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Partial Connectivity in Wireless Sensor Networks

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Partial Connectivity in Wireless Sensor Networks with Applications

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Abstract

Given a bounded subset $\mathcal{B} \subset \mathbb{R}^2$, a discrete set of nodes $\{x_k\}_{1 \leq k \leq n} = \mathcal{X}_n \subset \mathcal{B}$ is uniformly distributed throughout $\mathcal{B}$ and each $x_k \in \mathcal{X}_n$ is generated at each time instant $k \in \{1, 2, ..., n\}$ according to the Poisson distribution. For some fixed radius $r > 0$, two nodes $x, y \in \mathcal{X}_n$ are said to *connect* and form an *edge* if $d(x, y) \leq r$. The graph of all such connected nodes $x, y \in \mathcal{X}_n$ is denoted by $G(\mathcal{X}_n; r)$. Given $C \subseteq \mathcal{X}_n$, it is said that C forms a *connected cluster* if given $x, y \in C$, there exists a set of edges connecting x to y. If $\rho \in (\frac{1}{2}, 1)$ is chosen, then it is shown that there exists a radius $r_0 = r_0(n, \rho)$ such that the probability is $\frac{1}{2}$ for the occurrence of a connected cluster in $\mathcal{X}_n$ of order at least $N \leq n$ such that $\frac{N}{n} \geq \rho$. Furthermore, if $\mathbb{R}^2$ is partitioned into subsets of conjoined hexagons of a size chosen so that each can be inscribed into a circle of radius $\frac{r}{4} > 0$, then by defining $x, y \in \mathcal{X}_n$ to be *connected* if x and y are in the same or conjoined hexagons, the new graph denoted by $G(\mathcal{X}_n; *)$, is shown to be contained in the original graph, $G(\mathcal{X}_n; r)$. It is shown that there exists a radius $r_0^* = r_0^*(n, \rho)$ such that the probability is $\frac{1}{2}$ for the occurrence of a connected cluster in $\mathcal{X}_n$ of order at least $N \leq n$ in the hexagonal paradigm. It will be proven that $r_0 \leq r_0^*$ and subsequently, $r_0 = r_0^*$. By proving results similar to Cai [5] theorems 3.2.3 and 3.3.1, the length of the sharp threshold interval in the hexagonal paradigm can be estimated. Then, by using the relationship between the radii in the continuum and hexagonal paradigms, the length of the sharp threshold interval in the continuum paradigm is established.

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To myself, I would say, "Always remember, in all thy getting, get an understanding."

Contents

Chapter 1

Introduction

1.1 Motivation

The issues of redundancy, interference and isolation are of paramount importance in the study of coverage and connectivity amongst randomly deployed sensors in some large, bounded area. Each node in a network connects to others within its communications radius and transmits information across the network in a series of hops. If each node in the network has a common communications radius that is too large, then there will be a tendency for covered areas to overlap with multiple nodes transmitting the same information to multiple nodes within the network causing packet collisions and information loss $[12-14, 18]$. In contrast, if the common communications radius is too small, then large areas in the bounded region will be left unmonitored with information only being transmitted to other nodes within smaller, isolated subregions $[12, 18]$. Given a fixed number of nodes, the last statements imply that there is a smallest radius of connectivity r_c, such that if each of the nodes has a common communications radius $r < r_c$, then interconnectivity amongst nodes will be sparse and isolated, while full connectivity occurs in the event that $r > r_c$ $[16-17]$. Equivalently, given a common, fixed radius of connectivity, there is a smallest number of randomly distributed nodes n_c, such that interconnectivity amongst nodes will be sparse and isolated when $n < n_c$, while full connectivity occurs in the event that $n > n_c$ [23].

Because of concerns over the cost per unit for wireless sensor nodes, network lifetime and efficient utilization [21], it is prohibitively expensive to deploy a large number of sensor nodes in order to effectively provide failover for both coverage and connectivity in the event that certain subsets of nodes become inoperative. As such, instead of requiring full connectivity of all nodes, in [5], Cai, et.al. use a hexagonal lattice structure to model interconnectivity amongst randomly distributed nodes throughout some large, bounded region. Therein, connectivity ensues between nodes within the same or neighboring hexagons. With this setup, it is proven in [5] that there is a critical density of nodes in the lattice structure such that the probability is $\frac{1}{2}$ for the occurrence of the event that at least half of all the sensor nodes are connected. This probability of $\frac{1}{2}$ corresponds to the center of the interval in r about which the probability of the aforementioned event increases from some small positive value to a value very close to 1. An estimation of the length of this interval is given in [5]. This dissertation is concerned with providing a similar analysis without an assumed lattice structure.

1.2 Dissertation Plan

Estimation of the length of the interval about the critical radius in the continuum will be accomplished in several steps.

1) Define the concept of an open edge between nodes and the concept of a connected component about a node and an open edge.

2) Define a set of connected clusters with the property that the number of nodes is at least half of all of available nodes and show that the set is non-empty.

3) With a suitable probability measure, show that the probability that the set of connected clusters is non-empty is a continuous function of the distance between nodes.

4) With continuity established for the probability measure as a function of the distance between nodes, show that there is a radius such that the set of connected clusters is non-empty with probability $1/2$.

5) State the main theorem to be proven which regards an upper and lower bound on the probability that the set of connected clusters is not empty.

6) Define a hexagonal lattice structure where the common size of each hexagon is a function of the connection radius of an inscribed circle.

7) Show results similar to those in numbers $1-5$ *in the presence of the hexagonal structure.*

8) To prove the analogue to number 5 *in the presence of a hexagonal lattice structure, first define a torus structure on the lattice to eliminate any boundary connectivity conditions.*

9) In the presence of the torus, prove the hexagonal analogue to number 5*.*

10) Show an inequality relationship between radii from and the hexagonal analogue to 4 *and use this result to prove* 5*.*

1.3 Dissertation Organization

In chapter (2), results from selected related works are presented and summarized. In chapter (3), some needed background material and results from random graph theory, site percolation on a lattice and random field theory are presented. In chapter (4), results are presented showing that the probability measure is a continuous function of the radius r and the existence of a particular radius r_0 such that the probability is $\frac{1}{2}$ for the occurrence of the event that at least half of all nodes are connected in a cluster and the theorem regarding the length of the sharp threshold interval about r_0 is stated. Chapter (5) presents similar results as those in chapter (4) within a different connection model framework that is based upon partitioning the bounded region into hexagons and the length of the sharp threshold interval in this framework is stated and proven. Chapter (6) presents results that establish a relationship, $r_0 \leq r_0^*$ and the length of the sharp threshold interval about r_0 is established. In Chapter 7, an application is presented in the realm of data classification whereby an upper bound for a critical radius is found such that correlated data are classified into at least a fixed number of specified classes. Chapter (8) establishes a genetic algorithmic framework for continuation of the results presented in this dissertation. Lastly, the *Appendix* chapter contains supporting results about the increasing event and probability measure for both the continuum and hexagonal connectivity models.

Chapter 2

Related Work

2.1 Connectivity by Node Density and Radial Distance

In [5], Cai, et. al. define two events on a region of bounded area which will be assumed to be of area 1 without loss of generality. The first event is that there is a connected cluster of nodes totaling at least one half of all generated nodes. Therein, it is shown that there is a critical density of nodes λ_0 such that this event occurs with probability $\frac{1}{2}$. For densities $\lambda \geq \lambda_0$, it is shown that connected clusters of nodes in numbers slightly less than $\frac{1}{2}$ of all available nodes occur with high probability approaching 1 that depends upon the number of hexagons in the bounded region and the densities λ and λ_0. Likewise, for densities $\lambda \leq \lambda_0$, it is shown that connected clusters of nodes in numbers slightly more than $\frac{1}{2}$ of all available nodes occur with low probability approaching 0 that depends upon the number of hexagons in the bounded region and the densities λ and λ_0. The second event is that there is a connected cluster of nodes such that the total covered area by the occupied hexagons in the connected cluster is at least $\frac{1}{2}$ of the bounded region. Therein, it is shown that there is a critical density of occupied hexagons λ_0^H such that this event occurs with probability $\frac{1}{2}$. For densities $\lambda \geq \lambda_0^H$, it is shown that a connected cluster of hexagons of total area $\frac{1}{2}$ occurs with high probability approaching 1 that depends upon the number of hexagons in the bounded region and the densities λ and λ_0^H. Likewise, for densities $\lambda \leq \lambda_0^H$, it is shown that connected clusters of hexagons of total area $\frac{1}{2}$ occurs with low probability approaching 0 that depends upon the number of hexagons in the bounded region and the densities λ and λ_0^H. For both events defined, it is shown that the associated probability of occurrence sharply increases from some small positive value to a value close to 1 on some interval of constant length about the critical density. For both events, the length of the interval about the critical density is shown to be proportional to the reciprocal of the log of the area of a prototypical hexagon.

A similar analysis proceeds in this dissertation whereby similar events are defined in the context of random geometric graphs and a critical number of nodes, equivalently critical radius depending upon the number of nodes, is found for each event defined such that each event occurs with probability $\frac{1}{2}$. The length of the interval about the critical radii such that the probability of occurrence increases sharply is shown to be proportional to the product of the critical radii and a sublogarithmic factor of the number of nodes. It is also shown that the length of the interval about the critical radii is related to the minimax distance between two sets of $\frac{n}{2}$ nodes, when the set of n nodes is divided a certain way.

In [15], Bettstetter begins with the basic problem of obtaining a minimum node density such that every individual node has at least one neighbor to which it connects. Additionally, he investigates the problems of the minimum node density required such that each node has at least N neighbors and the minimum node density such that the network will have no operational isolated nodes with high probability in the event that a certain number of nodes fail. As such, a closed-form, analytical expression is derived for a lower bound on the critical radius of connectivity r_c, such that every node has at least one node to which it connects with a certain probability, p. Likewise, a closed form expression is found for the probability that each individual node has at least N neighbors.

Similarly, in this dissertation, a critical node density is found such that at least N nodes are connected with probability $\frac{1}{2}$. A radial connectivity distance is shown to exist such that this event occurs with said probability and an upper bound on the required radial distance is found as a function of the minimax distance between nodes in two partial subsets of all nodes.

2.2 Connectivity by Sector-Based SubConnectivity

In [11], Xue, et.al. use a sector-based strategy, and full connectivity within the sector, in order to address the problem of full connectivity within some bounded region. For any small, fixed $\delta > 0$, the subconnectivity function

$$\phi_n^+ = (1+\delta)\log_{\frac{2\pi}{2\pi-\theta}}(n)$$

is defined. They show that if each individual node is able to connect to each of its ϕ_n^+ nearest neighbors that lie within a sector of angle $\theta \in (0, 2\pi)$, with the exception of those nodes sufficiently close to the boundary of the sector defined by θ, then all nodes will be asymptotically connected such that an infinite cluster of connected nodes exists almost surely. Additionally, the subconnectivity function

$$\phi_n^- = (1-\delta)\log_{\frac{2\pi}{2\pi-\theta}}(n)$$

is defined and it is shown that the network of nodes is asymptotically disconnected such that only isolated connected clusters of nodes exist, almost surely. Clearly,

$$\begin{aligned}
\phi_n &\equiv \lim_{\delta\to 0^+}\phi_n^- \\
&= \lim_{\delta\to 0^+}(1-\delta)\log_{\frac{2\pi}{2\pi-\theta}}(n) \\
&= \log_{\frac{2\pi}{2\pi-\theta}}(n)
\end{aligned}$$

demarcates the critical phenomenon for full connectivity using this sector based strategy. Furthermore, using other geometric arguments, it is shown that the network will be fully connected if each individual node connects to its ϕ_n^π nearest neighbors, where

$$\begin{aligned}
\phi_n^\pi &\equiv \lim_{\theta\to\pi}\phi_n^+ \\
&= \lim_{\theta\to\pi}\lim_{\delta\to 0^+}\phi_n^+ \\
&= \lim_{\theta\to\pi}\lim_{\delta\to 0^+}(1+\delta)\log_{\frac{2\pi}{2\pi-\theta}}(n) \\
&= \lim_{\theta\to\pi}\log_{\frac{2\pi}{2\pi-\theta}}(n) \\
&= \log_2(n).
\end{aligned}$$

Therefore, if all nodes in either half of the region are fully connected, then with high probability, all nodes will be fully connected. This suggests some correlation between the events of full connectivity in either half plane. As is turns out, this is a true statement.

In this dissertation, lemma 2.1 from [7] is used. It states that if the maximum distance between two sets of independent, uniformly distributed nodes is small, with high probability, then the connectivity graph of nodes in one set is a subgraph of the connectivity of nodes in the other set, with high probability. Herein, the theorem is applied to both halfs of a set of n nodes which are divided evenly and it is shown that full connectivity occurs when the radius of connectivity is the sum of r_c plus half this maximum distance.

In [43], Han et. al. consider a point coverage problem whereby points are scattered throughout the two dimensional plane and the minimum number of sensors required to cover the scattered points is determined. If each node has a certain communications range r and a connectivity sector $s = s(r, \theta)$ for a given node is defined as a radial line of length r extending from the node together with a sweeping angle θ, then it is shown that a finite set of n points is completely covered by $O(\log(n))$ distinct connectivity sectors. In addition, it is shown that an infinite cluster of connected nodes exists if each connectivity sector s contains a hexagonal region h such that $\text{area}(h)/\text{area}(s) \geq .827$ and h defines the overlapping communications ranges of nodes contained therein.

In this dissertation, theorem 2.48 from [1] is used. It states that the probability of the occurrence of an increasing event will increase from some small positive value to a value very close to 1 on some small interval. Furthermore, the amount of the increase is on the order of $O(\log(n))$. As such, each node contributes $O(\log^{-1}(n))$ to the increase of the probability on the small interval. Much in the same way, each sector in [43] contributes $O(\log^{-1}(n))$ to the sharp increase of the probability of an increasing event on some small interval.

2.3 Other Works

In [44], Bhondekar, et. al. use a genetic algorithm for optimizing a self-organizing wireless sensor network design. At each step, the algorithm randomly selects individual nodes to be parent nodes according to a biological evolutionary process in order to produce child nodes in a connected cluster with said parent. A selection of an optimal node configuration is made, subject to some design constraint function f that is a function of the coverage area, node overlap errors, isolated node errors, master-node ratio and network energy. An integer lattice is assumed with identical nodes placed either randomly or deterministically at integer coordinates in some bounded, square region. The nodes can operate in one of three modes, X-mode, Y-mode and Z-mode. X nodes transmit information gathered from Y and Z nodes back to base stations via multihop communication. As such, the configured communications radius for X nodes is assumed to be much larger than that of either Y or Z nodes and the number of X nodes is assumed to be much smaller than the number of Y or Z nodes. The constraint function f is a measure of the quality of the network topology and is a weighted sum of the design parameters, coverage area, node overlap errors, isolated node errors, master-node ratio and network energy. Multiplying coefficients for the optimal design parameters were determined experimentally and found to be $-4.0, 0.5, 10.0, -1.0, 1.0$, respectively. Each node is randomly assigned a 2 bit binary representation so that a network configuration consists of a binary word. An optimal subset of 300 network configurations from the sample space are then changed randomly via a genetic mutations algorithm until a configuration is found that minimizes the constraint function, f. It was shown that each of the design parameters can be minimized in less than 3000 mutations to the subset of network configurations from the sample space.

A genetic algorithm paradigm will be used in a future extension of some of the results in this dissertation. As such, a little more detail will be given in a later section. In the interim, suppose that n nodes are uniformly distributed throughout some bounded region, $\mathcal{B}$. As in [44], the network is self-organizing and each of the nodes operates in one of X, Y or Z mode. Given fixed communications radii r_y and r_z on Y and Z nodes respectively, it will be shown that there is a critical radius $r_{0,x} = r_{0,x}(n, \rho, r_y, r_z)$ on X nodes such that at least half of all nodes are connected with probability $\frac{1}{2}$. Furthermore, an estimate of the length of the sharp threshold interval will be computed with techniques developed within the paradigm. Using a genetic algorithm, a representative node configuration will be found such that coverage area, node overlap errors, isolated node errors, master-node ratio and network energy are minimized.

Chapter 3

Review of Important Concepts

3.1 Random Geometric Graphs

Definition A *node process* is a mapping $X : \mathbb{R}^2 \to \mathbb{R}^2$ such that for $\mathcal{B} \subset \mathbb{R}^2$, there is an $n \in \mathbb{N}$ and a subset $\mathcal{X}_n = \{x_k\}_{1 \leq k \leq n} \subset \mathcal{B}$ such that $X(\mathcal{B}) = \mathcal{X}_n$.

Definition Suppose $\mathcal{B} \subset \mathbb{R}^2$ and X is a node process that randomly generates independent nodes $\mathcal{X}_n = \{x_k\}_{1 \leq k \leq n} \subset \mathcal{B}$ according to some probability distribution. Nodes $x, y \in \mathcal{X}_n$ are said to be *r-connected* and form an *r-open edge* if $d(x, y) \leq r$, for some fixed $r > 0$. Nodes $x, y \in \mathcal{X}_n$ are *r-disconnected* and form an *r-closed edge* otherwise.

