\vee\xi$ induces an automorphism σ of $E = E_\xi$. By the previous proposition, $\sigma = \lambda \mathrm{Id}_E + g\rho$ and we have "canonical lifts" of σ: $\{\lambda \cdot \mathrm{Id}_L + g \cdot S, \lambda \cdot \mathrm{Id}_{L^\vee(t)} - g \cdot S^\vee\}$. Then $\{\beta^\vee, \beta\}$ differ from the canonical lifts by a homotopy $\tau\colon L \to L^\vee(t)$, with equations

$$\beta^\vee = \lambda \cdot \mathrm{Id}_L + g \cdot S + \xi\tau, \qquad \beta = \lambda \cdot \mathrm{Id}_{L^\vee(t)} - g \cdot S^\vee + \tau\xi.$$

Therefore, $g \cdot S + \xi\tau = -g \cdot S - \xi\tau^\vee$. Then $-2g \cdot S = \xi(\tau + \tau^\vee)$. This indicates that $2g \cdot \rho$ is the zero endomorphism of E_ξ. Hence we may conclude that $g = 0$. Furthermore, $\xi(\tau + \tau^\vee) = 0$, hence $\tau^\vee = -\tau$ (as ξ is injective). Lastly, for convenience, we will take $\lambda = 1$.

So $\beta = \mathrm{Id}_{L^\vee(t)} + \tau\xi$, where $\tau^\vee = -\tau$. Since ξ is minimal, $\tau\xi$ is a minimal endomorphism of $L^\vee(t)$ and thus it is nilpotent. It follows that $(1+\tau\xi)^{1/2}$ is defined and invertible as a truncated power series in $\tau\xi$, and that $\xi(1+\tau\xi)^{1/2} = (1+\xi\tau)^{1/2}\xi$. Thus $\chi = \xi\beta = \xi(1+\tau\xi)^{1/2}(1+\tau\xi)^{1/2} = (1+\xi\tau)^{1/2}\xi(1+\tau\xi)^{1/2}$, where $((1+\xi\tau)^{1/2})^\vee = (1+\tau\xi)^{1/2}$. Hence χ and ξ are in the same orbit. □

Lastly, we analyze the stabilizer of an element $\xi \in U$.

Proposition 2.6.

Under the action of $\operatorname{Aut} L$ on U, $(\alpha, \phi) \mapsto \alpha \circ \phi \circ \alpha^\vee$, the stabilizer of $\xi \in U$ is the subgroup $\operatorname{stab}(E_\xi)$ with two connected components (corresponding to $\pm\mathrm{Id}_L$). The component $\operatorname{stab}^0(E_\xi)$ containing Id_L is described below:

(1) *When E_ξ is stable, $\operatorname{stab}^0(E_\xi)$ is*

$$\left\{\mathrm{Id}_L + \xi\tau \,:\, \tau \in \operatorname{Hom}(L, L^\vee(t)),\ \tau - \tau^\vee = \tau\xi\tau^\vee = \tau^\vee\xi\tau\right\}.$$

(2) *When E is unstable, $\operatorname{stab}^0(E_\xi)$ is*

$$\left\{\mathrm{Id}_L + gS + \xi\tau \,:\, g \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2a_1 - e)),\ \tau \in \operatorname{Hom}(L, L^\vee(t)),\ \tau - \tau^\vee = \tau\xi\tau^\vee = \tau^\vee\xi\tau\right\},$$

where S is as in Proposition 2.4.

Proof. Suppose $\xi = \alpha\xi\alpha^\vee$ for a point $\xi \in U$. With $E = E_\xi$, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L^\vee(t) & \xrightarrow{\ \xi\ } & L & \longrightarrow & E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\alpha^\vee)^{-1}} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} & & \\
0 & \longrightarrow & L^\vee(t) & \xrightarrow{\ \xi\ } & L & \longrightarrow & E & \longrightarrow & 0,
\end{array}$$

where $\sigma \in \operatorname{Aut} E$. Hence there are elements g, τ such that

$$\alpha = \lambda \cdot \mathrm{Id}_L + gS + \xi\tau, \qquad (\alpha^\vee)^{-1} = \lambda \cdot \mathrm{Id}_{L^\vee(t)} - g \cdot S^\vee + \tau\xi.$$

Rewrite the second as $\alpha^{-1} = \lambda \cdot \mathrm{Id}_L - g \cdot S - \xi\tau^\vee$. Since these equations are true modulo the ideal of linear forms on $\mathbb{P}^n$, $\lambda = \lambda^{-1}$. Hence $\lambda = \pm 1$, and we will proceed with the case $\lambda = 1$.

