4-26-2019

Knot Groups of Torus Knots

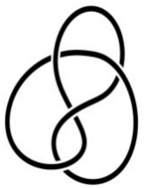
Alexander H. Galindo
University of Missouri-St. Louis, ahghyc@mail.umsl.edu

Knot Groups of Torus Knots

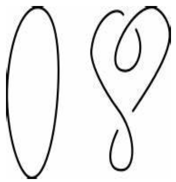
Presenter: Alexander Galindo (Department Math/CS, UMSL)
Adviser: Dr. Ronald Dotzel (Department Math/CS, UMSL)

What is a knot?

A Knot is an embedded circle in 3-dimensional space. In general it resembles a, possibly, tangled piece of string with the ends joined. A knot is "trivial" if it can be unraveled without cutting it. Two examples are pictured below.



Typical Knot



Two Unknots

What is a group?

A Group is a nonempty set together with a binary, associative operation.
It must also contain an "identity element" and each element of the group must have an inverse.

Example 1: A group is the set of integers under addition. The identity element is "0" and the inverse of an integer n is –n, because n+ (-n) = 0.

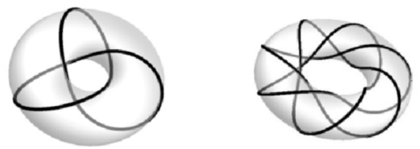
Example 2: The "integers mod m", {0, 1, 2, …,*m-1*} under the operation of "addition mod m". For instance if m = 6, then 3 + 5 = 2, 5 + 5 = 4 (i.e.. we take the remainder after the sum is divided by m). Again 0 is the identity and the inverse of k is *m-k*.

What is the fundamental group?

For any subset C of 3-dimensional space we can define the "Fundamental Group". This is a group constructed in such a way as to faithfully reflect some of the essential geometric features of the subset C. For example, if C has some holes in it. The fundamental group is the set of equivalence classes of closed paths or loops in C, two loops being "equivalent" if there is a "homotopy" which carries one loop to the other (intuitively, one loop can be distorted or morphed into the other without leaving the subset). The binary operation is "path addition", wherein the sum of two paths is simply the path resulting from following first one path then the other. The inverse of an equivalence class is the equivalence class of the "reverse" of a path in the original class. The identity element is the equivalence class of the "constant path" (which never leaves its base point).

How can we tell if a knot is really just the unknot?

The "complement of the knot" is 3- space minus the knot. Think of it a huge block of wood that has a "worm-hole" bored through it. An invariant of the knot is the fundamental group of the complement. The fundamental group of the unknot is "isomorphic" (ie. has the same structure) to the integers. This is a "free group" on one generator. A "torus knot" is a knot which can be drawn on the surface known as a torus (think inner tube). A torus knot winds a number of times in the longitudinal direction and a number of times in the meridinal direction. The fundamental group of a torus knot which winds p times around the longitude and q times around the meridian is a free group on two generators and subject to the single relation. Since this group is decidedly not isomorphic to the integers torus knots are not unknots.



Torus Knots