Definition Let E be the set of edges between nodes in $\mathcal{X}_n$. $G(\mathcal{X}_n; r)$ is defined to be the *r-graph* of the set of all r-open and r-closed edges from E between nodes in $\mathcal{X}_n$.

Definition Given nodes $x, y \in \mathcal{X}_n$, denote the edge between x and y as $< x, y >_r$. A subset of nodes $C \subseteq \mathcal{X}_n$ forms an *r-connected cluster* if and only if given any $x, y \in C$, there exists r-open edges $< x, a_1 >_r, < a_1, a_2 >_r, ..., < a_{k-1}, y >_r \in E$ connecting x to y, for nodes $\{a_1, a_2, ..., a_{k-1}\} \subseteq C$.

Definition Let A be a set of graphs of E and $G(\mathcal{X}_n; r) \in A$. A is said to be an *increasing property* if and only if for $r' \neq r$ such that $G(\mathcal{X}_n; r) \subseteq G(\mathcal{X}_n; r')$ it is true that $G(\mathcal{X}_n; r') \in A$.

Definition Let P be a probability measure on $(\Omega, \mathcal{F})$. If A is a monotone (increasing) property and $\epsilon \in (0, \frac{1}{2})$, define

$$r(n, \epsilon) = \inf\{r > 0 : P(G(\mathcal{X}_n; r) \in A) \geq \epsilon\} \tag{3.1}$$

and

$$\Delta(n, \epsilon) = r(n, 1 - \epsilon) - r(n, \epsilon). \tag{3.2}$$

If $\Delta(n,\epsilon) = o(1)$, then A has a *sharp threshold.*

Theorem 3.1.1 *(Theorem 1.1* [7]*) For increasing properties A consisting of graphs of nodes $\mathcal{X}_n \subset \mathbb{R}^2$,*

$$\Delta(n,\epsilon) = O(r_c \log^{1/4}(n))$$

where

$$r_c \propto \sqrt{\frac{\log n}{n}}.$$

According to theorem 3.1.1, there is a critical radius of connectivity $r_c(n) > 0$ such that $P(A) > 0$. As such, there exists a short interval of radii such that $P(A)$ increases from some small positive value, to a value close to 1, on this interval. This dissertation will be concerned, at least in part, with estimating the length of this critical interval for a particular event A, using this framework.

Definition Let X be a node process. Suppose X generates $\mathcal{X}_n$ and $\mathcal{Y}_n$ independently and uniformly in $\mathcal{B}$. A *bottleneck matching* between $\mathcal{X}_n$ and $\mathcal{Y}_n$ is a bijection $b : \mathcal{X}_n \to \mathcal{Y}_n$ such that $\max\{d(x,y) : x \in \mathcal{X}_n, y = b(x) \in \mathcal{Y}_n\}$ is minimized.

Theorem 3.1.2 *(Theorem 1.3* [7]*) Let M_n denote the bottleneck matching between $\mathcal{X}_n$ and $\mathcal{Y}_n$. Then,*

$$M_n = \Theta(r_c \log^{1/4}(n))$$

where r_c is given by theorem 3.1.1.

According to theorem 3.1.2, an estimate of the length of the bottleneck matching is on the order of the length of the sharp threshold interval.

Lemma 3.1.3 *(Theorem 2.1* [7]*) Let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be independent, random samples of a node process X and suppose $P(M_n > \gamma(n)) \leq p$ for some function $\gamma(n)$ and some $p \in (0, \frac{1}{2})$. For any radius $r > 0$, if $G^r_{\mathcal{X}_n}$ is distributed as $G(\mathcal{X}_n; r)$ and $G^{r+2\gamma(n)}_{\mathcal{Y}_n}$ is distributed as $G(\mathcal{Y}_n; r + 2\gamma(n))$, then*

$$P(G^r_{\mathcal{X}_n} \subseteq G^{r+2\gamma(n)}_{\mathcal{Y}_n}) \geq 1 - p.$$

Lemma 3.1.3 is a statement that if the probability is small for the event of the bottleneck matching between $\mathcal{X}_n$ and $\mathcal{Y}_n$ being larger than some predefined value, then the probability is large for the event that the corresponding graphs have a containment relation.

3.2 Site Percolation on a Lattice

Let $\mathbb{L}^2 = (V, E)$ be any lattice on $\mathbb{R}^2$, where V is a set of vertices and E is the corresponding set of edges between neighboring nodes on vertices in V. Let $\mathcal{X} = \{-1, 1\}$ be a state space of values on nodes across vertices in V and suppose nodes take the value 1 independently with probability $p \in (0, 1)$.

Definition Nodes on vertices $a, b \in V$ are said to be *connected* and the edge $e = < a, b > \in E$ is said to be *open* if and only if $x_a = 1 = x_b$ for states, $x_a, x_b \in \mathcal{X}$ on a and b, respectively.

Definition Nodes on a subset of vertices $C \subseteq V$ form a *connected cluster* if and only if given any $a, b \in C$, there exists open edges $e_1 = < a, a_1 >, e_2 = < a_1, a_2 >, ..., e_k = < a_{k-1}, b > \in E$ connecting a to b, for vertices $\{a_1, a_2, ..., a_{k-1}\} \subseteq C$.

Suppose nodes on vertices $C \subseteq V$ is any connected cluster containing the origin and $\omega : E \to \mathcal{X}_E = \{-1, 1\}$ is any function such that $\omega(e) = 1$ if $e \in E$ is open and $\omega(e) = 0$ otherwise, independently of all other edges $f \in E \backslash \{e\}$. For any edge $e \in E$, let μ_e be Bernoulli measure such that $\mu_e(\omega(e) = 1) = p$ and $\mu_e(\omega(e) = 0) = 1 - p$. If $\Omega = \{(\omega(e))_{e \in E}\}$ is the sample space of random outcomes on the edge space, then define a probability measure on Ω to be the product measure

$$P(\cdot) = \prod_{e \in E} \mu_e(\cdot). \tag{3.3}$$

For any event A consisting of elements from Ω, the probability $P(A)$ is defined to be

$$P(A) = \sum_{\omega \in A} P(\omega). \tag{3.4}$$

From [2], the principal quantity of interest is the *percolation probability* given by

$$\theta(p) = P(|C| = \infty) = 1 - \sum_{k=1}^{\infty} P(|C| = k). \tag{3.5}$$

If $\theta(p) > 0$, then with probability 1 there exists a unique connected cluster $C \subseteq V$ such that $|C| = \infty$.

Theorem 3.2.4 *(Theorem 1.11* [2]*) The probability $\psi(p)$ that there exists an infinite connected cluster satisfies $\psi(p) = 1$ if $\theta(p) > 0$ and $\psi(p) = 0$ otherwise.*

The central question that is motivated by the percolation probability of equation 3.5 is then, "What values of $p \in (0,1)$ guarantee that $\theta(p) > 0$?" According to [2], for a lattice on $\mathbb{R}^2$, there exists a unique value $p_c \in (0,1)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$, where

$$p_c = \sup\{p \in (0,1) : \theta(p) = 0\} \tag{3.6}$$

by definition.

Definition Let Π be a group of permutations of E and $\pi = (\pi_e)_{e \in E} \in \Pi$. For $\omega \in \Omega$, the permutation π acts on ω by $\pi\omega = \omega(\pi_e)$. Suppose each edge $e \in E$ is enumerated. If $\mathcal{A} \subseteq \Pi$ is any subgroup, then $\mathcal{A}$ *acts transitively on* E if there exists $\alpha = (\alpha_j)_{1 \leq j \leq |E|} \in \mathcal{A}$ such that $\alpha_j = k$ for all pairs $j, k \in E$.

$\pi \in \Pi$ is a re-ordering of all edges, $e \in E$. If $\mathcal{X}$ is a state space of values on the set of edges such that $\Omega = \mathcal{X}^E$, then $\pi\omega \in \Omega$ is the corresponding re-ordering of the element $\omega \in \Omega$ on all edges, $e \in E$.

Definition Let $\mathcal{F}$ be a set consisting of subsets of Ω such that $\emptyset, \Omega \in \mathcal{F}$. If the union of every countable collection of pairwise disjoint elements from $\mathcal{F}$ and the intersection of every finite collection of pairwise disjoint elements from $\mathcal{F}$ are both elements of $\mathcal{F}$, then $\mathcal{F}$ is called a *σ-algebra* of subsets of Ω.

Definition Let Π be any group of permutations of E and $\mathcal{A} \subseteq \Pi$ any subgroup. If $\mathcal{F}$ is a σ-algebra of subsets of Ω, then a probability measure P on $(\Omega, \mathcal{F})$ is called *$\mathcal{A}$-invariant* if

$$P(\omega) = P(\alpha\omega)$$

for all $\alpha \in \mathcal{A}$ and $\omega \in \Omega$. An event is called *$\mathcal{A}$-invariant* if $A = \alpha A$ for all $\alpha \in \mathcal{A}$ and $A \in \mathcal{F}$.

Definition Let $\mathcal{F}$ be a σ-algebra of subsets of Ω and $A \in \mathcal{F}$. If $\omega \in A$ and $\omega(e) \leq \omega'(e)$ for all $e \in E$ implies $\omega' \in A$, then A is called an *increasing event.*

Theorem 3.2.5 *(Theorem 2.48* [1]*) There exists a constant $c \in (0, \infty)$ such that the following holds. Let $\mathcal{F}$ be a σ-algebra of subsets of Ω and $A \in \mathcal{F}$ any increasing event. Suppose $N = |E| \geq 1$ and P is a strictly positive and monotone probability measure on $(\Omega, \mathcal{F})$. If $\mathcal{A} \subseteq \Pi$ is any subgroup acting transitively on E such that P and A are $\mathcal{A}$-invariant, then*

$$\frac{d}{dp}P(A) \geq \frac{cm_p}{p(1-p)} \min\{P(A), 1 - P(A)\} \log N$$

where $m_p = P(J_e)(1 - P(J_e))$ and J_e is the event that $e \in E$ is open.

If A is an increasing event and $P(A) = \epsilon \in (0, \frac{1}{2})$, then theorem 3.2.5 requires there to be a short interval, as a subset of $(p_c, 1)$, such that $P(A)$ increases sharply from ϵ to $1 - \epsilon$ on this interval. According to theorem 3.2.5, the length of this sharp threshold interval is $O(1/\log N)$.

3.3 Random Fields

3.3.1 Introduction

Let $\mathbb{L}^2 = (V, E)$ be any lattice on $\mathbb{R}^2$, where V is a set of vertices and E is the corresponding set of edges between nodes on vertices. Let $\mathcal{X} = \{-1, 1\}$ be a state space of values on V.

Definition If $\Lambda \subset V$ is finite and $\Omega(\Lambda) = \mathcal{X}^\Lambda$, then a *configuration* on vertices in Λ is an element, $\omega_\Lambda \in \Omega(\Lambda)$.

Definition An *energy function (Hamiltonian)* for configurations across vertices in V is defined as

$$H(\omega_\Lambda, {}_\Lambda\omega) = -(\sum_{s,s' \in \Lambda} J_{s,s'}\omega_s\omega_{s'} + \sum_{s \in \Lambda, s' \in V \setminus \Lambda} J'_{s,s'}\omega_s\omega_{s'}) \tag{3.7}$$

where the configuration ${}_\Lambda\omega \in \Omega(V \setminus \Lambda)$ is fixed and the values $J_{s,s'}, J'_{s,s'}$ are the *relative strengths* of the interactions between the nodes at vertices $s = (i, k), s' = (i', k') \in V$ such that

$$h(s, s') = |i - i'| + |k - k'| = 1. \tag{3.8}$$

Definition The nodes on a set of vertices satisfying the distance constraint given by equation 3.8 are *nearest neighbors* and the relative strengths of the interactions are defined to be zero when $s, s' \in V \setminus \Lambda$ or $h(s, s') > 1$.

Suppose Ψ is a finite index set and $\mathcal{V}_f$ is the set of all finite subsets of V. With $\Lambda_\psi \in \mathcal{V}_f$ for all $\psi \in \Psi$, the collection

$$\pi = \left\{ \frac{\exp(-\beta H(\omega_{\Lambda_\psi}, {}_{\Lambda_\psi}\omega))}{\sum_{\omega_{\Lambda_\psi} \in \Omega(\Lambda_\psi)} \exp(-\beta H(\omega_{\Lambda_\psi}, {}_{\Lambda_\psi}\omega))} \right\}_{\psi \in \Psi} \tag{3.9}$$

is a set of conditional probability measures. The question of the existence and uniqueness of a probability measure μ, with conditional probability measures given by elements of π, may now be asked.

3.3.2 General Problem Statement

Definition A *configuration* over V is an element, $\omega = (\omega_s)_{s \in V} \in \mathcal{X}^V = \Omega$. If $\mathcal{F}$ is a σ-algebra of subsets of Ω, then a *random field* μ on V is a probability measure on $(\Omega, \mathcal{F})$.

Suppose there is a family of conditional probability measures that are indexed by a collection of finite subsets of V. The general question to be asked is: "Does there exist a probability measure μ with conditional probability measures the same as the predefined family, π?"

3.3.3 Conditional Probability Measure

To answer the question just posed, suppose $V_1 \in \mathcal{V}_f$. Let μ be a random field, $B \in \mathcal{F}(V_1)$ and let $y \in \Omega(V \backslash V_1)$ be any fixed configuration on $V \backslash V_1$.

Proposition 3.3.6 *Given y, the conditional probability measure of μ on the finite subset $V_1 \subset V$ is well-defined when there exists $\psi \in \Psi$ such that $V_1 = \Lambda_\psi$ with*

$$\mu_{V_1}(B \mid y) = \sum_{\omega \in B} \pi_{V_1}(\omega, y) \tag{3.10}$$

and $\pi_{V_1} \in \pi$.

Proof Follows from the discussion in section 2.1.1 of [38]. ∎

3.3.4 Coherent Specifications

It is not enough that the conditional probability measures given by equation 3.10 be well-defined in order for μ to be a probability measure associated to the specification π. It must also be the case that for any $V_1, V_2 \in \mathcal{V}_f$ such that $V_1 \subset V_2$, it is true that $\mu_{V_2}|_{V_1} = \mu_V$.

Definition Suppose $V_1, V_2 \in \mathcal{V}_f$ such that $V_1 \subset V_2$. Let μ_{V_1} and μ_{V_2} be conditional probability measures of configurations on V_1 and $V_2 \backslash V_1$, respectively. If $B_1 \in \mathcal{F}(V_1)$, $B_2 \in \mathcal{F}(V_2 \backslash V_1)$ and $z \in V \backslash V_2$, then the composition of μ_{V_1} and μ_{V_2} is defined as

$$(\mu_{V_2}\mu_{V_1})(B_1 B_2 \mid z) = \int_{B_2} \mu_{V_1}(B_1 \mid yz)\mu_{V_2}(\mathcal{X}^{V_1}, dy \mid z) \tag{3.11}$$

where $B_1 B_2 \subset \mathcal{X}^{V_1} \times \mathcal{X}^{(V_2 \backslash V_1)}$ and

$$\mu_{V_2}(\mathcal{X}^{V_1}, dy \mid z) = \int_{\omega \in \Omega(V_1)} \mu_{V_2}(d\omega, dy \mid z).$$

A family of conditional probability measures $\{\mu_{V_1}\}_{V_1 \in \mathcal{V}_f}$ associated with the random field μ is called *coherent*, if

$$\mu_{V_2}\mu_{V_1} = \mu_{V_2} \tag{3.12}$$

for all $V_1, V_2 \in \mathcal{V}_f$.

For equation 3.10, it is true that μ is a discrete probability measure. Therefore, it is enough to show equation 3.11 applied to the specification of equation 3.10 for $B_1 = \{\omega\}$ and $B_2 = \{y\}$ to obtain

$$\begin{aligned}
\mu_{V_2}(\mathcal{X}^{V_1}, dy \mid z) &= \sum_{\omega_1 \in \Omega(V_1)} \frac{\exp\left(-\beta H(\omega_1 y, z)\right)}{\sum_{\omega_2 \in \Omega(V_2)} \exp\left(-\beta H(\omega_2, z)\right)} \\
&= \frac{\sum_{\omega_1 \in \Omega(V_1)} \exp\left(-\beta H(\omega_1 y, z)\right)}{\sum_{\omega_2 \in \Omega(V_2)} \exp\left(-\beta H(\omega_2, z)\right)}
\end{aligned}$$

which implies

$$\begin{aligned}
(\mu_{V_2}\mu_{V_1})(B_1 B_2 \mid z) &= \frac{\exp(-\beta H(\omega_1, yz))}{\sum_{\omega_1 \in \Omega(V_1)} \exp(-\beta H(\omega_1, yz))} \times \mu_{V_2}(\mathcal{X}^{V_1}, dy \mid z) \qquad (3.13) \\
&= \frac{\exp(-\beta H(\omega_1 y, z))}{\sum_{\omega_1 \in \Omega(V_1)} \exp(-\beta H(\omega_1 y, z))} \times \mu_{V_2}(\mathcal{X}^{V_1}, dy \mid z) \qquad (3.14) \\
&= \frac{\exp(-\beta H(\omega_1 y, z))}{\sum_{\omega_2 \in \Omega(V_2)} \exp(-\beta H(\omega_2, z))} \\
&= \mu_{V_2}(B_1 B_2 \mid z). \qquad \blacksquare
\end{aligned}$$

The first term of equation 3.13 is the conditional probability of a configuration $\omega_1 \in \Omega(V_1)$ given the $a^{'}$priori concatenated configuration, $yz \in \Omega(V \backslash V_1)$. The first term of equation 3.14 is the conditional probability of a concatenated configuration $\omega_1 y \in \Omega(V_2)$ given the $a^{'}$priori configuration, $z \in \Omega(V_1 \backslash V_2)$. As such

$$(\mu_{V_2}\mu_{V_1})(B_1 B_2 \mid z) = \mu_{V_2}(B_1 B_2 \mid z)$$

implies that the family of conditional probability measures given by equation 3.10 is coherent.