Recall that

$$S = \begin{bmatrix} 0 & q_2 & q_3 & \dots & q_{2s} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and ξS has the element $-f$ in its $(1,1)$ position and is zero elsewhere. Also if $\tau\colon \bigoplus \mathcal{O}_{\mathbb{P}^n}(a_j) \to \mathcal{O}_{\mathbb{P}^n}(-a_i + t)$ has some non-zero entry in its first row, then $-a_j - a_1 + t \geq 0$ for some j. But then $\xi_{1j} = 0$ (since ξ is a minimal matrix), and for degree reasons ξ has its entire j-th column equal to zero, contradicting the injectivity of ξ. Hence τ has its top row equal to zero, and by the symmetry of the degrees of τ, also its left column. In conclusion, $S^2 = 0$, $S\xi\tau^\vee = 0$, $\xi\tau S = 0$. The equation $\alpha\alpha^{-1} = \mathrm{Id}_L$ reduces to

$$\xi\tau - \xi\tau^\vee - \xi\tau\xi\tau^\vee = 0.$$

Since ξ is injective, we get $\tau - \tau^\vee = \tau\xi\tau^\vee$. Likewise, the equation $\alpha^{-1}\alpha = \mathrm{Id}_L$ reduces to $\tau - \tau^\vee = \tau^\vee\xi\tau$.

Conversely, given gS and τ such that $\tau - \tau^\vee = \tau\xi\tau^\vee = \tau^\vee\xi\tau$, we have $\alpha = \lambda \cdot \mathrm{Id}_L + gS + \xi\tau$ is invertible with $\alpha^{-1} = \lambda \cdot \mathrm{Id}_L - g \cdot S - \xi\tau^\vee$, and it is immediate that $\xi = \alpha\xi\alpha^\vee$. □

We will now perform a dimension count of all pairs (X, E), where E is an ACM bundle on a smooth quintic hypersurface X in $\mathbb{P}^4$, with a fixed set of numerical information $s, t, a_1, \dots, a_{2s}$, with the assumption that there is at least one such bundle on some smooth X (*i.e.* the open set U is non-empty in V). It turns out that the description of the stabilizer of $\xi \in U$ in all the cases that we need simplifies further, with $\tau\xi\tau^\vee = \tau^\vee\xi\tau = 0$ just because of the numerical information $s, t, a_1, \dots, a_{2s}$. Hence $\tau = \tau^\vee$ is the only restriction on τ and the stabilizer has an easily computed dimension.

Proposition 2.7.

In each of the cases of Proposition 2.3, provided $U \neq \emptyset$, the dimension of $U/\operatorname{Aut} L$ is 125, except in cases (3), (6) and (8), where the dimensions are 122, 123 and 124 respectively.

Proof. In cases (1), (2), (4), (7), (10), (12), (13), (17), $\operatorname{Hom}(L, L^\vee(t)) = 0$. As an example of the dimension count, the case (10) is a non-stable case, since $2a_1 - e = 6 - 4 \geq 0$. The stabilizer of any $\xi \in U$ has dimension $h^0(\mathcal{O}_{\mathbb{P}^4}(2)) = 15$. V, the vector space of minimal skew-symmetric maps $\mathcal{O}_{\mathbb{P}^4}(-4) \oplus 3\mathcal{O}_{\mathbb{P}^4}(-1) \to \mathcal{O}_{\mathbb{P}^4}(3) \oplus 3\mathcal{O}_{\mathbb{P}^4}$, has dimension 225. $\operatorname{Aut}(\mathcal{O}_{\mathbb{P}^4}(3) \oplus 3\mathcal{O}_{\mathbb{P}^4})$ has dimension 115. Then $U/\operatorname{Aut} L$ has dimension $225 - 115 + 15 = 125$.

