January 2024

Total Variation Flow in R^n Dimensions with Examples Relating to Perimeters of Level Sets

Luis Schneegans
University of Missouri-St. Louis, las9mr@mail.umsl.edu

Victoria Shumakovich
University of Maryland at College Park

Total Variation Flow in $\mathbb{R}^n$ Dimensions with Examples Relating to Perimeters of Level Sets

Luis Schneegans, Victoria Shumakovich

SUMaR 2022
Kansas State University
NSF Grant #DMS-1659123
Mentor: Dr. Marianne Korten

August 5, 2022

Our project focused on building upon the equations below, which were explored in 1 dimension [2]:

$$\frac{\partial u}{\partial t} = div\left(\frac{\nabla u}{|\nabla u|}\right) \tag{1}$$

$$u_t = sign(u)_{xx} \tag{2}$$

for $x \in \mathbb{R}^n$ and $t \in (0, \infty)$.

Equation (1) represents the Total Variation Flow and equation (2) represents the Sign Fast Diffusion Equation.

Introduction

Specifically, we were gaining insight into radial solutions in higher dimensions and how the perimeter of the level sets change.

Introduction

Specifically, we were gaining insight into radial solutions in higher dimensions and how the perimeter of the level sets change.

Applications:

- seeking radial solutions
- entropy solutions
- solving image denoising problems

Background

Theorem

For initial data $u(0,\rho)$, such as $\rho \cdot u(0,\rho)$ is monotone decreasing, there is a time, T, at which the function stabilizes to a constant or zero. This is called the ***extinction time*** *or* ***stabilization time****[4].*

Notation:

Some variables to know:

- $\rho =$ radius, $\rho \in (0, \infty)$
- $v =$ solution to Sign Fast Diffusion Equation (SFDE)
- $u =$ solution to Total Variation Flow (TVF)
- $n =$ dimension of $\mathbb{R}^n$
- $T =$ extinction time, or the time at which the solution stabilizes
- Note: $t \in [0, \infty)$ or $[0, T]$

Lemma 1

Let $u(t, x) = u(t, \rho)$, with $\rho := |x|$, be a radial, continuous, and piecewise smooth solution to the TVF, with $|u_\rho| > 0$ on sets of positive measure in $\mathbb{R}^n$ for each t. Let $n \geq 2$. Then, $\upsilon := u_\rho$ satisfies the SFDE

$$\upsilon_t = \left(\frac{(sign(\upsilon) \cdot \rho^{n-1})'}{\rho^{n-1}} \right)'. \tag{3}$$

Idea of Proof

1. Compute $\frac{\partial}{\partial x_i}\left(\frac{\nabla u}{|\nabla u|}\right) = \frac{u'(\rho)}{\rho \cdot |u'(\rho)|} \cdot \vec{x}$.
2. By taking the divergence we get $u_t = \frac{\left(sign(u'(\rho)) \cdot \rho^{n-1}\right)'}{\rho^{n-1}}$.
3. Remembering $\upsilon := u_\rho$ we get $\upsilon_t = \left(\frac{(sign(\upsilon) \cdot \rho^{n-1})'}{\rho^{n-1}}\right)'$.

Lemma 2

Let (t, ρ) vary in an open set. If $\rho \cdot u(0, \rho)$ is radially decreasing, then the solution to the TVF is [1]

$$u(t, \rho) = \left(u(0, \rho) - \frac{(n-1)}{\rho} t \right)_+ . \tag{4}$$

Idea of Proof

If $\rho \cdot u(0,\rho)$ is radially decreasing then $sign(u_\rho) = -1$ and

$$-\frac{(\rho^{n-1})'}{\rho^{n-1}} = -\frac{(n-1)\cdot\rho^{n-2}}{\rho^{n-1}} = -\frac{n-1}{\rho}$$

Idea of Proof

If $\rho \cdot u(0, \rho)$ is radially decreasing then $sign(u_\rho) = -1$ and

$$-\frac{(\rho^{n-1})'}{\rho^{n-1}} = -\frac{(n-1) \cdot \rho^{n-2}}{\rho^{n-1}} = -\frac{n-1}{\rho}$$

Thus, $u_t = -\frac{n-1}{\rho}$ (which is independent of t).

Idea of Proof

If $\rho \cdot u(0,\rho)$ is radially decreasing then $sign(u_\rho) = -1$ and

$$-\frac{(\rho^{n-1})'}{\rho^{n-1}} = -\frac{(n-1)\cdot\rho^{n-2}}{\rho^{n-1}} = -\frac{n-1}{\rho}$$

Thus, $u_t = -\dfrac{n-1}{\rho}$ (which is independent of t).