Definition A coherent family of conditional probability measures $\pi = \{\pi_{V_1}\}_{V_1 \in \mathcal{V}_f}$ is called a *conditional specification* and the set $G(\pi)$ is defined to be the set of fields μ that admit π as their conditional specification, i.e. $\mu \in G(\pi) \Longleftrightarrow \mu_{V_1} = \pi_{V_1}$ for all $V_1 \in \mathcal{V}_f$.

3.3.5 Interaction Potentials and Existence of Gibbs Measures

A random field, with conditionals given by equation 3.10, is associated to the specific Hamiltonian given by equation 3.7. In general, a coherent specification in the form of the one given by equation 3.9 can be defined through the use of a more general Hamiltonian that is modeled as the sum of functions giving a measure of the interaction between neighboring lattice vertices.

Definition An *interaction potential* measuring the amount of energy associated to interacting nodes at neighboring vertices $s, s' \in V_1 \in \mathcal{V}_f$ is a family

$$\phi = \{\phi_{V_1}\}_{V_1 \in \mathcal{V}_f}$$

of $\mathcal{F}(V_1)$-measurable functions $\phi_{V_1} \colon \Omega(V_1) \to \mathbb{R}$ such that for $\Lambda \in \mathcal{V}_f$ and $\omega \in \Omega$

$$U_\Lambda^\phi(\omega) = \sum_{V_1 \in \mathcal{V}_f : V_1 \cap \Lambda \neq \emptyset} \phi_{V_1}(\omega) < \infty. \tag{3.15}$$

$U_\Lambda^\phi(\omega)$ defines the *total energy* of the configuration $\omega \in \Omega$ across the finite set of vertices, $\Lambda \in \mathcal{V}_f$.

Implicit in the definition of the conditional probability measure, as stated in equation 3.10, is the measure of the changes from one configuration to the next. Since each configuration is discrete, this change is taken to be the integral count of all changes between configurations. However, in the more general case, a conditional probability measure of a continuum of configurations, utilizing the Hamiltonian of equation 3.15, will be well-defined provided that the associated normalizing constant is finite, when it is computed using an appropriate measure of changes in configurations.

Definition Suppose λ is a positive measure of sets in $\mathcal{F}(\Lambda)$ called a *reference measure*. It will be said that an interaction potential ϕ is λ-*admissible* if for all $\omega \in \Omega$

$$Z_\Lambda^\phi(\omega) = \int_{\Omega(\Lambda)} \exp U_\Lambda^\phi(\omega_\Lambda, {}_\Lambda\omega) \lambda^\Lambda(d\omega_\Lambda) < \infty \tag{3.16}$$

where ω_Λ and ${}_\Lambda\omega$ are configurations on Λ and $V \backslash \Lambda$, respectively and λ^Λ is the restriction of λ to Λ. Any λ-admissible potential ϕ is called a *Gibbs potential*.

If ϕ is λ-admissible, then by arguments in [4], the family

$$\pi^\phi = \left\{\pi_\Lambda^\phi\right\}_{\Lambda \in \mathcal{V}_f}$$

defined by

$$\pi_\Lambda^\phi(\omega) = (Z_\Lambda^\phi(\omega))^{-1} \exp U_\Lambda^\phi(\omega_\Lambda, {}_\Lambda\omega) \tag{3.17}$$

can be shown to be a coherent set of conditional probability measures.

Definition The family π^ϕ is called the *Gibbs specification associated to the Gibbs potential* ϕ. When ϕ is a Gibbs potential, $G(\pi^\phi)$ is denoted $G(\phi)$ and any measure $\mu \in G(\phi)$ is called a *Gibbs measure associated to* ϕ.

To show the existence of a Gibbs measure, it has to be shown that $G(\phi)$ is *non*-empty. One way to accomplish this task [4] is to define a coherent specification $\left\{\pi_{\Lambda_n}^\phi(\cdot|\omega)\right\}_{n\geq 1}$ associated to a λ-admissible potential, ϕ. This sequence of conditional probability measures is indexed by a monotonically increasing sequence of finite subsets $(\Lambda_n \in \mathcal{V}_f)_{n\geq 1}$ such that $\Lambda_n \uparrow V$. By defining $\mathcal{P}(\Omega, \mathcal{F})$ to be the space of all probability measures on $(\Omega, \mathcal{F})$, the existence of a convergent subsequence is guaranteed, since every uniformly bounded, infinite sequence of probability measures contains a weakly convergent subsequence [6]. As such, $\left\{\pi_{\Lambda_n}^\phi(\cdot|\omega)\right\}_{n\geq 1}$ is relatively compact in $\mathcal{P}(\Omega, \mathcal{F})$ and the limit of its convergent subsequence is an element of $G(\phi)$.

Define the space of *summable potentials* as

$$B^V = \left\{ \phi : |\, \|\phi\|\, |_s = \sum_{A \in \mathcal{V}_f : A \ni s} \|\phi_A\|_\infty < \infty, s \in V \right\}. \tag{3.18}$$

By arguments in [4], it can be shown that a summable potential ϕ is λ-admissible if and only if λ is finite on the state space, $\mathcal{X}$. Furthermore, suppose $\mathcal{X}$ is a complete, separable metric space. If the reference measure λ is positive and finite, then for a summable potential ϕ, it is true that $G(\phi)$ is non-empty and compact by arguments in [4]. This last statement is also due to $G(\phi)$ being closed in the compact space, $\mathcal{P}(\Omega, \mathcal{F})$ [38].

Returning to the specification given in equation 3.9, assume that a conditional specification has been defined and that it is yet to be determined if an associated *Gibbs* measure exists. Clearly, $\mathcal{X} = \{-1, 1\}$ coupled with any metric will form a complete, separable metric space. Taking the reference measure λ as a count of changes in configurations on finite subsets of V, it is clear that λ will be positive and finite. Lastly, the total energy on a finite subset $\Lambda \in \mathcal{V}_f$ takes the form of $H(\omega_\Lambda, {}_\Lambda\omega)$, given by equation 3.7. Hence, $H(\omega_\Lambda, {}_\Lambda\omega)$ must be a finite number, requiring that any potential used to define the total energy must be summable. Thus, $G(\phi)$ is non-empty for any potential ϕ associated to $H(\omega_\Lambda, {}_\Lambda\omega)$. Therefore, at least one Gibbs measure exists for the conditional specification defined by equation 3.9. As it turns out, for the example of equation 3.9, any Gibbs measure μ associated to the Gibbs specification π^{ϕ} is not always unique, by the discussion in the next section.

3.3.6 Relationship Between Random Fields and Site Percolation

Proposition 3.3.7 *(Theorem 1.16 [1]) Suppose $\mathbb{L}^2 = (V, E)$ is a finite hexagonal lattice such that for nodes on vertices $s, t \in V$, $< s, t >_{\mathbb{L}^2}$ and $< C_{\{s,t\}} >_{\mathbb{L}^2}$ are the edge between and a set of edges connecting s and t, respectively. With λ being the density of nodes in $\mathcal{B}$, let*

$$p_\lambda = 1 - e^{-A_{h^r}\lambda} \tag{3.19}$$

be the common probability that any hexagon will contain at least one of n uniformly distributed nodes, where A_{h^r} is the common area of each hexagon, which can be inscribed into a circle of radius r, for some $r > 0$. If $\mathcal{X} = \{-1, 1\}$ is a state space of values on V such that for $s \in V$, it is true that $x_s = 1$ with probability p_λ, then for vertices $s_1, s_2 \in V$, the following holds

$$P(< C_{\{s_1,s_2\}} >_{\mathbb{L}^2}) = \frac{2 \times \exp(A_{h^r}\lambda \sum_{<s,t>_{\mathbb{L}^2} \in < C_{\{s_1,s_2\}} >_{\mathbb{L}^2}} x_s x_t)}{Z_\lambda} - 1 \tag{3.20}$$

where

$$Z_\lambda = \sum_{(x_s)_{s\in V} \in \mathcal{X}^V} \exp\left(A_{h^r}\lambda \sum_{<s,t>_{\mathbb{L}^2} \in E} x_s x_t\right).$$

Proof This is just a restatement of theorem 1.16 of [1] for the special case of $q = 2$ and $\beta = A_{h^r}\lambda$. ∎

By equation 3.19, each hexagon in the lattice has an equal probability of containing at least one of the n nodes so that the set of hexagons containing a node is uniformly distributed throughout the partition. Equation 3.20, refers to the probability measure that will be used throughout the remainder of the thesis.

Chapter 4

Continuum Model

4.1 Procedure

Let $\mathcal{B} \subset \mathbb{R}^2$ be a bounded region containing the origin $\hat{0} = (0, 0)$ and let X be a node process such that $X(\mathcal{B}) = \mathcal{X}_n$ is a set of n nodes uniformly distributed spatially throughout $\mathcal{B}$, where n is a Poisson random variable which takes a particular value (denoted as n) with density parameter $\lambda = \lambda(n)$. For some fixed $r > 0$, nodes in $\mathcal{X}_n$ will connect if their mutual distance is within r. For fixed $\rho \in (\frac{1}{2}, 1)$, define $\mathcal{A}_{n,\rho,r}$ to be the set of all subsets of $\mathcal{X}_n$ containing at least $100\rho\%$ of all generated nodes which form a connected subset containing $\hat{0}$.

Let $\epsilon > 0$ be given and let $r(n, \rho, \epsilon)$ be the least connectivity radius $r > 0$ such that $P(\mathcal{A}_{n,\rho,r}) \geq \epsilon$. It will be shown that $P(\mathcal{A}_{n,\rho,r})$ is an increasing function of the connection radius r. The aim is to estimate the length of the interval of connectivity radii such that the occurrence of $\mathcal{A}_{n,\rho,r}$ increases in probability from a value of ϵ to a value of $1 - \epsilon$ on the interval of radii.

As an integral step in estimating the length of the interval of radii, continuity in r and ρ of $P(\mathcal{A}_{n,\rho,r})$ will be shown. As such, by continuity in ρ, for small $\delta > 0$, the probability of $\mathcal{A}_{n,\rho,r}$ is close to the probabilities of $\mathcal{A}_{n,\rho+\delta,r}$ and $\mathcal{A}_{n,\rho-\delta,r}$. Furthermore, by continuity and the increasing nature of $P(\mathcal{A}_{n,\rho,r})$ in r, there exists $r_0 = r_0(n, \rho, r)$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$. An upper bound for $P(\mathcal{A}_{n,\rho+\delta,r})$ is found when $r \leq r_0$ and a lower bound for $P(\mathcal{A}_{n,\rho-\delta,r})$ is found when $r \geq r_0$. For $\epsilon > 0$, this upper and lower bound is used to estimate the length of the interval of radii such that $P(\mathcal{A}_{n,\rho,r})$ increases from ϵ to $1 - \epsilon$.

4.2 Definitions

Definition Given a fixed node, $y \in \mathcal{X}_n$, an *r-connected component* containing y is the subset of nodes $< C_y >_r \subseteq \mathcal{X}_n$ containing y and every $x \in \mathcal{X}_n \backslash \{y\}$ having a chain of r-open edges connecting x to y.

Definition Given an r-open edge, $e = < x, y >_r \in G(\mathcal{X}_n; r)$, an *r-connected component* containing e is the subset of nodes $< C_e >_r \subseteq \mathcal{X}_n$ containing x and y together with every $z \in \mathcal{X}_n \backslash \{x, y\}$ having a chain of r-open edges connecting z to both x and y.

Definition Let $\mathcal{E}$ be any σ-algebra of subsets of E. Suppose $\{\eta_k\}_{k \geq 1}$ is a sequence of random variables on E with values in $\mathbb{R}$. It will be said that η_k *converges weakly* to a random variable $\eta : E \to \mathbb{R}$ (written $\eta_k \Rightarrow \eta$), if

$$\begin{aligned} \lim_{k \to \infty} F_k(x) &= \lim_{k \to \infty} P(\eta_k \leq x) \\ &= P(\eta \leq x) \\ &= F(x) \end{aligned}$$

for all $x \in \mathbb{R}$.

4.3 The Event

4.3.1 Bounded Number of Nodes

Let $< C >_r \subseteq \mathcal{X}_n$ be an r-connected component containing $\hat{0}$ such that $| < C >_r | = N$ and define $\rho_n(C) = \frac{N}{n}$. Define the graph property of all connected components containing at least $100\rho\%$ of all available nodes by

$$\mathcal{A}_{n,\rho,r} = \{< C >_r \subseteq \mathcal{X}_n : \rho_n(C) \geq \rho\}. \tag{4.1}$$

As in [7], for $\epsilon \in (0, \frac{1}{2})$, define

$$r(n, \rho, \epsilon) = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon\} \tag{4.2}$$

to be the critical radius at which $\mathcal{A}_{n,\rho,r}$ occurs with probability at least ϵ and define

$$\Delta(n, \rho, \epsilon) = r(n, \rho, 1 - \epsilon) - r(n, \rho, \epsilon) \tag{4.3}$$

to be the length of the continuum of radii upon which $\mathcal{A}_{n,\rho,r}$ increases in probability from $\epsilon > 0$ to $1-\epsilon > 0$.

4.3.2 Unbounded Number of Nodes

In the event that n is unbounded, define the corresponding graph property to be

$$\mathcal{A}_r = \{< C >_r \subseteq \mathcal{X}_\infty : | < C >_r | = \infty\}. \tag{4.4}$$

4.4 Continuity Results

In order to prove the existence of $r_0 > 0$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$, it will be shown that $P(\mathcal{A}_{n,\rho,r})$ is a continuous function of r. By properties of probabilities measures, $P(\mathcal{A}_{n,\rho,r}) \in [0,1]$ and by proposition $A.1.50$, it is true that $P(\mathcal{A}_{n,\rho,r})$ is non-decreasing as a function of $r > 0$. By theorem 3.1.1, it is true that $P(\mathcal{A}_{n,\rho,r})$ increases from $\epsilon > 0$ to $1-\epsilon > 0$ for fixed $\epsilon \in (0, \frac{1}{2})$. Then, by continuity, there exists $r_0 > 0$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$. If I is any continuum of radii and $P(\mathcal{A}_{n,\rho,I})$ is defined to be the set $\{P(\mathcal{A}_{n,\rho,r}) : r \in I\}$, then it is easily seen that r_0 is in the interior of any compact interval of radii I_ϵ such that $P(\mathcal{A}_{n,\rho,I_\epsilon}) = [\epsilon, 1-\epsilon]$. Seeking a contradiction, suppose r_0 is in the boundary of I_ϵ. Since I_ϵ is compact and $P(\mathcal{A}_{n,\rho,r})$ is continuous in r, then $P(\mathcal{A}_{n,\rho,r_0}) = \epsilon$ or $P(\mathcal{A}_{n,\rho,r_0}) = 1-\epsilon$. Therefore, $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$ implies $\epsilon = \frac{1}{2}$. This is a contradiction since $\epsilon \in (0, \frac{1}{2})$. Thus, r_0 is in the interior of I_ϵ. Q.E.D.

Now, if it can be shown that r_0 is independent of ϵ, then $r_0 \in I_\epsilon$ for all $\epsilon \in (0, \frac{1}{2})$. Note that $r_0 \in I = \bigcap_k I_{\epsilon_k}$ for any sequence $\epsilon_k \to \frac{1}{2}$. Clearly I is compact so that r_0 is in the interior of I. Therefore, either I is an interval or $I = \{r_0\}$. Suppose I is an interval of radii. Since r_0 is in the interior of I, then there exists $r_0' < r_0 \in I$. Now, since $\epsilon_k \to \frac{1}{2}$, then $P(\mathcal{A}_{n,\rho,r_0'}) = \frac{1}{2}$ and $r_0' < r_0 = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) = \frac{1}{2}\}$. This is a contradiction. Therefore, $I = \{r_0\}$ so that r_0 is unique. Q.E.D.