In the remaining cases, elements $\tau \in \operatorname{Hom}(L, L^\vee(t))$ exist. Except in case (6), τ has a block decomposition

$$\tau = \begin{bmatrix} 0 & 0 \\ 0 & T \end{bmatrix},$$

where T is a square submatrix, and fortunately ξ has a corresponding block decomposition

$$\xi = \begin{bmatrix} A & B \\ -B^\vee & 0 \end{bmatrix}.$$

Then $\tau\xi\tau^\vee$ is zero by default. So, for example, in case (3), which is a stable case since $2a_1 - e < 0$, τ must be symmetric, hence the stabilizer of such a $\xi \in U$ has dimension 6. The vector space V of minimal skew-symmetric maps ξ has dimension 225, and $\operatorname{Aut}(5\mathcal{O}_{\mathbb{P}^4}(1) \oplus 3\mathcal{O}_{\mathbb{P}^4})$ has dimension 109. So $U/\operatorname{Aut} L$ has dimension $225 - 109 + 6 = 122$. Likewise in case (8), again stable, since τ is symmetric, the stabilizer of $\xi \in U$ has dimension 3. V has dimension 210, $\operatorname{Aut}(\mathcal{O}_{\mathbb{P}^4}(2) \oplus 3\mathcal{O}_{\mathbb{P}^4}(1) \oplus 2\mathcal{O}_{\mathbb{P}^4})$ has dimension 89, and $U/\operatorname{Aut} L$ has dimension $210 - 89 + 3 = 124$.

In case (6), with $\xi\colon 3\mathcal{O}_{\mathbb{P}^4}(-2)\oplus 2\mathcal{O}_{\mathbb{P}^4}(-1)\oplus\mathcal{O}_{\mathbb{P}^4} \to 3\mathcal{O}_{\mathbb{P}^4}(1)\oplus 2\mathcal{O}_{\mathbb{P}^4}\oplus\mathcal{O}_{\mathbb{P}^4}(-1)$, we see that

$$\xi = \begin{bmatrix} A & B & C \\ -B^\vee & D & 0 \\ -C^\vee & 0 & 0 \end{bmatrix},$$

with D skew-symmetric, while

$$\tau = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & N & T \end{bmatrix}.$$

Then

$$\tau\xi\tau^\vee = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & NDN^\vee \end{bmatrix}.$$

$NDN^\vee$ is a skew-symmetric 1×1 matrix, hence zero. So $\tau\xi\tau^\vee = 0$. As an example of the computation, the case (6) is a stable case, hence g plays no role in the stabilizer. τ must be symmetric, and hence the stabilizer of ξ has dimension 7. V has dimension 215, $\operatorname{Aut} L$ has dimension 99, whence $U/\operatorname{Aut} L$ has dimension 123. □

Corollary 2.8.

On the general quintic threefold in $\mathbb{P}^4$, the only non-split rank two ACM bundles, if they exist at all, must have invariants from the list in Proposition 2.3, excluding cases (3), (6) and (8).

Proof. Indeed, the dimension of the space of quintic hypersurfaces in $\mathbb{P}^4$ is 125. As an amusement, it is possible to see the inadequacy of the cases (3), (6) and (8) from another point of view. A non-minimal skew-symmetric map ϕ in case (3) reduces to a minimal skew-symmetric map in case (9), hence bundles with minimal resolutions of type (3) appear as limits of bundles of type (9). Likewise, the others. □

Remark 2.9.

Chiantini and Madonna [5] describe all possible indecomposable ACM rank two bundles on a quintic threefold X as listed below. We show how their list corresponds to the type of bundle from the list in the above corollary, for a general quintic threefold. Note that each of the descriptions of cases $\{1,\dots,17\}\setminus\{3,6,8\}$ in Proposition 2.3 gives a bundle E with first Chern class equal to 5 or 4 according as $t=0$ or -1, together with a description of a minimal resolution on $\mathbb{P}^4$. Chiantini and Madonna adopt the convention of normalizing E by twisting by a line bundle to get E' such that $H^0(E')\neq 0$, $H^0(E'(-1))=0$. Then they list of all possible values of $c_1(E')$ and $c_2(E')$.

Case number	1	2	4	5	7	9	10	11	12	13	14	15	16	17
$c_1(E')$	4	2	0	2	2	3	−2	0	0	2	−1	1	1	1
$c_2(E')$	30	14	5	11	13	20	1	3	4	12	2	4	6	8

3. Existence of ACM bundles on some smooth quintic threefolds or the non-emptiness of U

The next task is to show that in each of the cases $\{1,\dots,17\}\setminus\{3,6,8\}$, there is a skew-symmetric minimal matrix $\phi\colon L^\vee(t)\to L$ whose Pfaffian is a smooth quintic threefold in $\mathbb{P}^4$. Cases (10) through (17) are elementary: the skew-symmetric 4×4 matrix ϕ has all six upper triangular entries a,b,c,d,e,f nonzero, and the Pfaffian has the form $af-be+cd$, which is a smooth hypersurface in $\mathbb{P}^4$ when the forms are chosen generally.