Hence, $\displaystyle\int_0^t u_s = -\frac{n-1}{\rho}t$, and

$$u(t,\rho) = \left(u(0,\rho) - \frac{(n-1)}{\rho}t\right)_+ .$$

Lemma 3

Assume $\rho \cdot u(0,\rho)$ is a monotone decreasing function. Then,

$$u(t,\rho) = \left(u(0,\rho) - \frac{(n-1)}{\rho} t\right)_+ .$$

is a solution to $u_t = \frac{\left(sign(u'(\rho)) \cdot \rho^{n-1}\right)'}{\rho^{n-1}}$ with the initial datum $u(0,\rho)$. Then,

1. For all t, $u(t,\rho) \leq u(0,\rho)$ for each ρ.

Lemma 3

Assume $\rho \cdot u(0,\rho)$ is a monotone decreasing function. Then,

$$u(t,\rho) = \left(u(0,\rho) - \frac{(n-1)}{\rho} t \right)_+ .$$

is a solution to $u_t = \frac{\left(sign(u'(\rho)) \cdot \rho^{n-1}\right)'}{\rho^{n-1}}$ with the initial datum $u(0,\rho)$. Then,

1. For all t, $u(t,\rho) \leq u(0,\rho)$ for each ρ.
2. As t increases, $u(t,\rho)$ decreases for fixed ρ.

Lemma 3

Assume $\rho \cdot u(0, \rho)$ is a monotone decreasing function. Then,

$$u(t, \rho) = \left(u(0, \rho) - \frac{(n-1)}{\rho} t \right)_+ .$$

is a solution to $u_t = \frac{\left(sign(u'(\rho)) \cdot \rho^{n-1} \right)'}{\rho^{n-1}}$ with the initial datum $u(0, \rho)$. Then,

1. For all t, $u(t, \rho) \leq u(0, \rho)$ for each ρ.
2. As t increases, $u(t, \rho)$ decreases for fixed ρ.
3. $u(0, \rho)$ has a finite extinction time if and only if $\rho \cdot u(0, \rho)$ is bounded [4].

Lemma 3

Assume $\rho \cdot u(0,\rho)$ is a monotone decreasing function. Then,

$$u(t,\rho) = \left(u(0,\rho) - \frac{(n-1)}{\rho}t\right)_+ .$$

is a solution to $u_t = \frac{\left(sign(u'(\rho))\cdot\rho^{n-1}\right)'}{\rho^{n-1}}$ with the initial datum $u(0,\rho)$. Then,

1. For all t, $u(t,\rho) \leq u(0,\rho)$ for each ρ.
2. As t increases, $u(t,\rho)$ decreases for fixed ρ.
3. $u(0,\rho)$ has a finite extinction time if and only if $\rho \cdot u(0,\rho)$ is bounded [4].
4. If $u(t,\rho) = \left(u(0,\rho) - \frac{(n-1)}{\rho}t\right)_+$ has a global maximum, then the profile of $u(t,\rho)$ decreases to 0 as t increases from 0 to T.

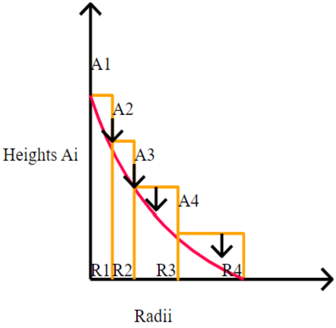


Figure: The following figure is a visual representation of how the maxima changes with time in a decreasing function. The horizontal axis shows the radius, R_i, and the vertical axis shows the heights, A_i. The pink curve is showing the monotone decreasing function with the orange rectangles representing $u_u(0, \rho)$ (the upper staircase above the curve). The arrows pointing downward represent how the maxima decrease with time. In addition, as the radii increase, the steps decrease at a slower rate.

Example #1:

Example

Let $\rho \cdot u(0,\rho)$ be monotone decreasing. We will show how $u(t,\rho)$ changes over time and point out how there is one case where no extrema are found. Let $u(0,\rho) = c_1 \frac{1}{\rho^2}$. Additionally, $n > 2$.

Outline of Work:

1. Using Equation (4), take $\dfrac{\partial u}{\partial \rho}$.
2. Set $u_\rho(t,\rho) = 0$ to find any potential extrema within the function.

Example #1:

Example

Let $\rho \cdot u(0, \rho)$ be monotone decreasing. We will show how $u(t, \rho)$ changes over time and point out how there is one case where no extrema are found. Let $u(0, \rho) = c_1 \frac{1}{\rho^2}$. Additionally, $n > 2$.

Outline of Work:

1. Using Equation (4), take $\dfrac{\partial u}{\partial \rho}$.
2. Set $u_\rho(t, \rho) = 0$ to find any potential extrema within the function.
3. Solving for ρ, we get: $\rho = -\dfrac{2c_1}{(n-1)t}$.

Example #2:

Example

Let $\rho \cdot u(0, \rho)$ be monotone decreasing. We will show how $u(t, \rho)$ changes over time for a generalized example and point out how this is one case for all $n > N > 1$, where $N \in \mathbb{Q}$, where no extrema are found. Let $u(0, \rho) = c_1 \frac{1}{\rho^N}$.

Outline of Work

1. Using Equation (4), take $\dfrac{\partial u}{\partial \rho}$.
2. Set $u_\rho(t, \rho) = 0$ to find any potential extrema within the function.

Example #2:

Example

Let $\rho \cdot u(0,\rho)$ be monotone decreasing. We will show how $u(t,\rho)$ changes over time for a generalized example and point out how this is one case for all $n > N > 1$, where $N \in \mathbb{Q}$, where no extrema are found. Let $u(0,\rho) = c_1 \frac{1}{\rho^N}$.