Continuity of $P(\mathcal{A}_r)$ in r is proven in [3] and can be used for proving continuity of $P(\mathcal{A}_{n,\rho,r})$ in r as follows. Let $\partial\mathcal{B}$ denote the boundary of $\mathcal{B}$ and define $\mathcal{A}_r^{\mathcal{B}} = \{\hat{0} \leftrightarrow \partial\mathcal{B}\}$ to be the event that there is an r-connected cluster containing $\hat{0}$ and a node in $\partial\mathcal{B}$. By arguments in [3], continuity of $P(\mathcal{A}_r)$ in r is equivalent to continuity of $P(\mathcal{A}_r^{\mathcal{B}})$ in r for all bounded regions $\mathcal{B}$ containing $\hat{0}$. Clearly, $P(\mathcal{A}_r^{\mathcal{B}}) = P(\mathcal{A}_r^{\mathcal{B}} - \mathcal{A}_{n,\rho,r}) + P(\mathcal{A}_r^{\mathcal{B}} \cap \mathcal{A}_{n,\rho,r})$ so that continuity of $P(\mathcal{A}_r^{\mathcal{B}})$ in r implies continuity of $P(\mathcal{A}_r^{\mathcal{B}} \cap \mathcal{A}_{n,\rho,r})$ in r. Now, there exists $r_0' > 0$ such that $P(\mathcal{A}_r^{\mathcal{B}}) = 1$ for all $r \geq r_0'$. Then, it follows that $P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}_r^{\mathcal{B}} \cap \mathcal{A}_{n,\rho,r})$ is continuous when $r \geq r_0'$. In particular, $P(\mathcal{A}_{n,\rho,r})$ is continuous at r_0'. So, there is $\delta > 0$ such that $P(\mathcal{A}_{n,\rho,r})$ is continuous upon $[r_0' - \delta, r_0' + \delta]$. Continuing this argument, continuity of $P(\mathcal{A}_{n,\rho,r})$ extends until $r_0' - \delta = 0$ so that $P(\mathcal{A}_{n,\rho,r})$ is continuous for all $r \geq 0$. Q.E.D.

Theorem 4.4.8 *(Theorem 3.8 [3]) Suppose* $\{r_k\}_{k\geq 1}$ *is a sequence of radii such that* $0 < r_k \leq R$ *for some* $R > 0$ *and* $\{\eta_k\}_{k\geq 1}$ *is a sequence of random variables which take values* r_k *with probability* 1. *If* $0 < r \leq R$ *and* η *is a random variable taking the value* r *with probability* 1 *such that* $\eta_k \Rightarrow \eta$ *as* $k \to \infty$. *Then,* $P(\mathcal{A}_{\eta_k}) \to P(\mathcal{A}_\eta)$ *as* $k \to \infty$.

Proof This is just a restatement of theorem 3.8 of [3] for the special case of random variables η_k and η such that $P(\eta_k = r_k) = 1 = P(\eta = r)$ for all $k \geq 1$. ∎

Corollary 4.4.9 *(to Theorem* 4.4.8*)* $P(\mathcal{A}_{n,\rho,r})$ *is a continuous function of* r.

Proof Continuity of $P(\mathcal{A}_r)$ in r follows from theorem 4.4.8. Therefore, the result follows by the discussion preceding the statement of theorem 4.4.8. ∎

Theorem 4.4.10 $r = r(n, \rho, \epsilon)$ *is a continuous function of* ϵ *if and only if* $P(\mathcal{A}_{n,\rho,r})$ *is a continuous function of* r.

Proof Suppose $r(n, \rho, \epsilon)$ is a continuous function of ϵ and let $\{\epsilon_k \in (0, \frac{1}{2})\}_{k\geq 1}$ be a sequence of positive real numbers such that $\epsilon_k \to \epsilon_0$ as $k \to \infty$. Let $\{X(e)\}_{e\in G(\mathcal{X}_n;r)}$ be a finite sequence of uniformly distributed random variables with values in $[0, 1]$ and define a sequence of random variables $\{\eta_k\}_{k\geq 1}$ by $\eta_k(e) = r(n, \rho, \epsilon_k) \equiv r_k$ when $X(e) < 1$ and 0 otherwise. Clearly, $\eta_k = r_k$ with probability 1 for all $k \geq 1$. Likewise, define a random variable η_0 by $\eta_0(e) = r(n, \rho, \epsilon_0) \equiv r_0$ when $X(e) < 1$ and 0 otherwise so that $\eta_0 = r_0$ with probability 1. Since $r(n, \rho, \epsilon)$ is continuous in ϵ, then $r_k \to r_0$ as $k \to \infty$ so that $\eta_k \Rightarrow \eta_0$ as $k \to \infty$. Now, define $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$. By lemma $A.1.56$, $0 < \eta_k \leq R$ for all $k \geq 0$. Therefore, $P(\mathcal{A}_{n,\rho,\eta_k}) \to P(\mathcal{A}_{n,\rho,\eta_0})$ as $k \to \infty$ by corollary 4.4.9 since $r_k \to r_0$ as $k \to \infty$. Thus, $P(\mathcal{A}_{n,\rho,r})$ is a continuous function of r. Conversely, suppose $P(\mathcal{A}_{n,\rho,r})$ is a continuous function of r and let $\{\epsilon_k \in (0, \frac{1}{2})\}_{k\geq 1}$ be any convergent sequence such that $\epsilon_k \to \epsilon_0$. Define $r_k = r(n, \rho, \epsilon_k)$ and $r_0 = r(n, \rho, \epsilon_0)$. Given $\xi > 0$, it is true that $\Xi \equiv \{k \geq 1 : |P(\mathcal{A}_{n,\rho,r_k}) - P(\mathcal{A}_{n,\rho,r_0})| \geq \xi\}$ is a set of measure zero by the continuity assumption. Therefore, $r_k \to r_0$ as $k \to \infty$ by proposition $A.1.57$. Thus, suppose that $\Xi \neq \emptyset$. Then, Ξ is at most countable so that $\Xi = \emptyset$ a.s. Hence, $r_k \to r_0$ as $k \to \infty$ by proposition $A.1.57$ so that $r(n, \rho, \epsilon)$ is a continuous function of ϵ. ∎

4.5 Continuum Giant Component

Theorem 4.5.11 *There exists $r_0 = r_0(n, \rho) < \infty$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$.*

Proof Let $\epsilon \in (0, \frac{1}{2})$ be given. Since $\mathcal{A}_{n,\rho,r}$ is an increasing property in r by proposition $A.1.47$, theorem 3.1.1 applies. Thus, there exists an interval I_ϵ of length $\Delta(n, \rho, \epsilon)$ such that $P(\mathcal{A}_{n,\rho,r}) \in [\epsilon, 1-\epsilon]$ for $r \in I_\epsilon$. Since $P(\mathcal{A}_{n,\rho,r})$ is a continuous function of r by corollary 4.4.9 and non-decreasing in r by proposition $A.1.50$ and $\frac{1}{2} \in [\epsilon, 1-\epsilon]$, then there exists $r_0 \in I_\epsilon$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$. If $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$, then by lemma $A.1.56$, it is true that $0 < r_0(n, \rho, \epsilon) \leq R < \infty$. It remains to be shown that $r_0 = r_0(n, \rho)$, independent of ϵ. ∎

Lemma 4.5.12 *$r_0 = r_0(n, \rho, \epsilon)$ is independent of ϵ.*

Proof Let $\epsilon_1, \epsilon_2 \in (0, \frac{1}{2})$ and suppose $r_{0,1} = r_0(n, \rho, \epsilon_1), r_{0,2} = r_0(n, \rho, \epsilon_2)$ such that

$$P(\mathcal{A}_{n,\rho,r_{0,1}}) = \frac{1}{2} = P(\mathcal{A}_{n,\rho,r_{0,2}}). \tag{4.5}$$

It has to be shown that $r_{0,1} = r_{0,2}$. Let $\{\epsilon_k\}_{k\geq1}$ be a sequence such that $\epsilon_k = \epsilon_1$ for all $k \geq 1$ and define $r_{0,k} = r_0(n, \rho, \epsilon_k)$. Then, for arbitrary $\xi > 0$, it is true that

$$\Xi \equiv \{k \geq 1 : |P(\mathcal{A}_{n,\rho,r_{0,k}}) - P(\mathcal{A}_{n,\rho,r_{0,2}})| \geq \xi\} = \emptyset \tag{4.6}$$

since $r_{0,k} = r_{0,1}$ for all $k \geq 1$. Hence, by proposition $A.1.57$, $r_{0,k} \to r_{0,2}$ as $k \to \infty$. But, $r_{0,k} = r_{0,1}$ for all $k \geq 1$ so that $r_{0,1} = r_{0,2}$. Thus, $r_0 = r_0(n, \rho)$, independent of ϵ. ∎

Corollary 4.5.13 *Given $r > 0$, there exists a density of nodes $\lambda_0 = \lambda(n_0)$ such that*

$$P(\mathcal{A}_{n_0,\rho,r}) = \frac{1}{2}.$$

Proof By lemma 4.5.12, let $n_0 = n_0(r, \rho)$ be the minimum of all positive (real) solutions to $r = r_0(n, \rho)$ for some fixed $r > 0$. It follows that $P(\mathcal{A}_{n_0,\rho,r}) = \frac{1}{2}$. ∎

Theorem 4.5.11 is the statement and proof of the existence of a particular radius r_0 at which the probability is $\frac{1}{2}$ for the occurrence of the event of at least half of the nodes in $\mathcal{X}_n$ will form a connected cluster. Furthermore, lemma 4.5.12 proves that r_0 is independent of $\epsilon \in (0, \frac{1}{2})$. Lastly, given a fixed radius $r > 0$, corollary 4.5.13 is a statement and proof that the radius r_0 guaranteed by theorem 4.5.11 can be used to find a critical density of nodes such that the probability is $\frac{1}{2}$ for the occurrence of the event of at least $100\rho\%$ of the nodes in $\mathcal{X}_n$ forming a connected cluster.

4.6 Continuum Sharp Threshold Interval Length

Given the particular radius guaranteed by theorem 4.5.11, then theorem 3.1.1 can be used to find an estimate of the length of the sharp threshold interval such that $P(\mathcal{A}_{n,\rho,r})$ increases sharply from some $\epsilon \in (0, \frac{1}{2})$ to $1-\epsilon$. By lemma 4.5.12, it is true that r_0 is independent of any particular ϵ. Thus, the interval and its length must be fixed given n and $\rho \in (\frac{1}{2}, 1)$.

Theorem 4.6.14 $\Delta(n, \rho) = O(r_0 \log^{\frac{1}{4}} n)$.

Proof For $\delta \in (0, \frac{1}{2})$, let $\epsilon_\delta = \frac{1}{2} - \delta$. By theorem 3.1.1 and theorems 4.5.11 and 4.5.12,

$$\begin{aligned} \Delta(n, \rho) &= \lim_{\delta \to 0^+} \Delta(n, \rho, \epsilon_\delta) \\ &= \lim_{\delta \to 0^+} O(r(n, \rho, \epsilon_\delta) \log^{\frac{1}{4}} n) \\ &= O(r_0 \log^{\frac{1}{4}} n). \quad \blacksquare \end{aligned}$$

Theorem 4.6.14 gives an expected result, given theorem 3.1.1 above. According to theorem 3.1.2 above, this length turns out to be the same as the length of the bottleneck matching, M_n. In [47], it is shown that an algorithm used to compute M_n will need $O(n \log(n))$ computations. However, in [5], a much more practical estimate of this length is obtained after the bounded region is partitioned by hexagons of a known size. If M is the number of these hexagons in the bounded region, then it is shown that a good estimate of the sharp interval length is a polynomial in $\frac{1}{M}$.

Theorem 4.6.15 *There is a constant $c > 0$, independent of M, such that for all $\epsilon_1 > 0$ and every fixed small $\delta > 0$*

$$P(\mathcal{A}_{n,\rho+\delta,r}) \leq (\frac{1}{2} + \epsilon_1) M^{-c(r_0 - r)} \tag{4.7}$$

for all $r \leq r_0$ and

$$P(\mathcal{A}_{n,\rho-\delta,r}) \geq 1 - (\frac{1}{2} + \epsilon_1) M^{-c(r - r_0)} \tag{4.8}$$

for all $r \geq r_0$.

Theorem 4.6.16 $P(\mathcal{A}_{n,\rho,r})$ *is a continuous function of ρ.*

The proof of theorem 4.6.16 requires theorem 4.6.15 which will be proven later. For now, the result of theorem 4.6.16 is assumed. By theorem 4.6.16, for small $\delta > 0$,

$$P(\mathcal{A}_{n,\rho-\delta,r}) \approx P(\mathcal{A}_{n,\rho,r}) \approx P(\mathcal{A}_{n,\rho+\delta,r}).$$

In this light, theorem 4.6.15 asserts that if $r_1 < r_0 < r_2$ and for some $\epsilon \in (0, \frac{1}{2})$ it is true that $P(\mathcal{A}_{n,\rho,r_1}) = \epsilon$ and $P(\mathcal{A}_{n,\rho,r_2}) = 1 - \epsilon$, then $r_2 - r_1$ is an estimate of the sharp threshold interval length for the event, $\mathcal{A}_{n,\rho,r}$.

Chapter 5

Hexagonal Partition Model

It was seen in the previous chapter that $r_0 > 0$ exists such that the probability is $\frac{1}{2}$ for the event of at least $100\rho\%$ of all nodes to connect. By theorem 3.1.1,

$$r_c = r_c(n) \propto \sqrt{\frac{\log n}{n}} \leq r_0(n) = r_0 \tag{5.1}$$

where r_c defines the critical radius at which the same event occurs with arbitrarily small positive probability.

For fixed $r \in (r_c, r_0)$, let h^r be the largest hexagon that can be inscribed into a circle of radius $\frac{r}{4} > 0$. Let $H(r)$ be a countably infinite collection of copies of h^r such that

$$\mathbb{R}^2 = \bigcup_{h^r_{i,j} \in H(r)} h^r_{i,j} \tag{5.2}$$

and for $h^r_{i,j}, h^r_{i',j'} \in H(r)$, it is true that $h^r_{i,j} \neq h^r_{i',j'}$ whenever $|i - i'| + |j - j'| \neq 0$. Connectivity between $x, y \in \mathcal{X}_n$ is then defined as x and y both lying in the same hexagon or neighboring hexagons.

With the bounded region $\mathcal{B}$ partitioned into hexagons contained within $\mathcal{B} \cap H(r)$, the analysis proceeds whereby the original problem of estimating the sharp threshold interval length in the continuum is now replaced by the problem of estimating the length in the hexagonal partition framework. As such, definitions of connectivity and the increasing event are defined in the new framwork. Then, the continuity and existence results are shown to still hold in the new framework. Later, an analogue to theorem 4.6.15 is stated and proven.

5.1 Definitions

Definition A *hexagonal partition* of $\mathcal{B}$ is a finite collection of hexagons from $H(r)$ such that $\mathcal{B}$ is a union of all hexagons in the finite collection.

Definition The *Hamming distance* between elements, $h^r_{i,j}, h^r_{i',j'} \in H(r)$ is defined to be the quantity

$$h(h^r_{i,j}, h^r_{i',j'}) = |i - i'| + |j - j'|.$$

Definition Nodes $x, y \in \mathcal{X}_n$ are $H(r)$-*connected* and $< x, y >_{H(r)}$ is an $H(r)$-*open edge*, if there exists $h^r_{i_x,j_x}, h^r_{i_y,j_y} \in H(r)$ such that $x \in h^r_{i_x,j_x}$ and $y \in h^r_{i_y,j_y}$ where $h(h^r_{i_x,j_x}, h^r_{i_y,j_y}) \leq 2$ with $|i_x - i_y| \leq 1$ and $|j_x - j_y| \leq 1$. Nodes in $\mathcal{X}_n$ are $H(r)$-*disconnected* and form an $H(r)$-*closed edge* otherwise.

Definition Given a $y \in \mathcal{X}_n$, an $H(r)$-*connected component* containing y is the subset of nodes $< C_y >_{H(r)} \subseteq \mathcal{X}_n$ containing y and every $x \in \mathcal{X}_n \backslash \{y\}$ having an $H(r)$-open set of edges connecting x to y.

Definition Given an $H(r)$-connected edge, $e =< x, y >_{H(r)}$, an $H(r)$-*connected component* containing e is the subset of nodes $< C_e >_{H(r)} \subseteq \mathcal{X}_n$ containing x and y and every $z \in \mathcal{X}_n \backslash \{x, y\}$ having an $H(r)$-open set of edges connecting z to both x and y.