For the cases $s\geq 3$, we use a Bertini type theorem of Okonek [17] to produce arithmetically Gorenstein curves in $\mathbb{P}^4$.

Proposition 3.1.

Let L_1 be a sum of $2s-1$ line bundles on $\mathbb{P}^4$ such that $(\wedge^2 L_1)\otimes\mathcal{O}_{\mathbb{P}^4}(-t)$ is generated by its global sections. Then the general section α defines, through its ideal of Pfaffians, a smooth arithmetically Gorenstein curve C in $\mathbb{P}^4$:

$$0 \to \mathcal{O}_{\mathbb{P}^4}(st-\det L_1) \to L_1^\vee(t) \xrightarrow{\alpha} L_1 \to \mathcal{I}_C\otimes\mathcal{O}_{\mathbb{P}^4}\big(\det L_1-(s-1)t\big) \to 0.$$

Any smooth hypersurface X of degree d containing C supports an ACM rank two vector bundle E (with a global section whose zero scheme is C) which is determined by a $2s\times 2s$ skew symmetric map $L^\vee(t') \xrightarrow{\phi} L$, where $L = L_1(\det L_1 - d - st)\oplus\mathcal{O}_{\mathbb{P}^4}$, $t' = 2\det L_1-(2s-1)t$.

Proof. For the first part, see [17, Section 3]. C is arithmetically Gorenstein since L_1 is a sum of line bundles. $\omega_C = \mathcal{O}_C\big(2\det L_1-(2s-1)t-5\big)$. If X is a smooth hypersurface of degree d containing C, since C is sub-canonical, by Serre's lemma, it is the zero-scheme of a section of some rank two vector bundle E on X. E is ACM from

$$0\to\mathcal{O}_X\to E\to\mathcal{I}_{C/X}(e)\to 0,$$

where $e = 2\det L_1-d-(2s-1)t$. Lifting sections from $\mathcal{I}_{C/X}$ to E, we see that there is a surjection $L_1\big(-\det L_1+(s-1)t+e\big)\oplus\mathcal{O}_{\mathbb{P}^4} \to E \to 0$, and this gives the result. Note that in this proposition we are ignoring whether the skew-symmetric maps created are minimal or non-minimal. However, it is clear that if α is minimal (and general), and if the polynomial defining X in $\mathbb{P}^4$ is not a minimal generator of the ideal $I(C)$ in $\mathbb{P}^4$, then the resulting ϕ is also a minimal skew-symmetric map. □

Proposition 3.2.

Every type of ACM bundle in the list $\{1,\dots,17\}\setminus\{3,6,8\}$ of Proposition 2.3 exists on at least one smooth quintic threefold.

Proof. We have already mentioned the cases (10) and above. In the other cases, let $L_1 = \bigoplus_{i=1}^{2s-1}\mathcal{O}_{\mathbb{P}^4}(a_i)$. Then the bundle $(\wedge^2 L_1)(-t)$ is globally generated, and the general map $L_1^\vee(t)\xrightarrow{\alpha} L_1$ is a minimal map yielding a smooth curve C. By (1) of Lemma 2.1, with $d=5$, we get $\det L_1 = st+5-a_{2s}$. Hence we get $L_1 \twoheadrightarrow \mathcal{I}_C(5+t-a_{2s})$, for some smooth curve C. Since the largest minimal generator of the ideal $I(C)$ has degree $5+t-a_{2s}-a_{2s-1}$, whenever $t<a_{2s}+a_{2s-1}$, the general quintic hypersurface X containing C will be smooth with defining polynomial non-minimal in $I(C)$. This includes all cases except for case (5), and hence we are done in all of these cases.