Outline of Work

1. Using Equation (4), take $\dfrac{\partial u}{\partial \rho}$.
2. Set $u_\rho(t,\rho) = 0$ to find any potential extrema within the function.
3. Solving for ρ, we get: $\rho = \sqrt[N-1]{-\dfrac{Nc_1}{(n-1)t}}$.

Example #3 (Volcano Case):

Example

Keeping the level ℓ fixed, as t increases, the perimeter of the level sets does not change as t increases until the graph of $u(x,t)$ no longer intersects the plane $u(x,t)=\ell$. When ℓ is not fixed, the perimeter of level sets increases with t.

We will use the particular solution of the equation below found in [4].

$$u(x,t)=\begin{cases} \frac{(n-1)(T-t)}{|x|} & |x| > \left(\frac{1}{(n-1)(T-t))^{\frac{1}{(n-1)}}}\right) \\ [(n-1)(T-t)]^{\frac{n}{n-1}} & |x| \leq \frac{1}{[(n-1)(T-t)]}^{\frac{1}{n-1}} \end{cases}.$$

Example #3 (Volcano Case):

Key Notes

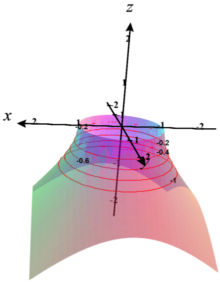
1. If $\ell_1 < \ell_2$, then the level curve $u(x, t_0) = \ell_1$ has a larger perimeter than the level curve $u(x, t_0) = \ell_2$ as long as the sets are not empty.

Example #3 (Volcano Case):

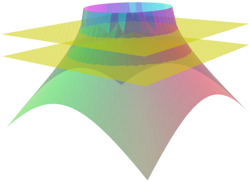
Key Notes

1. If $\ell_1 < \ell_2$, then the level curve $u(x, t_0) = \ell_1$ has a larger perimeter than the level curve $u(x, t_0) = \ell_2$ as long as the sets are not empty.
2. For $\ell_2 = \frac{1}{[(n-1)(T-t_0)]}^{\frac{1}{n-1}}$ the intersection of the profile of $u(|x|, t_0)$ with the plane $u(x, t_0) = \ell_2$ is a disc. For $\ell_2 > \frac{1}{[(n-1)(T-t_0)]}^{\frac{1}{n-1}}$ the intersection of the profile of $u(|x|, t_0)$ with the plane $u(x, t_0) = \ell_2$ is the empty set.

Example #3 (Volcano Case):



(a) The following is a 3D representation of the "volcano" shape with contour lines. As you go down the volcano, the circle contours become larger.



(b) Analogous to the circle contours, the perimeters of the level sets become larger when ℓ decreases.

Example #3 (Volcano Case):

Work:

1. The perimeter of a level set is equal to $2\pi r(t)$ where $r(t) = \frac{1}{[(n-1)(T-t)]^{\frac{1}{n-1}}}$.
2. Thus, the perimeter at a specific ℓ is $\frac{2\pi}{[(n-1)(T-t)]^{\frac{1}{n-1}}}$.

Example #3 (Volcano Case):

Work:

1. The perimeter of a level set is equal to $2\pi r(t)$ where $r(t) = \frac{1}{[(n-1)(T-t)]^{\frac{1}{n-1}}}$.
2. Thus, the perimeter at a specific ℓ is $\frac{2\pi}{[(n-1)(T-t)]^{\frac{1}{n-1}}}$.
3. Lastly, you can observe the behavior of the level sets given the equation.

Acknowledgements

We would like to thank Dr. Marianne Korten for being our mentor for this project. We would also like to extend our thanks to Dr. David Yetter and Dr. Kim Klinger-Logan, in addition to Dr. Korten, for running this REU and giving us this opportunity.

Lastly, we would like to thank the NSF foundation for funding the SUMaR REU, under the NSF Grant #DMS-1659123.

Bibliography

D.Herrera, M.Korten, N.Saal.
New Approximation Results and Barriers for the Total Variation Flow. *Preprint*, 2016.

A. J. Davis, M. K. Korten, A. J. Talbert, and R. C. Tenaglia.
Stabilization Times for the Total Variation Flow and the Sign Fast Diffusion Equation. *Preprint*, 2019.

M. Bonforte and A. Figalli.
Total variation flow and sign fast diffusion in one dimension. *Journal of Differential Equations*, 252(8):4455–4480, 2012.

M. Korten,C. Moore, P. Salazar.
Extinction Times of the Total Variation Flow. *Preprint*, 2010.

D. Herrera, M. Korten, J. Luong, R. McConnell, N. Saal, and J. Vesta.
New Approximation Results and Barriers For the Total Variation Flow.
Preprint, 2018.

F. Andreu-Vaillo, V. Caselles, J. M. Mazón.
Parabolic Quasilinear Equations Minimizing Linear Growth Functions *Birkhäuser Verlag*, 2004.

Questions?

Thank you!