5.2 The Increasing Property

5.2.1 Bounded Number of Nodes

Let $< C >_{H(r)} \subseteq \mathcal{X}_n$ be an r-connected component such that $| < C >_{H(r)} | = N$ and define $\rho^*_n(C) = \frac{N}{n}$. Define the graph property of all connected components containing at least $100\rho\%$ of all available nodes by

$$\mathcal{A}^*_{n,\rho,r} = \{< C >_{H(r)} \subseteq \mathcal{X}_n : \rho^*_n(C) \geq \rho\}. \tag{5.3}$$

As in [7], for $\epsilon \in (0, \frac{1}{2})$, define

$$r^*(n, \rho, \epsilon) = \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon\} \tag{5.4}$$

to be the critical radius at which $\mathcal{A}^*_{n,\rho,r}$ occurs with probability at least ϵ and define

$$\Delta^*(n,\rho,\epsilon) = r^*(n,\rho,1-\epsilon) - r^*(n,\rho,\epsilon) \tag{5.5}$$

to be the length of the continuum of radii upon which $\mathcal{A}^*_{n,\rho,r}$ increases in probability from $\epsilon > 0$ to $1-\epsilon > 0$.

5.2.2 Unbounded Number of Nodes

In the event that n is unbounded, define the corresponding graph property and associated event to be

$$\mathcal{A}^*_r = \{< C >_{H(r)} \subseteq \mathcal{X}_\infty : | < C >_{H(r)} | = \infty\}. \tag{5.6}$$

5.3 Continuity Results

In order to prove the existence of $r_0^* > 0$ such that $P(\mathcal{A}^*_{n,\rho,r_0}) = \frac{1}{2}$, it will be shown that $P(\mathcal{A}^*_{n,\rho,r})$ is a continuous function of r. By properties of probabilities measures, $P(\mathcal{A}^*_{n,\rho,r}) \in [0,1]$ and by proposition $A.2.63$, it is true that $P(\mathcal{A}^*_{n,\rho,r})$ is non-decreasing as a function of $r > 0$. By theorem 3.1.1, it is true that $P(\mathcal{A}^*_{n,\rho,r})$ increases from $\epsilon > 0$ to $1-\epsilon > 0$ for fixed $\epsilon \in (0, \frac{1}{2})$. Then, by continuity, there exists $r_0^* > 0$ such that $P(\mathcal{A}^*_{n,\rho,r_0^*}) = \frac{1}{2}$. If I is any continuum of radii and $P(\mathcal{A}^*_{n,\rho,I})$ is defined to be the set $\{P(\mathcal{A}^*_{n,\rho,r}) : r \in I\}$, then it is easily seen that r_0^* is in the interior of any compact interval of radii I_ϵ such that $P(\mathcal{A}^*_{n,\rho,I_\epsilon}) = [\epsilon, 1-\epsilon]$. Seeking a contradiction, suppose r_0^* is in the boundary of I_ϵ. Since I_ϵ is compact and $P(\mathcal{A}^*_{n,\rho,r})$ is continuous in r, then $P(\mathcal{A}^*_{n,\rho,r_0^*}) = \epsilon$ or $P(\mathcal{A}^*_{n,\rho,r_0^*}) = 1-\epsilon$. Therefore, $P(\mathcal{A}^*_{n,\rho,r_0^*}) = \frac{1}{2}$ implies $\epsilon = \frac{1}{2}$. This is a contradiction since $\epsilon \in (0, \frac{1}{2})$. Thus, r_0^* is in the interior of I_ϵ. Q.E.D.

Now, if it can be shown that r_0^* is independent of ϵ, then $r_0^* \in I_\epsilon$ for all $\epsilon \in (0, \frac{1}{2})$. Note that $r_0^* \in I = \bigcap_k I_{\epsilon_k}$ for any sequence $\epsilon_k \to \frac{1}{2}$. Clearly I is compact so that r_0^* is in the interior of I. Therefore, either I is an interval or $I = \{r_0^*\}$. Suppose I is an interval of radii. Since r_0^* is in the interior of I, then there exists $r_0' < r_0^* \in I$. Now, since $\epsilon_k \to \frac{1}{2}$, then $P(\mathcal{A}^*_{n,\rho,r_0'}) = \frac{1}{2}$ and $r_0' < r_0^* = \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) = \frac{1}{2}\}$. This is a contradiction. Therefore, $I = \{r_0^*\}$ so that r_0^* is unique. Q.E.D.

Continuity of $P(\mathcal{A}^*_r)$ in r is proven in [3] and can be used for proving continuity of $P(\mathcal{A}^*_{n,\rho,r})$ in r as follows. Let $\partial\mathcal{B}$ be defined as in chapter 4 and define $\mathcal{A}^{\mathcal{B}*}_r = \{\hat{0} \leftrightarrow \partial\mathcal{B}\}$ to be the event that there is an $H(r)$-connected cluster containing both $\hat{0}$ and a node in $\partial\mathcal{B}$. By arguments in [3], continuity of

$P(\mathcal{A}_r^*)$ in r is equivalent to continuity of $P(\mathcal{A}_r^{\mathcal{B}*})$ in r for all bounded regions $\mathcal{B}$ containing $\hat{0}$. Clearly, $P(\mathcal{A}_r^{\mathcal{B}*}) = P(\mathcal{A}_r^{\mathcal{B}*} - \mathcal{A}_{n,\rho,r}^*) + P(\mathcal{A}_r^{\mathcal{B}*} \cap \mathcal{A}_{n,\rho,r}^*)$ so that continuity of $P(\mathcal{A}_r^{\mathcal{B}*})$ in r implies continuity of $P(\mathcal{A}_r^{\mathcal{B}*} \cap \mathcal{A}_{n,\rho,r}^*)$ in r. Now, there exists $r_0' > 0$ such that $P(\mathcal{A}_r^{\mathcal{B}*}) = 1$ for all $r \geq r_0'$. Then, it follows that $P(\mathcal{A}_{n,\rho,r}^*) = P(\mathcal{A}_r^{\mathcal{B}*} \cap \mathcal{A}_{n,\rho,r}^*)$ is continuous when $r \geq r_0'$. In particular, $P(\mathcal{A}_{n,\rho,r}^*)$ is continuous at r_0'. So, there is $\delta > 0$ such that $P(\mathcal{A}_{n,\rho,r}^*)$ is continuous upon $[r_0' - \delta, r_0' + \delta]$. Continuing this argument, continuity of $P(\mathcal{A}_{n,\rho,r}^*)$ extends until $r_0' - \delta = 0$ so that $P(\mathcal{A}_{n,\rho,r}^*)$ is continuous for all $r \geq 0$. Q.E.D.

Corollary 5.3.17 *(to Theorem 4.4.8) $P(\mathcal{A}_{n,\rho,r}^*)$ is a continuous function of r.*

Proof Note that $H(r)$-connected nodes are within distance r of one another. Therefore, continuity of $P(\mathcal{A}_r^*)$ in r follows from theorem 4.4.8. Hence, the result follows by the discussion preceding the statement of corollary 5.3.17. ∎

Theorem 5.3.18 *$r = r^*(n, \rho, \epsilon)$ is a continuous function of ϵ if and only if $P(\mathcal{A}_{n,\rho,r}^*)$ is a continuous function of r.*

Proof Suppose $r^*(n, \rho, \epsilon)$ is a continuous function of ϵ and let $\{\epsilon_k \in (0, \frac{1}{2})\}_{k \geq 1}$ be a sequence of positive real numbers such that $\epsilon_k \to \epsilon_0$ as $k \to \infty$. Let $\{X(e)\}_{e \in G(\mathcal{X}_n; H(r))}$ be a finite sequence of uniformly distributed random variables with values in $[0, 1]$ and define a sequence of random variables $\{\eta_k^*\}_{k \geq 1}$ by $\eta_k^*(e) = r^*(n, \rho, \epsilon_k) \equiv r_k^*$ when $X(e) < 1$ and 0 otherwise. Clearly, $\eta_k^* = r_k^*$ with probability 1 for all $k \geq 1$. Likewise, define a random variable η_0^* by $\eta_0^*(e) = r^*(n, \rho, \epsilon_0) \equiv r_0^*$ when $X(e) < 1$ and 0 otherwise so that $\eta_0^* = r_0$ with probability 1. Since $r(n, \rho, \epsilon)$ is continuous in ϵ, then $r_k \to r_0$ as $k \to \infty$ so that $\eta_k^* \Rightarrow \eta_0^*$ as $k \to \infty$. Now, define $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$. By lemma $A.2.69$, $0 < \eta_k^* \leq R$ for all $k \geq 0$. Therefore, $P(\mathcal{A}_{n,\rho,\eta_k^*}^*) \to P(\mathcal{A}_{n,\rho,\eta_0^*}^*)$ as $k \to \infty$ by corollary 5.3.17 since $r_k^* \to r_0^*$ as $k \to \infty$. Thus, $P(\mathcal{A}_{n,\rho,r}^*)$ is a continuous function of r. Conversely, suppose $P(\mathcal{A}_{n,\rho,r}^*)$ is a continuous function of r and let $\{\epsilon_k \in (0, \frac{1}{2})\}_{k \geq 1}$ be any convergent sequence such that $\epsilon_k \to \epsilon_0$. Define $r_k^* = r^*(n, \rho, \epsilon_k)$ and $r_0^* = r^*(n, \rho, \epsilon_0)$. Given $\xi > 0$, it is true that $\Xi \equiv \{k \geq 1 : |P(\mathcal{A}_{n,\rho,r_k^*}^*) - P(\mathcal{A}_{n,\rho,r_0^*}^*)| \geq \xi\}$ is a set of measure zero by the continuity assumption. Therefore, $r_k^* \to r_0^*$ as $k \to \infty$ by proposition $A.2.70$. Thus, suppose that $\Xi \neq \emptyset$. Then, Ξ is at most countable so that $\Xi = \emptyset$ a.s. Hence, $r_k^* \to r_0^*$ as $k \to \infty$ by proposition $A.2.70$ so that $r^*(n, \rho, \epsilon)$ is a continuous function of ϵ. ∎

5.4 Hexagonal Giant Component

Theorem 5.4.19 *There exists $r_0^* = r_0^*(n, \rho) < \infty$ such that $P(\mathcal{A}_{n,\rho,r_0^*}^*) = \frac{1}{2}$.*

Proof Let $\epsilon \in (0, \frac{1}{2})$ be given. Since $\mathcal{A}^*_{n,\rho,r}$ is an increasing property in r by proposition $A.2.60$, theorem 3.1.1 applies. Thus, there exists an interval I_ϵ of length $\Delta^*(n, \rho, \epsilon)$ such that $P(\mathcal{A}^*_{n,\rho,r}) \in [\epsilon, 1-\epsilon]$ for $r \in I_\epsilon$. Since $P(\mathcal{A}^*_{n,\rho,r})$ is a continuous function of r by corollary 5.3.17 and non-decreasing in r by proposition $A.2.63$ and $\frac{1}{2} \in [\epsilon, 1-\epsilon]$, then there exists $r_0^* \in I_\epsilon$ such that $P(\mathcal{A}^*_{n,\rho,r_0^*}) = \frac{1}{2}$. If $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$, then by lemma $A.2.69$, it is true that $0 < r_0^*(n, \rho, \epsilon) \leq R < \infty$. It remains to be shown that $r_0^* = r_0^*(n, \rho)$, independent of ϵ. ∎

Lemma 5.4.20 *$r_0^* = r_0^*(n, \rho, \epsilon)$ is independent of ϵ.*

Proof Let $\epsilon_1, \epsilon_2 \in (0, \frac{1}{2})$ and suppose $r^*_{0,1} = r_0^*(n, \rho, \epsilon_1), r^*_{0,2} = r_0^*(n, \rho, \epsilon_2)$ such that

$$P(\mathcal{A}^*_{n,\rho,r^*_{0,1}}) = \frac{1}{2} = P(\mathcal{A}^*_{n,\rho,r^*_{0,2}}). \tag{5.7}$$

It has to be shown that $r^*_{0,1} = r^*_{0,2}$. Let $\{\epsilon_k\}_{k\geq 1}$ be a sequence such that $\epsilon_k = \epsilon_1$ for all $k \geq 1$ and define $r^*_{0,k} = r_0^*(n, \rho, \epsilon_k)$. Then, for arbitrary $\xi > 0$, it is true that

$$\Xi \equiv \{k \geq 1 : |P(\mathcal{A}^*_{n,\rho,r^*_{0,k}}) - P(\mathcal{A}^*_{n,\rho,r^*_{0,2}})| \geq \xi\} = \emptyset \tag{5.8}$$

since $r^*_{0,k} = r^*_{0,1}$ for all $k \geq 1$. Hence, by proposition $A.2.70$, $r^*_{0,k} \to r^*_{0,2}$ as $k \to \infty$. But, $r^*_{0,k} = r^*_{0,1}$ for all $k \geq 1$ so that $r^*_{0,1} = r^*_{0,2}$. Thus, $r_0^* = r_0^*(n, \rho)$, independent of ϵ. ∎

Corollary 5.4.21 *Given $r > 0$, there exists a density of nodes, $\lambda_0^* = \lambda(n_0^*)$, such that*

$$P(\mathcal{A}^*_{n_0^*,\rho,r}) = \frac{1}{2}.$$

Proof By lemma 5.4.20, let $n_0^* = n_0^*(r, \rho)$ be the minimum of all positive (real) solutions to $r = r_0^*(n, \rho)$ for some fixed $r > 0$. It follows that $P(\mathcal{A}^*_{n_0^*,\rho,r}) = \frac{1}{2}$. ∎

Theorem 5.4.19 is the statement and proof of the existence of a particular radius r_0^* at which the probability is $\frac{1}{2}$ for the occurrence of the event of at least half of the nodes in $\mathcal{X}_n$ will form a connected cluster. Furthermore, lemma 5.4.20 proves that r_0^* is independent of $\epsilon \in (0, \frac{1}{2})$. Lastly, given a fixed radius $r > 0$, corollary 5.4.21 is a statement and proof that the radius r_0^* guaranteed by theorem 5.4.19 can be used to find a critical density of nodes such that the probability is $\frac{1}{2}$ for the occurrence of the event of at least half of the nodes in $\mathcal{X}_n$ will form a connected cluster.

5.5 Hexagonal Sharp Threshold Interval Length

Given the particular radius guaranteed by theorem 5.4.19, then theorem 3.1.1 can be used to find an estimate of the length of the sharp threshold interval such that $P(\mathcal{A}^*_{n,\rho,r})$ increases sharply from some $\epsilon \in (0, \frac{1}{2})$ to $1-\epsilon$. By lemma 5.4.20, it is true that r_0^* is independent of any particular ϵ. Thus, the interval and its length must be fixed given n and $\rho \in (\frac{1}{2}, 1)$.

Theorem 5.5.22 $\Delta^*(n, \rho) = O(r_0^* \log^{\frac{1}{4}} n)$.

Proof For $\delta \in (0, \frac{1}{2})$, let $\epsilon_\delta = \frac{1}{2} - \delta$. By theorem 3.1.1 and theorems 5.4.19 and 5.4.20,

$$\begin{aligned} \Delta^*(n, \rho) &= \lim_{\delta \to 0^+} \Delta^*(n, \rho, \epsilon_\delta) \\ &= \lim_{\delta \to 0^+} O(r^*(n, \rho, \epsilon_\delta) \log^{\frac{1}{4}} n) \\ &= O(r_0^* \log^{\frac{1}{4}} n). \qquad \blacksquare \end{aligned}$$

As in theorem 4.6.14 above, theorem 5.5.22 gives an expected result, given theorem 3.1.1 above. Likewise, a similar result to theorem 3.3.1 of [5] can be stated and later proven, as in the case of theorem 4.6.15.

Theorem 5.5.23 *There is a constant $c > 0$, independent of M, such that for all $\epsilon_1 > 0$ and every fixed small $\delta > 0$*

$$P(\mathcal{A}^*_{n,\rho+\delta,r}) \leq (\frac{1}{2} + \epsilon_1) M^{-c(r_0^* - r)}$$

for all $r \leq r_0^*$ and

$$P(\mathcal{A}^*_{n,\rho-\delta,r}) \geq 1 - (\frac{1}{2} + \epsilon_1) M^{-c(r - r_0^*)} \tag{5.9}$$

for all $r \geq r_0^*$.

Let M^2 be the number of hexagons partitioning the region $\mathcal{B}$ and let $H_{\mathcal{B}}(r) = H(r) \cap \mathcal{B}$. Given $< C >_{H(r)} \subseteq \mathcal{X}_n$, define $H_C = \{h^r_{\mathcal{B}} \in H_{\mathcal{B}}(r) : h^r_{\mathcal{B}} \cap < C >_{H(r)} \neq \emptyset\}$ to be the connected cluster of hexagons such that each hexagon contains at least one node from the connected cluster of nodes, $< C >_{H(r)}$.

Lemma 5.5.24 $E[\ \rho^*_n(C)\ ] = \frac{E[\ |H_C|\ ]}{M^2}$.