In case (5), we take a different approach. To build the required matrix

$$\phi = \begin{bmatrix} 0 & c_{12} & c_{13} & c_{14} & l_{15} & l_{16}\\ -c_{12} & 0 & c_{23} & c_{24} & l_{25} & l_{26}\\ -c_{13} & -c_{23} & 0 & c_{34} & l_{35} & l_{36}\\ -c_{14} & -c_{24} & -c_{34} & 0 & l_{45} & l_{46}\\ -l_{15} & -l_{25} & -l_{35} & -l_{45} & 0 & 0\\ -l_{16} & -l_{26} & l_{36} & l_{46} & 0 & 0\end{bmatrix},$$

where c_{ij} are cubic forms, l_{ij} linear forms, start with a general choice for the 4×2 sub-matrix of linear forms. The 2 by 2 minors define the ideal of a smooth rational normal quartic curve in $\mathbb{P}^4$. A general quintic hypersurface containing this curve will be smooth with a defining polynomial given as a combination of the six minors with cubic coefficients. With appropriate signs and locations, these will be the six cubic forms c_{ij} in the matrix. Then ϕ will define the quintic threefold by its Pfaffian. □

4. Dominance of U over the space of quintics and counting problems

Beauville in [2] showed that the map $U_1 \to \mathcal{S}$, from the space $U_1 = \{(X, E) : E \text{ in case } (1)\}$ to $\mathcal{S}$ the parameter space of smooth quintic threefolds is dominant, with finite generic fibre. The method (indicated below) goes back to [1] and was a computation in the appendix in [2] by F.-O. Schreyer. For any rank two non-split ACM bundle E on a quintic threefold X, the module $\bigoplus_v H^2(\mathrm{End}(E)(v))$ is a module of finite length over the polynomial ring $S = k[X_0, \dots, X_4]$, with a unique generator in degree -5. It is isomorphic to $S(5)/\mathrm{Pfaf}_{n-2}(\phi)$, where ϕ is the skew-symmetric matrix giving (E, X), and $\mathrm{Pfaf}_{n-2}(\phi)$ is the ideal generated by the $(n-2)$-Pfaffians of ϕ, see [15]. To establish the dominance of the map, it suffices to show that $H^2(\mathrm{End}\, E) = 0$, or algebraically, that the vector space of all quintic polynomials is contained in the ideal $\mathrm{Pfaf}_{n-2}(\phi)$, when ϕ is a general skew-symmetric matrix in U_1. This was done by Schreyer using a computation with a computer algebra system.

As a consequence, the general quintic threefold X supports a finite number of rank two non-split ACM bundles E from case (1). Beauville points out that the exact number, constant for all quintic threefolds in a non-empty open subset of U_1, has not been calculated. (He calls it an instance of a generalized Casson invariant of X defined by Thomas [18].)

Schreyer's computer calculations can be repeated for each of the cases $\{1,\dots,17\}\setminus\{3,6,8\}$. This was done using the computer algebra system Macaulay2 [8], and we report the result as

Computational Fact.

The general quintic threefold in $\mathbb{P}^4$ supports a finite positive number of rank two ACM bundles, for each of the cases $\{1,\dots,17\}\setminus\{3,6,8\}$. (See Appendix.)

While the dominance of U over the space of quintics follows from this elementary machine computation, many cases have received a more qualitative proof. In addition, in some cases, the count of the number of bundles of each type on a general X has also been completed by enumerative methods. We review the literature.

Cases (10) and (15). A bundle E_{10} on a quintic threefold $X\colon f = 0$ for case (10) has a minimal resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-4) \oplus 3\mathcal{O}_{\mathbb{P}^4}(-1) \xrightarrow{\phi_{10}} \mathcal{O}_{\mathbb{P}^4}(3) \oplus 3\mathcal{O}_{\mathbb{P}^4} \to E_{10} \to 0.$$

An E_{15} has resolution

$$0 \to \mathcal{O}_{\mathbb{P}^4}(1) \oplus 3\mathcal{O}_{\mathbb{P}^4}(-2) \xrightarrow{\phi_{15}} \mathcal{O}_{\mathbb{P}^4}(-1) \oplus 3\mathcal{O}_{\mathbb{P}^4}(2) \to E_{15} \to 0.$$

The ACM bundle E_{10} on X gives rise to a matrix factorization of f, $\phi_{10}\psi = f \cdot I_L$. It is evident that ψ has the form ϕ_{15}. Hence these bundles come in pairs on the same X. In fact, restricting the resolution of E_{10} to X, we get $0 \to E_{15}(-3) \to L\restriction_X \to E_{10} \to 0$. Thus on a smooth quintic threefold X, there is a one-to-one correspondence between bundles $\{E_{10}\}$ and bundles $\{E_{15}\}$ on X.