Proof Let $< C >_{H(r)} \subseteq \mathcal{X}_n$ be an $H(r)$-connected cluster and let K_{H_C} be a random variable taking as values the number of nodes in the region R_{H_C} defined by the hexagons in H_C. Since the n nodes are uniformly distributed spatially and $\mathcal{B}$ is partitioned into M^2 copies of the prototypical hexagon h^r, then

$$\begin{aligned} E[\ K_{H_C}\ ] &= n\frac{E[\ area(R_{H_C})\ ]}{area(\mathcal{B})} \\ &= n\frac{E[\ |H_C|\ ] \times area(h^r)}{M^2 \times area(h^r)} \\ &= n\frac{E[\ |H_C|\ ]}{M^2}. \end{aligned}$$

But, $E[\ K_{H_C}\ ] = E[\ |< C >_{H(r)}|\ ]$. Therefore,

$$E[\ |< C >_{H(r)}|\ ] = n\frac{E[\ |H_C|\ ]}{M^2}$$

implies

$$E[\ \rho_n^*(C)\ ] = \frac{E[\ |H_C|\ ]}{M^2}. \qquad \blacksquare$$

Define $\mathcal{D}_{n,\rho,r} = \{H_C \subseteq H_{\mathcal{B}}(r) : E[\ \rho_n^*(C)\ ] \geq \rho\}$. With $\mathcal{D}_{n,\rho,r}$ defined as such, the original problem of estimating the length of the sharp threshold for the event $\mathcal{A}_{n,\rho,r}$ in the continuum is now recast as a site percolation problem on a hexagonal lattice. As will be defined later, a site in the lattice will be deemed open if the corresponding hexagon is occupied by at least one of the nodes from $\mathcal{X}_n$ and it will be deemed closed otherwise. Likewise, two sites are connected and belong to the same connected cluster if both sites are open and their hamming distance is less than or equal to one. Later, a torus on the lattice will be formed by defining a countable collection of permutations of the hexagons in the partition so that the length of the sharp threshold for the event $\mathcal{D}_{n,\rho,r}$ can be approximated by the length for another event $\mathcal{D}^*_{n,\rho,r}$ on the torus. In this way, boundary connection issues for sites in the partition of $\mathcal{B}$ are mitigated and the length of the sharp threshold interval for the event $\mathcal{D}^*_{n,\rho,r}$ approximates the length for $\mathcal{D}_{n,\rho,r}$, which approximates the length for $\mathcal{A}^*_{n,\rho,r}$, which finally approximates the length for $\mathcal{A}_{n,\rho,r}$, the original event in the continuum.

Theorem 5.5.25 *There is a constant $c > 0$, independent of M, such that*

$$P(\mathcal{D}_{n,\rho,r}) \leq \frac{1}{2}M^{-c(r_0^*-r)}$$

for all $r \leq r_0^*$*. Similarly, for some fixed small* $\delta > 0$ *and for all* $\epsilon_1 > 0$*, there is an* $M_0(\delta, \epsilon_1)$ *such that for all* $M > M_0(\delta, \epsilon_1)$

$$P(\mathcal{D}_{n,\rho-\delta,r}) \geq 1 - (\frac{1}{2} + \epsilon_1)M^{-c(r-r_0^*)}$$

for all $r \geq r_0^*$*.*

An important part of the proof of theorem 5.5.25 relies upon the sharp threshold inequality results of [39] and [40]. In order to apply these results, connectivity in the hexagon lattice structure should be extended to the case of a torus, whereby any boundary connectivity issues are mitigated. As such, make $H_{\mathcal{B}}(r)$ into a torus by identifying $h_{i,j} \in H_{\mathcal{B}}(r)$ with an element $h_{i',j'}$ in a copy of $H_{\mathcal{B}}(r)$, if $i' = i \bmod M$ and $j' = j \bmod M$. For every $k, l \in \mathbb{Z}$, the mapping $\tau_{k,l} : h_{i,j} \to h_{i+k,j+l}$ defines a shift translation. In this way, a subgroup of automorphisms $\tau = \{\tau_{k,l} : k, l \in \mathbb{Z}\}$ with the transitivity property is formed. Thus, any hexagon $h_{i,j}$ can be shifted to any other hexagon $h_{i',j'}$ with the translation, $\tau_{i'-i,j'-j}$. Now, hexagons in the 1st row (column) are allowed to be joined in a connected cluster with hexagons in the Mth row (column), provided that all hexagons in question are occupied.

Proposition 5.5.26 *Define* $\tau(H_{\mathcal{B}}(r))$ *to be the torus created by translations of hexagons in* $H_{\mathcal{B}}(r)$ *under the action of permutations in* τ *and define* $\mathcal{D}^*_{n,\rho,r} = \{H_C \subseteq \tau(H_{\mathcal{B}}(r)) : E[\ \rho_n^*(C)\ ] \geq \rho\}$*. Then,* $\mathcal{D}_{n,\rho,r} \subset \mathcal{D}^*_{n,\rho,r}$ *and* $\mathcal{D}_{n,\rho,r} \neq \mathcal{D}^*_{n,\rho,r}$*.*

Proof Since $\mathcal{D}^*_{n,\rho,r}$ contains all of the connected hexagons from $\mathcal{D}_{n,\rho,r}$ and any connections between the 1st and Mth rows (columns) while $\mathcal{D}_{n,\rho,r}$ contains no connection between the 1st and Mth rows (columns), then the result follows. ∎

Definition To each hexagon in the partition of $\mathcal{B}$, associate a *site* $i \in \{1, 2, ..., M^2\}$ as the center of the hexagon. For sites $i \in \{1, 2, ..., M^2\}$, define $s_i \in \{0, 1\}$ to be the *state on site* i. A site i is said to be *open* if $s_i = 1$ and *closed* otherwise. There exists an *edge* $e_{\{i,j\}}$ between sites $i, j \in \{1, 2, ..., M^2\}$ if and only if there exists a hexagon $h^r_{i,j} \ni i, j$ or there exists neighboring hexagons $h^r_i \ni i$ and $h^r_j \ni j$ in the partition of $\mathcal{B}$. Define $e_{\{i,j\}}$ to be *open* if and only if $s_i = 1 = s_j$ and *closed* otherwise.

Definition The *conditional influence of* i on the event $\mathcal{D}^*_{n,\rho,r}$ is defined to be

$$I(i) = P(\mathcal{D}^*_{n,\rho,r} \mid s_i = 1) - P(\mathcal{D}^*_{n,\rho,r} \mid s_i = 0)$$

and it is a measure of the change in the probability of $\mathcal{D}^*_{n,\rho,r}$ due to a state change from $s_i = 0$ to $s_i = 1$ at site, i.

For completeness, a theorem from [5] is stated without proof, which gives an upper bound on the change in $P(\mathcal{D}^*_{n,\rho,r})$ as a function of the node density λ. Utilizing the chain rule for derivatives, a lower bound on the change in $P(\mathcal{D}^*_{n,\rho,r})$ as a function of r is found and the resulting inequality relationship is used to estimate upper and lower bounds on $P(\mathcal{D}^*_{n,\rho,r})$, which will approximate the inequality results of theorem 5.5.25.

Lemma 5.5.27 *(Lemma 4.1.1 [5]) There is a constant $z > 0$, independent of M and λ, such that*

$$\frac{d}{d\lambda}P(\mathcal{D}^*_{n,\rho,r}) \leq z^*(\lambda)\min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\}\log M$$

where A_{h^r} is the area of the prototypical hexagon h^r and $z^*(\lambda) = -zA_{h^r}e^{-A_{h^r}\lambda}$.

Lemma 5.5.28 *There is a constant $c > 0$, independent of M and λ, such that*

$$\frac{d}{dr}P(\mathcal{D}^*_{n,\rho,r}) \geq c^*(\lambda)\min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\}\log M$$

where A_{h^r} is the area of the prototypical hexagon h^r and $c^*(\lambda) = c(\lambda)A_{h^r}e^{-A_{h^r}\lambda}$, with $c(\lambda) = -cg(\lambda)$ for some function $g(\lambda)$.

Proof As in corollary 5.4.21, let n^* be the inverse of r^* and seeking a contradiction, suppose $dr/d\lambda = 0$. Let $\epsilon \in (0, \frac{1}{2})$. By lemma 5.5.27, $dP/d\lambda$ exists. Now, the existence of dP/dr will be shown by proving a Lipschitz condition on the probability distribution $P(\mathcal{D}^*_{n,\rho,r})$ as a function of r. Assume area$(\mathcal{B}) = 1$. Without loss of generality, it can be assumed that $r \in [0, 1]$. Without further loss of generality, let $r_1, r_2 \in [0, 1]$ such that r^*_0 is the midpoint of $[r_1, r_2]$, i.e. $r^*_0 = (r_2 - r_1)/2$. Then, by theorem 5.5.22,

$$|P(\mathcal{D}^*_{n,\rho,r_2}) - P(\mathcal{D}^*_{n,\rho,r_1})| \quad \leq \quad 1 = (\Delta^*(n,\rho))^{-1}|r_2 - r_1|.$$

Therefore, $P(\mathcal{D}^*_{n,\rho,r})$ is Lipschitz continuous with respect to r. Hence, dP/dr exists. Now, since $dP/d\lambda$, dP/dr and $dr/d\lambda$ all exist, then the Chain Rule for derivatives yields,

$$\frac{d}{d\lambda}P(\mathcal{D}^*_{n,\rho,r}) = \frac{d}{dr}P(\mathcal{D}^*_{n,\rho,r}) \times \frac{dr}{d\lambda}.$$

Note that the existence of dP/dr requires that $|dP/dr| < \infty$. Therefore, since $dr/d\lambda = 0$, then

$$\frac{d}{d\lambda}P(\mathcal{D}^*_{n,\rho,r}) = \frac{d}{dr}P(\mathcal{D}^*_{n,\rho,r}) \times 0 = 0.$$

As a result, $P(\mathcal{D}^*_{n,\rho,r})$ is constant as a function of λ. So, suppose that $0 < n < n^*$. Then, $P(\mathcal{D}^*_{n,\rho,r}) = 0$, which implies that $P(\mathcal{D}^*_{n,\rho,r}) \equiv 0$. This is a contradiction, since $P(\mathcal{D}^*_{n,\rho,r})$ is a probability distribution. Hence, $dr/d\lambda \neq 0$. Now, by theorem 3.2.5, there is a constant $c > 0$, independent of M and λ, such that

$$I(i) \geq c \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\}\frac{\log M}{M^2}.$$

Under the action of τ, each hexagon in the bounded region $\mathcal{B}$ is translated to another hexagon in a copy of $\mathcal{B}$. Therefore, $\mathcal{D}^*_{n,\rho,r}$ and $P(\mathcal{D}^*_{n,\rho,r})$ are invariant under the action of τ. Hence, $I(i) = I(j)$ whenever, $\tau(i) = j$, where $\tau(i)$ is defined to be the translation of the hexagon $h^r_i \ni i$ to the hexagon $h^r_j \ni j$ in the copy of the partition of $\mathcal{B}$. From [5], in the proof of theorem 5.5.27, the following identity holds

$$\begin{aligned} \frac{d}{d\lambda}P(\mathcal{D}^*_{n,\rho,r}) &= \frac{d}{d\lambda}\left(e^{-A_{h^r}\lambda}\sum_{i=1}^{M^2} I(i)\right) \\ &= -A_{h^r}e^{-A_{h^r}\lambda}\sum_{i=1}^{M^2} I(i). \end{aligned} \tag{5.10}$$

For $r > 0$ and $k > 0$, any r-connected component in $\mathcal{X}_n$ containing at least $\frac{n+k}{2}$ nodes will inherently contain an r-connected component of size at least $\frac{n}{2}$. Hence, $\mathcal{A}^*_{n+k,\rho,r} \subseteq \mathcal{A}^*_{n,\rho,r}$. It follows that $P(\mathcal{A}^*_{n+k,\rho,r}) \leq P(\mathcal{A}^*_{n,\rho,r})$. Therefore, $r^*(n,\rho,\epsilon) \in \{r > 0 : P(\mathcal{A}^*_{n+k,\rho,r}) \geq \epsilon\}$, which implies $r^*(n + k, \rho, \epsilon) \leq r^*(n,\rho,\epsilon)$ for $k > 0$. Hence,

$$r^*(n+k,\rho,\epsilon) - r^*(n,\rho,\epsilon) \leq 0. \tag{5.11}$$

Since node density λ is proportional to node count n for any bounded region $\mathcal{B}$, then using inequality 5.11 yields

$$\frac{dr}{d\lambda} = \lim_{k\to 0}\frac{r^*(n+k,\rho,\epsilon) - r^*(n,\rho,\epsilon)}{k} \leq 0.$$

Since $dr/d\lambda \neq 0$, it follows that

$$\frac{dr}{d\lambda} < 0.$$

Since $dr/d\lambda$ exists, then $|dr/d\lambda| < \infty$. Thus, by substituting

$$I(i) \geq c \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\} \frac{\log M}{M^2}$$

into 5.10, it follows that

$$\begin{aligned}
\frac{d}{d\lambda} P(\mathcal{D}^*_{n,\rho,r}) &= -A_{h^r} e^{-A_{h^r}\lambda} \sum_{i=1}^{M^2} I(i) \\
&\leq -cA_{h^r} e^{-A_{h^r}\lambda} \sum_{i=1}^{M^2} \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\} \frac{\log M}{M^2} \\
&= -cA_{h^r} e^{-A_{h^r}\lambda} \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\} \log M.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{d}{d\lambda} P(\mathcal{D}^*_{n,\rho,r}) &= \frac{d}{dr} P(\mathcal{D}^*_{n,\rho,r}) \times \frac{dr}{d\lambda} \\
&\leq -cA_{h^r} e^{-A_{h^r}\lambda} \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\} \log M \qquad (5.12)
\end{aligned}$$

so that

$$\frac{d}{dr} P(\mathcal{D}^*_{n,\rho,r}) \geq -cA_{h^r} e^{-A_{h^r}\lambda} \left(\frac{dr}{d\lambda}\right)^{-1} \min\{P(\mathcal{D}^*_{n,\rho,r}), 1 - P(\mathcal{D}^*_{n,\rho,r})\} \log M. \qquad (5.13)$$

Defining $g(\lambda) = (dr/d\lambda)^{-1}$, the result follows. ∎

By inequality 5.13, $P(\mathcal{D}^*_{n,\rho,r})$ is increasing as a function of r and by inequality 5.12, $P(\mathcal{D}^*_{n,\rho,r})$ is decreasing as a function of λ.

Lemma 5.5.29 *Let $c > 0$ be as in theorem 5.5.28. Then, there exists r_0^*, independent of M, such that*

$$P(\mathcal{D}^*_{n,\rho,r}) \leq \frac{1}{2} M^{-c(r_0^* - r)}$$

for all $r \leq r_0^*$ *and*

$$P(\mathcal{D}^*_{n,\rho,r}) \geq 1 - \frac{1}{2}M^{-c(r-r_0^*)}$$

for all $r \geq r_0^*$.