E_{10} has a unique section in degree -3 which gives a line in X as zero-scheme, and conversely, given a line in X, there is a unique bundle of type E_{10}, by the Serre correspondence. (Since E_{10} is the image of the map $\phi_{15}\restriction_X$, we can read off the entries of the corresponding column of ϕ_{15} to find the zero-scheme.) Hence, on the generic quintic threefold, the number of bundles of type (10) is 2875 since it is well-known that the number of lines on a general X is 2875. Hence also, there are 2875 bundles of type (15) on X.

Cases (11) and (14). These cases are paired similarly on a fixed X. E_{14} has a unique section in degree -3 which gives a plane conic, and conversely, each conic on X gives rise to a unique E_{14}. Katz has proved that the general quintic threefold contains 609, 250 smooth conics (and no singular conics). Hence the number of such bundles of type (14) is 609, 250. Likewise, the number of bundles of type (11) is also 609, 250.

Cases (12) and (16). E_{12} has a unique section in degree -2 which gives a $(1, 2, 2)$ curve (of degree 4 and arithmetic genus 1). Kley [11] (see also [12]) shows that the general quintic threefold contains a finite number of such curves, all smooth elliptic quartics. Hence the general X supports a finite number of bundles E_{12}. Ellingsrud–Strømme have announced in [7] that using the description of the Hilbert scheme given by Vainsencher–Avritzer [19], they have made a

Chern class computation and shown that the number of smooth elliptic $(1,2,2)$ curves on a general quintic threefold is 3718024750. Hence also the number of ACM bundles in each of the cases (12) and (16).

Cases (13) and (17). E_{17} has a unique section in degree -2 which gives a $(2,2,2)$ curve on X. Clemens–Kley have proved in [6] that the general quintic contains at least one infinitesimally rigid curve of type $(2,2,2)$. Hence the number of bundles of type (13) or (17) on a general X depends on a count of the number of canonical curves of genus 5, degree 8 that lie on X. This has not yet been calculated.

Case (9). E_9 has no unique section in lowest degree. No qualitative proof of dominance is available. Nor a count.

Case (7). E_7 has two sections in lowest degree -1. No qualitative proof of dominance is available. Nor a count.

Case (5). While E_5 has two sections in lowest degree, we can still extract a unique curve from it. A construction of E_5 was given in the proof of Proposition 3.2. Any rational normal curve C in $\mathbb{P}^4$ is the zero locus of the 2 by 2 minors of a 4×2 matrix of linear forms. Given a smooth quintic hypersurface X containing C, the matrix ϕ_5 can be recreated (non-uniquely). Thus if $C\subset X$, we get a bundle E_5 on X. ϕ is non-unique since the same quintic may be expressed as a combination of the minors in different ways, because of the relations among the minors. Rather than directly show the bundles created for a pair $C\subset X$ are all the same by analyzing the action of $\operatorname{Aut} L$ on ϕ, we defer the issue, assuming that there is possible a continuous family of bundles E_5 all with the same $C\subset X$.

Given a smooth X with equation $f=0$, a matrix ϕ_5 and bundle E_5, the minors of the 4×2 matrix of linear forms must define a curve in X (of degree 4 and arithmetic genus 0). Indeed, if the zero locus of the minors contains a surface component S, since $S\subset X$ and $\operatorname{Pic} X=\mathbb{Z}$, S is given by $f=g=0$ in $\mathbb{P}^4$. Then the minors are all multiples of g, hence so is f, a contradiction. Likewise, if there is a threefold component to the zero locus of the minors.

Using Clemens' method of degenerations, Katz [10] has proved that the general quintic threefold X contains a finite number of rational quartic curves, all smooth and infinitesimally rigid in X, while Kleiman–Johnsen [9] point out that they will all be rational normal curves. Then each rational normal curve C will give rise to a pair (X,E_5) and vice-versa. Since we also have proved that the bundles E_5 are rigid on a general quintic threefold, this correspondence is one-to-one, settling the issue deferred earlier. Hence the count of ACM bundles on a general X in the case (5) equals the number of rational quartics on X, which has been calculated as 242467530000 by Kontsevich [13].

Case (4). E_4 has a unique section in degree -2 giving as zero-scheme a curve of degree 5, arithmetic genus 1. Kley [11] (see also [12]) has proved that the general quintic threefold X contains an isolated smooth elliptic curve C of degree 5. Since the parameter space of smooth elliptic quintic curves in $\mathbb{P}^4$ is irreducible, a dimension count using the incidence variety that parametrizes $C\subset X$ shows that on the general X, C must be arithmetically normal. Hence the general X contains a finite number of elliptic curves, in one-to-one correspondence with bundles on X of type (4). There is a conjectural count of the number of such curves in Bershadky *et al.* [3].

Case (2). E_2 has a unique section in degree -1. The zero-scheme C is a curve given by the ideal of Pfaffians of a 7×7 skew-symmetric matrix of linear forms, and has degree 14, arithmetic genus 15. According to Knutsen [12], the general quintic threefold X contains isolated smooth curves with these invariants. It is not clear to us whether these curves are arithmetically Gorenstein curves of the type we would require.

Case (1). E_1 has no unique section in lowest degree.

Appendix

In this appendix, the algebra software package Macaulay2 is used to perform the calculations mentioned in the beginning of Section 4. R will be the polynomial ring in five variables over the prime field with 32003 elements. We will describe the calculations of Schreyer, in `#1` below. The same arguments will apply to a reading of the calculations given below for the other cases in Section 4.

In `#1` below, a random 10 by 10 matrix of the type in case (1) of Proposition 2.3 is produced by the command 'random' in Macaulay2 followed by degree conventions as indicated in `#1`. It is skew-symmetrized to give the matrix M. Next, the ideal I of 8 by 8 Pfaffians of the matrix M is calculated. When $\dim I$ returns zero, we know that I has finite co-length and M defines a rank two bundle on some quintic hypersurface. The final step is to show that I contains all quintic polynomials. This is done by calculating the Betti numbers in a minimal free resolution of R/I. The output printed below shows, in Macaulay2's degree convention, that the last term of this free resolution is the graded free module consisting of 25 copies $R(-9)$. After sheafifying this exact sequence, the dimension of R/I in degree 5 is seen to be zero, since $R(-9+5)$, when sheafified, has no fourth cohomology.

```
R = ZZ/32003[a..e];

#1
A = random(R^{0,0,0,0,0,0,0,0,0,0},R^{-1,-1,-1,-1,-1,-1,-1,-1,-1,-1})
M= A - transpose A
I = pfaffians(8,M);
dim I
C = resolution I
i187 : betti C
              0  1   2   3  4  5
o187 = total: 1 45 124 145 90 25
           0: 1  .   .   .  .  .
           1: .  .   .   .  .  .
           2: .  .   .   .  .  .
           3: . 45  99  55  .  .
           4: .  .  25  90 90 25
```