Proof Arguing as in the proof to theorem 5.4.19, there exists r_0^* such that $P(\mathcal{D}^*_{n,\rho,r_0^*}) = \frac{1}{2}$. Arguing similarly to corollary 5.3.17, $P(\mathcal{D}^*_{n,\rho,r})$ is continuous in r. Therefore, $P(\mathcal{D}^*_{n,\rho,r}) \leq 1 - P(\mathcal{D}^*_{n,\rho,r})$ for $r \leq r_0^*$ and $P(\mathcal{D}^*_{n,\rho,r}) \geq 1 - P(\mathcal{D}^*_{n,\rho,r})$ for $r \geq r_0^*$. Thus, the result of lemma 5.5.28 takes the form

$$\frac{d}{dr}P(\mathcal{D}^*_{n,\rho,r}) \geq c^*(\lambda)P(\mathcal{D}^*_{n,\rho,r}) \log M$$

for $r \leq r_0^*$ and

$$\frac{d}{dr}P(\mathcal{D}^*_{n,\rho,r}) \geq c^*(\lambda)(1 - P(\mathcal{D}^*_{n,\rho,r})) \log M$$

for $r \geq r_0^*$. The last two inequalities can be written

$$\frac{d}{dr}\log P(\mathcal{D}^*_{n,\rho,r}) \geq c^*(\lambda) \log M$$

for $r \leq r_0^*$ and

$$\frac{d}{dr}\log\left(1 - P(\mathcal{D}^*_{n,\rho,r})\right) \leq -c^*(\lambda) \log M$$

for $r \geq r_0^*$, respectively. Consider $r \leq r_0^*$. Both sides of

$$\frac{d}{dr}\log P(\mathcal{D}^*_{n,\rho,r}) \geq c^*(\lambda) \log M$$

are integrated in the direction of increasing node density since $P(\mathcal{D}^*_{n,\rho,r})$ decreases as a function of node density λ by the proof to lemma 5.5.28. It was also shown that $dr/d\lambda < 0$, i.e. r is decreasing as a function of node density. Therefore, the integration limits for the interval $[r, r_0^*]$ are from r_0^* to r. Noting that the inequality is reversed for backward integration, the following is obtained for $c > 0$ and some $K_1(\lambda) \geq 0$,

$$\log P(\mathcal{D}^*_{n,\rho,r}) \leq K_1(\lambda) \log M^{c(r-r_0^*)}$$

which can be rewritten as

$$\log P(\mathcal{D}^*_{n,\rho,r}) \leq K_1(\lambda) \log M^{-c(r_0^*-r)}.$$

This implies

$$P(\mathcal{D}^*_{n,\rho,r}) \leq K_2(\lambda) M^{-c(r_0^*-r)}$$

for some $K_2(\lambda) \geq 0$. Therefore, using the initial condition $P(\mathcal{D}^*_{n,\rho,r_0^*}) = \frac{1}{2}$ yields $K_2(\lambda) = \frac{1}{2}$. Thus,

$$P(\mathcal{D}^*_{n,\rho,r}) \leq \frac{1}{2} M^{-c(r_0^*-r)}.$$

Now, consider $r \geq r_0^*$. Similary, both sides of

$$\frac{d}{dr} \log\left(1 - P(\mathcal{D}^*_{n,\rho,r})\right) \leq -c^*(\lambda) \log M$$

are integrated in the direction of increasing connection radii on $[r_0^*, r]$ since $P(\mathcal{D}^*_{n,\rho,r})$ increases as a function of connection radii r by the proof to lemma 5.5.28. Therefore, the integration limits are from r_0^* to r. The following is obtained for $c > 0$ and some $K_3(\lambda) \geq 0$,

$$\log\left(1 - P(\mathcal{D}^*_{n,\rho,r})\right) \leq -K_3(\lambda) \log M^{c(r-r_0^*)}$$

which can be rewritten as

$$\begin{aligned}\log\left(1 - P(\mathcal{D}^*_{n,\rho,r})\right) &\leq -K_3(\lambda) \log M^{-c(r_0^*-r)} \\ &= K_3(\lambda) \log M^{-c(r-r_0^*)}.\end{aligned}$$

This implies

$$1 - P(\mathcal{D}^*_{n,\rho,r}) \leq K_4(\lambda) M^{-c(r-r_0^*)}$$

for some $K_4(\lambda) \geq 0$. Therefore, using the initial condition $P(\mathcal{D}^*_{n,\rho,r_0^*}) = \frac{1}{2}$ yields $K_4(\lambda) = \frac{1}{2}$. Hence,

$$P(\mathcal{D}^*_{n,\rho,r}) \geq 1 - \frac{1}{2} M^{-c(r-r_0^*)}. \quad \blacksquare$$

By proposition 5.5.26, there are cases when $\mathcal{D}_{n,\rho,r} \subset \mathcal{D}^*_{n,\rho,r}$, but $\mathcal{D}_{n,\rho,r} \neq \mathcal{D}^*_{n,\rho,r}$ so that the occurrence of $\mathcal{D}^*_{n,\rho,r}$ does not imply the occurrence of $\mathcal{D}_{n,\rho,r}$. To exclude these possibilities, the arguments of [5] are followed whereby a slightly larger event $\mathcal{D}_{n,\rho-\delta,r}$ is considered for some small $\delta > 0$ such that the occurrence of $\mathcal{D}^*_{n,\rho,r}$ implies the occurrence of $\mathcal{D}_{n,\rho-\delta,r}$.

As in [5], let $\phi(M)$ be any M-dependent integer such that $\phi(M) \to \infty$ as $M \to \infty$ and

$$\phi(M) = o(c(r - r_0^*) \log M).$$

Choose a coordinate system so that $\mathcal{B}$ has its lower left corner at the origin. Define the top, bottom, left and right boundary strips of $\mathcal{B}$ as $H_i, i = 1, 2, 3, 4$ with sizes $\phi(M) \times M$, $\phi(M) \times M$, $M \times \phi(M)$ and $M \times \phi(M)$ by

$$H_1 = \{H_{i,j} : i = M - \phi(M) + 1, ..., M, j = 1, ..., M\}$$

$$H_2 = \{H_{i,j} : i = 1, ..., \phi(M), j = 1, ..., M\}$$

$$H_3 = \{H_{i,j} : i = 1, ..., M, j = 1, ..., \phi(M)\}$$

$$H_4 = \{H_{i,j} : i = 1, ..., M, j = M - \phi(M) + 1, ..., M\}.$$

Let E_i be the event that there is a connected path of *occupied* hexagons crossing H_i long way.

Lemma 5.5.30 *For $i = 1, 2, 3, 4$, there are constants $c_i > 0$ such that for large M and $r \geq r_0^*$*

$$P(E_i) \geq 1 - e^{-c_i \phi(M)}.$$

Proof As in [5], by the duality property, the occurrence of $E_i, i = 1, 2, 3, 4$ is equivalent to the non-occurrence of the event that there is a connected path of *unoccupied* hexagons crossing $H_i, i = 1, 2, 3, 4$ short way. The rest of the proof follows [5] with the edge probability as a function of node density $p(\lambda_0)$ replaced by r_0^* and the critical probability for the occurrence of an infinite cluster of occupied hexagons p_c replaced by $r^*(n, \rho, \epsilon)$. ∎

Proof (*Theorem* 5.5.25) By proposition 5.5.26, $\mathcal{D}_{n,\rho,r} \subset \mathcal{D}^*_{n,\rho,r}$ so that $P(\mathcal{D}_{n,\rho,r}) \leq P(\mathcal{D}^*_{n,\rho,r})$. To estimate $P(\mathcal{D}_{n,\rho-\delta,r})$ for $r > r_0$ and any given $\delta > 0$, let $E = E_1 \cap E_2 \cap E_3 \cap E_4$ and consider $F = \mathcal{D}^*_{n,\rho,r} \cap E$. Since $P(F) = P(F \cap \mathcal{D}_{n,\rho-\delta,r}) + P(F - \mathcal{D}_{n,\rho-\delta,r})$, then

$$P(\mathcal{D}_{n,\rho-\delta,r}) \geq P(F) - P(F - \mathcal{D}_{n,\rho-\delta,r}).$$

Noting that $P(E_1) = P(E_2)$ and $P(E_3) = P(E_4)$, then the FKG inequality of [2] yields

$$P(F) \geq P(\mathcal{D}^*_{n,\rho,r})P^2(E_1)P^2(E_3).$$

By lemma 5.5.30, there exists $b > 0$ such that for all sufficiently large M,

$$P(F) \geq 1 - \frac{1}{2}M^{-c(r-r_0^*)} - O(e^{-b\phi(M)}).$$

Using $\phi(M) = o(c(r - r_0^*)\log M)$, this implies that for any given $\epsilon_1 > 0$ and all sufficiently large M depending upon ϵ_1,

$$P(F) \geq 1 - (\frac{1}{2} + \epsilon_1)M^{-c(r-r_0^*)}.$$

It is now claimed that $F - \mathcal{D}_{n,\rho-\delta,r} = \emptyset$, requiring that $P(F - \mathcal{D}_{n,\rho-\delta,r}) = 0$ for all large M. Following [5], the occurrence of F implies that there is a connected path of hexagons which encloses the sub-lattice given by $H_{\mathcal{B}}(r) - \bigcup_{i=1}^4 H_i$. Because the nodes in $\mathcal{X}_n$ are uniformly distributed, then there is a connected cluster of hexagons within the original lattice totaling at least $\rho M^2 - (2M\phi(M) + 2\phi(M)(M - 2\phi(M)))$ hexagons, where ρM^2 is a lower bound on the number of occupied hexagons in the largest connected cluster and $2M\phi(M) + 2\phi(M)(M - 2\phi(M))$ is the total number of hexagons in the strips, $H_i, i = 1, 2, 3, 4$. Let $\delta_1 = (2M\phi(M) + 2\phi(M)(M - 2\phi(M)))/M^2$. It follows that $F \subset \mathcal{D}_{n,\rho-\delta_1,r}$ since F occurs in those hexagons of $\mathcal{B}$ that are not near the boundary of $\mathcal{B}$ by a simple translation τ of hexagons $h \in \bigcup_{i=1}^4 H_i$ to hexagons $h \in H_{\mathcal{B}}(r) - \bigcup_{i=1}^4 H_i$. Thus, if M is large enough so that $\delta_1 < \delta$, then $F \subset \mathcal{D}_{n,\rho-\delta_1,r} \subset \mathcal{D}_{n,\rho-\delta,r}$. ∎

Proof (*Theorem* 5.5.23) Consider $r \leq r_0^*$. Since

$$P(\mathcal{A}^*_{n,\rho+\delta,r}) = P(\mathcal{A}^*_{n,\rho+\delta,r}, \mathcal{D}_{n,\rho,r}) + P(\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r})$$

then

$$P(\mathcal{A}^*_{n,\rho+\delta,r}) \leq P(\mathcal{D}_{n,\rho,r}) + P(\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r}).$$

It will be shown that $P(\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r}) = o(M^{-c(r_0^*-r)})$. Let x be a configuration of states across hexagons in $H_{\mathcal{B}}(r)$ and let $\mathcal{C}(x) = \{C_1, ..., C_K\}$ be the set of clusters in x. For $i = 1, ..., K$, let N_{C_i} be the number of nodes in the cluster, C_i. Then, $\{N_{C_i} \mid \mathcal{C}(x), n\} \sim B(n, \frac{|H_{C_i}|}{M^2})$. Suppose $C_{i_0} \in \mathcal{C}(x)$ is any cluster such that $\rho^*_n(C_{i_0}) \geq \rho + \delta$. Since the occurrence of the event $\mathcal{D}^c_{n,\rho,r}$ implies $\frac{|H_{C_{i_0}}|}{M^2} < \rho$, then

$$\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r} \subset \{\rho^*_n(C_{i_0}) \geq \rho + \delta, \frac{|H_{C_{i_0}}|}{M^2} < \rho\}.$$

By arguments in [48] and [5], there is an $\alpha = \alpha(\rho, \delta) > 0$ such that

$$P(\rho^*_n(C_{i_0}) \geq \rho + \delta \mid \{\frac{|H_{C_{i_0}}|}{M^2} < \rho\}, \mathcal{C}(x), n) \leq e^{-\alpha(\rho,\delta)n}.$$

Let K be a random variable which takes the number of nodes generated in $\mathcal{B}$ as values. It follows that

$$\begin{aligned}
P(\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r}) &\leq P(\{\rho^*_n(C_{i_0}) \geq \rho + \delta\}, \{\frac{|H_{C_{i_0}}|}{M^2} < \rho\}) \\
&= P(\rho^*_n(C_{i_0}) \geq \rho + \delta \mid \frac{|H_{C_{i_0}}|}{M^2} < \rho) \times P(\frac{|H_{C_{i_0}}|}{M^2} < \rho) \\
&\leq P(\rho^*_n(C_{i_0}) \geq \rho + \delta \mid \frac{|H_{C_{i_0}}|}{M^2} < \rho) \qquad (5.14) \\
&= E[P(\rho^*_n(C_{i_0}) \geq \rho + \delta \mid \{\frac{|H_{C_{i_0}}|}{M^2} < \rho\}, \mathcal{C}(x), n)] \\
&\leq E[e^{-\alpha n}] \\
&= e^{-\alpha n} \times P(K = n) \\
&\leq e^{-\alpha n} \qquad (5.15)
\end{aligned}$$

where inequality 5.14 follows since $P(\frac{|H_{C_{i_0}}|}{M^2} < \rho) \leq 1$ and inequality 5.15 follows since $P(K = n) \leq 1$.

Now, since $\alpha n > d \log M$ implies $e^{-\alpha n} < M^{-d}$, then for any $d > 0$ and every fixed $\delta > 0$, it follows that $P(\mathcal{A}^*_{n,\rho+\delta,r} - \mathcal{D}_{n,\rho,r})$ decays to zero at a rate faster than M^{-d} for n large enough. The case of $r \geq r^*_0$ is proven with similar arguments. ∎

Theorem 5.5.31 *$P(\mathcal{A}^*_{n,\rho,r})$ is a continuous function of ρ.*

Proof Let $\sigma = 1 - \rho$ in equation 5.3. Then, $\mathcal{A}^*_{n,\sigma,r}$ is an increasing property in σ for increasing $\rho \in (\frac{1}{2}, 1)$. Therefore, by theorem 3.2.5, it is true that $\mathcal{A}^*_{n,\sigma,r}$ has a sharp threshold in σ and hence in ρ. Thus, $P(\mathcal{A}^*_{n,\rho,r})$ is differentiable in ρ which implies that $P(\mathcal{A}^*_{n,\rho,r})$ is continuous as a function of ρ. ∎

By theorem 5.5.31, for small $\delta > 0$,

$$P(\mathcal{A}^*_{n,\rho-\delta,r}) \approx P(\mathcal{A}^*_{n,\rho,r}) \approx P(\mathcal{A}^*_{n,\rho+\delta,r}).$$

In this light, theorem 5.5.23 asserts that if $r_1^* < r_0^* < r_2^*$ and for some $\epsilon \in (0, \frac{1}{2})$ it is true that $P(\mathcal{A}^*_{n,\rho,r_1^*}) = \epsilon$ and $P(\mathcal{A}^*_{n,\rho,r_2^*}) = 1 - \epsilon$, then $r_2^* - r_1^*$ is an estimate of the sharp threshold interval length for the event, $\mathcal{A}^*_{n,\rho,r}$.

Chapter 6

Proof (Main Theorem 4.6.15)

Recall from section 3.1 that if $\mathcal{X}_n$ is a set of nodes generated by a node process $X : \mathbb{R} \to \mathbb{R}$, then for $r > 0$, $G(\mathcal{X}_n; r)$ is defined to be the r-graph of the set of r-open and r-closed edges between nodes in $\mathcal{X}_n \subset \mathcal{B}$. As before, let $H(r)$ be a partition of $\mathbb{R}^2$ into copies of the prototypical hexagon h^r.

Definition $G(\mathcal{X}_n; H(r))$ is defined to be the $H(r)$-*graph* of all $H(r)$-open and $H(r)$-closed edges between nodes in $\mathcal{X}_n \subset \mathcal{B}$.

Lemma 6.0.32 $G(\mathcal{X}_n; H(r)) \subseteq G(\mathcal{X}_n; r)$.

Proof Suppose $< x, y >_{H(r)} \in G(\mathcal{X}_n; H(r))$ is any $H(r)$-connected edge. Without loss of generality, choose a coordinate system on $\mathbb{R}^2$ so that $< x, y >_{H(r)}$ lies on a coordinate axis with $\hat{0} = (0, 0)$ defined such that $d(x, \hat{0}) = \frac{d(x,y)}{2} = d(\hat{0}, y)$. Since $x, y \in \mathcal{X}_n \subset \mathcal{B}$ and $H(r)$ is a partition of $\mathcal{B}$, then there exists $h^r_{i_x,j_x}, h^r_{i_y,j_y} \in H(r)$ such that $x \in h^r_{i_x,j_x}, y \in h^r_{i_y,j_y}$ and $h(h^r_{i_x,j_x}, h^r_{i_y,j_y}) \leq \max\{|i_x - i_y|, |j_x - j_y|\} \leq 1$. Each of $h^r_{i_x,j_x}$ and $h^r_{i_y,j_y}$ are copies of h^r and can be inscribed into copies of a circle of radius $\frac{r}{4}$. Therefore, $d(x, y) = d(x, \partial h^r_{i_x,j_x}) + d(\partial h^r_{i_y,j_y}, y) \leq \frac{r}{2} + \frac{r}{2} = r$ so that $x, y \in \mathcal{X}_n$ are r-connected. Thus, $< x, y >_{H(r)} \in G(\mathcal{X}_n; r)$, which shows that $G(\mathcal{X}_n; H(r)) \subseteq G(\mathcal{X}_n; r)$. ∎

Lemma 6.0.33 $P(\mathcal{A}^*_{n,\rho,r}) \leq P(\mathcal{A}_{n,\rho,r})$.

Proof By lemma 6.0.32, it is true that $\mathcal{A}^*_{n,\rho,r} \subseteq \mathcal{A}_{n,\rho,r}$. Therefore, since P is non-decreasing by properties of probability measures, the lemma follows. ∎

Lemma 6.0.34 $r_0 \leq r^*_0$.