The calculations in the remaining cases $\{2,\ldots,17\}\setminus\{3,6,8\}$ are listed in columns below. For typographical reasons, the arguments for the 'random' command are broken over two lines. It will be noticed that in each case, the last term of the free resolution is always a number of copies of $R(-9)$. Hence the same conclusion as in #1 above will hold, namely that the ideal of appropriately sized Pfaffians will contain all quintic polynomials.

```
#2
A = random(R^{1,0,0,0,0,0,0,0},
            R^{-2,-1,-1,-1,-1,-1,-1,-1})
M= A - transpose A
I = pfaffians(6,M);
dim I
C = resolution I
i181 : betti C
              0  1  2   3  4  5
o181 = total: 1 28 84 113 77 21
           0: 1  .  .   .  .  .
           1: .  .  .   .  .  .
           2: .  7  7   .  .  .
           3: . 21 49  28  .  .
           4: .  . 28  85 77 21

#4
A = random(R^{2,0,0,0,0,0},
            R^{-3,-1,-1,-1,-1,-1})
M= A - transpose A
I = pfaffians(4,M);
dim I
C = resolution I
i175 : betti C
              0  1  2  3  4  5
o175 = total: 1 15 40 51 35 10
           0: 1  .  .  .  .  .
           1: .  5  5  .  .  .
           2: .  .  .  1  .  .
           3: . 10 25 15  .  .
           4: .  . 10 35 35 10
```

```
#5
A = random(R^{1,1,1,1,-1,-1},
            R^{-2,-2,-2,-2,0,0})
M= A - transpose A
I = pfaffians(4,M);
dim I
C = resolution I
i169 : betti C
              0  1  2  3  4 5
o169 = total: 1 14 35 45 32 9
           0: 1  .  .  .  . .
           1: .  6  8  3  . .
           2: .  .  .  .  . .
           3: .  8 19  8  . .
           4: .  .  8 34 32 9

#7
A = random(R^{1,1,0,0,0,0},
            R^{-2,-2,-1,-1,-1,-1})
M= A - transpose A
I = pfaffians(4,M);
dim I
C = resolution I
i163 : betti C
              0  1  2  3  4  5
o163 = total: 1 15 52 84 63 17
           0: 1  .  .  .  .  .
           1: .  1  .  .  .  .
           2: .  8  8  .  .  .
           3: .  6 19 10  .  .
           4: .  . 25 74 63 17
```

```
#9
A = random(R^{1,1,1,1,1,0},
           R^{-1,-1,-1,-1,-1,0})
M= A - transpose A
I = pfaffians(4,M);
dim I
C = resolution I
i151 : betti C
               0  1  2  3  4  5
o151 = total: 1 15 55 96 75 20
           0: 1  .  .  .  .  .
           1: .  .  .  .  .  .
           2: . 10  5  1  .  .
           3: .  5 25  5  .  .
           4: .  . 25 90 75 20
```

```
#10
A = random(R^{3,0,0,0},R^{-4,-1,-1,-1})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i145 : betti C
               0 1  2  3 4 5
o145 = total: 1 6 14 16 9 2
           0: 1 3  3  1 . .
           1: . .  .  . . .
           2: . .  .  . . .
           3: . 3  9  9 3 .
           4: . .  2  6 6 2
```

```
#11
A = random(R^{2,1,1,-1},R^{-3,-2,-2,0})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i133 : betti C
               0 1  2  3  4 5
o133 = total: 1 6 16 22 15 4
           0: 1 2  1  .  . .
           1: . 1  2  1  . .
           2: . 1  2  1  . .
           3: . 2  5  4  1 .
           4: . .  6 16 14 4
```

```
#12
A = random(R^{2,1,0,0},R^{-3,-2,-1,-1})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i127 : betti C
               0 1  2  3  4 5
o127 = total: 1 6 20 33 25 7
           0: 1 1  .  .  . .
           1: . 2  2  .  . .
           2: . 2  3  1  . .
           3: . 1  5  4  . .
           4: . . 10 28 25 7
```

```
#13
A = random(R^{1,1,1,0},R^{-2,-2,-2,-1})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i121 : betti C
               0 1  2  3  4  5
o121 = total: 1 6 28 58 48 13
           0: 1 .  .  .  .  .
           1: . 3  .  .  .  .
           2: . 3  3  .  .  .
           3: . .  9  1  .  .
           4: . . 16 57 48 13
```

```
#14
A = random(R^{3,1,1,0},R^{-3,-1,-1,0})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i115 : betti C
               0 1  2  3  4 5
o115 = total: 1 6 16 22 15 4
           0: 1 2  1  .  . .
           1: . 1  2  1  . .
           2: . 1  2  1  . .
           3: . 2  5  4  1 .
           4: . .  6 16 14 4
```

```
#15
A = random(R^{2,2,2,-1},R^{-2,-2,-2,1})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i109 : betti C
               0 1  2  3 4 5
o109 = total: 1 6 14 16 9 2
           0: 1 3  3  1 . .
           1: . .  .  . . .
           2: . .  .  . . .
           3: . 3  9  9 3 .
           4: . .  2  6 6 2
```

```
#16
A = random(R^{2,2,1,0},R^{-2,-2,-1,0})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i97 : betti C
              0 1  2  3  4 5
o97 = total: 1 6 20 33 25 7
          0: 1 1  .  .  . .
          1: . 2  2  .  . .
          2: . 2  3  1  . .
          3: . 1  5  4  . .
          4: . . 10 28 25 7
```

```
#17
A = random(R^{2,1,1,1},R^{-2,-1,-1,-1})
M= A - transpose A
I = pfaffians(2,M);
dim I
C = resolution I
i103 : betti C
                0 1  2  3  4  5
o103 = total: 1 6 28 58 48 13
           0: 1 .  .  .  .  .
           1: . 3  .  .  .  .
           2: . 3  3  .  .  .
           3: . .  9  1  .  .
           4: . . 16 57 48 13
```

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