Proof Seeking a contradiction, suppose $r_0 > r_0^*$. Then,

$$\begin{aligned}
\frac{1}{2} &= P(\mathcal{A}_{n,\rho,r_0}) && (6.1)\\
&\geq P(\mathcal{A}_{n,\rho,r_0^*}) && (6.2)\\
&\geq P(\mathcal{A}^*_{n,\rho,r_0^*}) && (6.3)\\
&= \frac{1}{2} && (6.4)
\end{aligned}$$

where equality 6.1 follows by theorem 4.5.11, inequality 6.2 follows by properties of probability measures and by hypothesis, inequality 6.3 follows by lemma 6.0.33 and equality 6.4 follows by theorem 5.4.19. It follows that $P(\mathcal{A}_{n,\rho,r_0^*}) = \frac{1}{2}$. Therefore, $r_0^* \in \{r > 0 : P(\mathcal{A}_{n,\rho,r}) = \frac{1}{2}\}$ and $r_0^* < r_0 = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) = \frac{1}{2}\}$. This is a contradiction. Thus, $r_0 \leq r_0^*$. ∎

Proof *(Theorem* 4.6.15*)* Since $P(\mathcal{A}_{n,\rho,r})$ and $P(\mathcal{A}^*_{n,\rho,r})$ are continuous functions of r, then by theorem 5.5.23 and lemma 6.0.34, for every $r \in [0, r_0]$ there exists $r' \leq r$ such that

$$\begin{aligned}
P(\mathcal{A}_{n,\rho+\delta,r'}) &\leq P(\mathcal{A}^*_{n,\rho+\delta,r}) && (6.5)\\
&\leq (\frac{1}{2} + \epsilon_1)M^{-c(r_0^*-r)}\\
&\leq (\frac{1}{2} + \epsilon_1)M^{-c(r_0-r)}. && (6.6)
\end{aligned}$$

Consider $r_0 \in [0, r_0]$. Then, continuity of $P(\mathcal{A}_{n,\rho+\delta,r})$ in r and the non-decreasing property of $P(\mathcal{A}_{n,\rho+\delta,r})$ in r implies inequality 6.6 for all $r \in [0, r']$. It is claimed that $r' = r_0$. Seeking a contradiction if $r' < r_0$, suppose $P(\mathcal{A}_{n,\rho+\delta,r}) \leq (\frac{1}{2}+\epsilon_1)M^{-c(r_0-r)}$ for all $r \in [0, r']$ and $P(\mathcal{A}_{n,\rho+\delta,r}) > (\frac{1}{2} + \epsilon_1)M^{-c(r_0-r)}$ for all $r > r'$. By hypothesis, $r_0 > r'$ so that when $r = r_0$, it follows that $P(\mathcal{A}_{n,\rho+\delta,r_0}) > \frac{1}{2}$. Now, since for any connected cluster $< C >_r$ such that $\rho_n(C) \geq \rho + \delta$ for $\delta > 0$, the statement $\rho_n(C) \geq \rho$ is implied, then $\mathcal{A}_{n,\rho+\delta,r} \subseteq \mathcal{A}_{n,\rho,r}$ for all $r \in [0, r_0]$. Hence, $r' < r_0$ leads to

$$P(\mathcal{A}_{n,\rho,r_0}) \geq \limsup_{\delta\to 0^+} P(\mathcal{A}_{n,\rho+\delta,r_0}) \geq P(\mathcal{A}_{n,\rho+\delta,r_0}) > \frac{1}{2}. \quad (6.7)$$

In particular, inequality 6.7 gives $P(\mathcal{A}_{n,\rho,r_0}) > \frac{1}{2}$. This is a contradiction since $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$ by theorem 4.5.11. It follows that $r' = r_0$ and

$$P(\mathcal{A}_{n,\rho+\delta,r}) \leq (\frac{1}{2} + \epsilon_1)M^{-c(r_0-r)}$$

for $r \leq r_0$. A similar argument is used to prove

$$P(\mathcal{A}_{n,\rho-\delta,r}) \geq 1 - (\frac{1}{2} + \epsilon_1)M^{-c(r-r_0)}$$

for $r \geq r_0$. ∎

The implication of the proof to theorem 4.6.15 is that $P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}^*_{n,\rho,r})$ for $r \in [0, r_0]$. As such, when $r \in [0, r_0)$, probabilities in the continuum are computed by partitioning the bounded region and computing probabilities using the random cluster measure. This notion will be of great use in the next chapter when the concern is to not have one large cluster and several smaller ones. Rather, it is desired to have only smaller clusters.

Theorem 6.0.35 $P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}^*_{n,\rho,r})$ *for* $r \in [0, r_0]$.

Proof By continuity in ρ of $P(\mathcal{A}^*_{n,\rho,r})$ as given by theorem 5.5.31, it is true that

$$\lim_{\delta \to 0^+} P(\mathcal{A}^*_{n,\rho+\delta,r}) = P(\mathcal{A}^*_{n,\rho,r}).$$

Suppose $\delta_1 > \delta_2$ such that $\rho+\delta_1, \rho+\delta_2 \in (\frac{1}{2}, 1)$ and let $< C >_r \in \mathcal{A}_{n,\rho+\delta_1,r}$. Then, $\rho_n(C) \geq \rho+\delta_1 > \rho + \delta_2$ so that $< C >_r \in \mathcal{A}_{n,\rho+\delta_2,r}$. Hence, $\mathcal{A}_{n,\rho+\delta_1,r} \subseteq \mathcal{A}_{n,\rho+\delta_2,r}$. By properties of probability measures, $P(\mathcal{A}_{n,\rho,r})$ is monotone non-decreasing as a function of decreasing ρ. By inequality 6.5, it follows that $P(\mathcal{A}_{n,\rho+\delta,r}) \leq P(\mathcal{A}^*_{n,\rho+\delta,r})$ for all $r \in [0, r']$ so that

$$\limsup_{\delta \to 0^+} P(\mathcal{A}_{n,\rho+\delta,r}) \leq \limsup_{\delta \to 0^+} P(\mathcal{A}^*_{n,\rho+\delta,r}) = P(\mathcal{A}^*_{n,\rho,r}). \tag{6.8}$$

From the proof of theorem 4.6.15, it was shown that $r' = r_0$. Therefore, inequality 6.8 holds for all $r \in [0, r_0]$. Now, the monotone convergence theorem [51] gaurantees that $P(\mathcal{A}_{n,\rho+\delta,r}) \to P(\mathcal{A}_{n,\rho,r})$ as $\delta \to 0^+$. Therefore, inequality 6.8 becomes

$$P(\mathcal{A}_{n,\rho,r}) = \limsup_{\delta \to 0^+} P(\mathcal{A}_{n,\rho+\delta,r}) \leq P(\mathcal{A}^*_{n,\rho,r}). \tag{6.9}$$

In particular, $P(\mathcal{A}_{n,\rho,r}) \leq P(\mathcal{A}^*_{n,\rho,r})$ so that with the result of lemma 6.0.33, namely $P(\mathcal{A}^*_{n,\rho,r}) \leq P(\mathcal{A}_{n,\rho,r})$, the theorem follows. ∎

Corollary 6.0.36 $P(\mathcal{A}_r) = P(\mathcal{A}^*_r)$ *for* $r \in [0, r_0]$.

Proof By theorem 6.0.35, it is true that $P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}^*_{n,\rho,r})$ for all $r \in [0, r_0]$ and all $n \geq 1$. By proposition $A.2.64$, it follows that $P(\mathcal{A}^*_r) \leq P(\mathcal{A}^*_{n,\rho,r}) = P(\mathcal{A}_{n,\rho,r})$. In particular, $P(\mathcal{A}^*_r) \leq$

$P(\mathcal{A}_{n,\rho,r})$. Without loss of generality, assume that area$(\mathcal{B}) = 1$. From [5], differentiability of $P(\mathcal{A}_{n,\rho,r})$ in $\lambda = \lambda(n) = E[n]$ implies continuity of $P(\mathcal{A}_{n,\rho,r})$ in λ so that the following holds

$$\lim_{E[n]\to\infty} P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}_r). \tag{6.10}$$

Therefore, $P(\mathcal{A}_r^*) \leq P(\mathcal{A}_{n,\rho,r})$ and equation 6.10 implies $P(\mathcal{A}_r^*) \leq P(\mathcal{A}_r)$. Similarly, $P(\mathcal{A}_r) \leq P(\mathcal{A}_r^*)$ so that the corollary follows. ∎

Corollary 6.0.37 $r_0 = r_0^*$.

Proof By theorem 6.0.35, it is true that $1/2 = P(\mathcal{A}_{n,\rho,r_0}) = P(\mathcal{A}^*_{n,\rho,r_0})$. In particular, $1/2 = P(\mathcal{A}^*_{n,\rho,r_0})$. Since $1/2 = P(\mathcal{A}^*_{n,\rho,r_0^*})$ by theorem 5.4.19 and r_0^* is unique, then $r_0 = r_0^*$. ∎

Proof *(Theorem* 4.6.16*)* Follows directly from theorem 6.0.35 and theorem 5.5.31. ∎

What is implied by theorem 6.0.35 and corollary 6.0.37 is that the problem of estimating the probabilities and length of the sharp threshold interval in the continuum can be re-cast as problems of estimation in the presence of a hexagonal partition of the bounded region. As such, tools from percolation and the random cluster model theories can readily be employed. This fact will be of paramount importance in the applications to data classification where a data set consisting of multi-dimensional points is partitioned into disjoint, connected subsets. As it is advantageous to not have one connected cluster containing at least $100\rho\%$ of all points, since otherwise there may exist a single cluster containing almost all points by lemma A.1.46, the connection radius for points in the continuum must be in the sub-critical range $r \in [0, r_0)$ when classifying data into more than 2 classes. Since $P(\mathcal{A}_{n,\rho,r}) = P(\mathcal{A}^*_{n,\rho,r})$ for $r \in [0, r_0]$, disjoint clusters of points in the continuum are equivalent to disjoint clusters of occupied hexagons in the hexagonal partition of the bounded region containing all points. As such, multi-dimensional points in the continuum can be thought to belong to the same class if they are within a certain Euclidean distance of one another. As a result, the multi-dimensional points will have representatives belonging to occupied, connected hexagons in the 2-dimensional, bounded, partitioned region. All representatives in connected clusters of hexagons form the members of a class.

Chapter 7

Application to Data Classification

7.1 Motivation

Suppose a car dealership has a database of customer information containing age, marital status, current employer, credit history, etc. and it is desirable to be able to predict the class of vehicle that a new customer would buy based upon the historical car-purchasing patterns of other customers. In order to be able to address this problem, a model of the historical patterns should be devised which "predicts", with high accuracy, the class of vehicle that was purchased by previous customers given the historical data. As such, suppose there are M_1^2 data points, sampled from the population of historical data according to some probability distribution. Further suppose that these data points will be used to build the predictive model while other data points sampled from the population will be used to validate the predictive model. Following the run of validation points against the model, the predicted classifications are checked against the known class of cars bought by the actual customers in the validation set.

Suppose there are N_1^2 classes into which the M_1^2 data points will be classified and that M_1^2 is much greater than N_1^2. For this description, each class can be thought of as a combination of price range and type (sedan, sports car, SUV). Each customer will be classified by price range of vehicle according to the attributes of the customer as detailed in the data point. Each customer attribute is scored with a number in the interval between 0 and 1 so that if there are m attributes in each data point, then each data point maps to a point in the m-dimensional unit cube. Buyers fit into one of the N_1^2 classes if its measure against a current member of the class is "close". Since each data point maps to a point (node) in the unit cube, then a good measure of the "closeness" of 2 customers can be the Euclidean distance (or a constant multiple) between the corresponding points in the m-dimensional unit cube. It is shown below that there is a minimal upper bound on this distance that will guarantee at least N_1^2 classes. For now, denote this upper bound by $B(M_1, N_1)$, which depends upon the parameters M_1 and N_1. Now, define $e(x, y)$ to be the (possibly modified) Euclidean distance between m-dimensional points x and y in the unit cube. If x and y are 2 distinct data points with y already belonging to one of the N_1^2 classes, then x also belongs to the same class if $e(x, y) < B(M_1, N_1)$.

It is worthwhile to note that there is a trade-off between limiting the size of M_1 when determining the clusters of data points versus computing reliable values for the sample mean for each of the classes. In probability theory (see [9]), the Law of Large Numbers states that when the sample size is large, the sample mean is a very good estimate for the true mean of the distribution of data. However, as the sample size grows, computation of the distances for determining the classes grows more than linearly since each data point must be compared to every other data point in order to determine if they belong in the same class.

Suppose all M_1^2 data points have been classified and new points are to be classified. Rather than classify new data points by comparing its distance to every data point in each class to determine the appropriate classification, new data points are placed into the class where the distance from the data point to the mean of the class is less than $B(M_1, N_1)$.

7.2 Procedure

The idea is to partition $\mathcal{B}$ into M_1^2 hexagons and find N_1^2 contiguous clusters of hexagons such that each of the clusters are mutually disjoint. Into one and only one hexagon of a given cluster will each data point be mapped to form a node in the connected cluster. As such, the connected clusters of hexagons will be the N_1^2 classes containing a representative node associated to one and only one data point.

Begin by proving that there is a minimum number of hexagons in any partition of $\mathcal{B}$ containing N_1^2 disjoint subsets of hexagons with M_1^2 equaling the sum total of all hexagons in the disjoint sets. This minimum number is important in so far as the size of the hexagons will be predetermined, which has the effect of determining the maximum distance between interconnected nodes in the partition.

7.3 Partition

Theorem 7.3.38 *Assume that there are M_1^2 samples and N_1^2 classifications for the samples. The minimum number of hexagons required to partition the unit square into N_1^2 disjoint regions such that M_1^2 is the sum total of all hexagons in the disjoint regions is given by*

$$S(M_1, N_1) = M_1^2 + (N_1 - 1)^2 + 2M_1N_1.$$

Proof Since $M_1^2 >> N_1^2$ by hypothesis, then the total number of hexagons required to partition $\mathcal{B}$ into disjoint regions of contiguous hexagons is $O(M_1^2)$. Label the disjoint regions $A_1, A_2, ..., A_{N_1^2}$ and let k be any integer such that $1 \leq k \leq N_1^2$. Since the total number of hexagons partitioning $\mathcal{B}$ is $O(M_1^2)$, then the number of hexagons in A_k is proportional to M_1^2. Likewise, the total number of hexagons in boundary(A_k) is proportional to area(A_k). Since area(A_k) is proportional to M_1^2, then the number of hexagons in boundary(A_k) is proportional to M_1^2. Note that each A_k shares a portion of its separating boundary with each of its neighboring clusters of hexagons. Let A_j be a neighboring cluster of A_k such that $j \neq k$ and $1 \leq j \leq N_1^2$. Since this portion of the separating boundary is proportional to both area(A_k) and area(A_j), then it is proportional to a common area of size area(A_{kj}). Repeating this same logic for all integers k and j such that $1 \leq k \leq N_1^2$ and $1 \leq j \leq N_1^2$, the total number of hexagons in the entire separating boundaries is proportional to a common area of size area(A). Since minimizing the total number of hexagons in $\mathcal{B}$ is tantamount to minimizing the area(A), then each of the N_1^2 disjoint clusters of connected hexagons is the same size and must be a square sub-region of $\mathcal{B}$ containing M_1^2/N_1^2 hexagons. The minimum number of hexagons that are required to enclose N_1^2 sub-regions of $\mathcal{B}$ containing M_1^2/N_1^2 hexagons is exactly $(N_1 - 1)^2 + 2M_1N_1$. Therefore, the minimum number of hexagons required to partition $\mathcal{B}$ into N_1^2 disjoint regions such that M_1^2 is the sum total of all hexagons in the disjoint regions is given by

$$S(M_1, N_1) = M_1^2 + (N_1 - 1)^2 + 2M_1N_1. \quad \blacksquare$$

Using theorem 7.3.38, a maximum radius for connectivity of nodes in the bounded region can be obtained such that, with probability 1, there are N_1^2 disjoint clusters in existence.

7.4 Interval about the Critical Radius

From [2], theorem 3.2.4 states that there is a critical probability of connection between hexagons containing a node such that it is no longer possible to have disjoint clusters when this critical probability of connection is exceeded. Hence, all occupied hexagons will be connected into one cluster, which is not what is desired in this case. Since the size of $\mathcal{B}$ is fixed, then to decrease the probability of connection while maintaining N_1^2 disjoint contiguous clusters of hexagons containing a node, the size of each hexagon must decrease while increasing the number of hexagons in the boundaries. In this way, the ratio of the total number of occupied hexagons to the total number of hexagons will be less than this critical probability of connection. Note that the minimum number of hexagons required for separation is given by theorem 7.3.38 so that the common radius of the circle which can inscribe any one of these hexagons is of size

$$R(M_1, N_1) = \frac{1}{2\sqrt{S(M_1, N_1)}} \tag{7.1}$$

thereby necessarily indicating that

$$B(M_1, N_1) = 2R(M_1, N_1).$$

In order to NOT exceed the critical probability of connection, which means maintaining the N_1^2 classes of M_1^2 data points, the radial size of each hexagon must be less than or equal to $R(M_1, N_1)$. By theorem 3.2.4 from [2], the clusters will be disjoint with probability 1. Hence, the following corollary to theorem 7.3.38 follows from these statements.

Corollary 7.4.39 *Let h^r be a hexagon of size such that it can be inscribed into a circle of radius $r = r(M_1, N_1) > 0$ where*

$$0 < r \leq R(M_1, N_1).$$

If $\mathcal{B}$ is partitioned into copies of h^r, then with probability 1, the region $\mathcal{B}$ will contain (at least) N_1^2 disjoint regions of contiguous hexagons that are occupied by the M_1^2 nodes associated to data points in the classes. ∎

As stated previously, it is desirable to have the ratio of the M_1^2 occupied hexagons to the total number of hexagons $S(M_1, N_1)$ to be less than the critical probability of connection for the hexagonal lattice i.e.

$$p(M_1, N_1) = \frac{M_1^2}{S(M_1, N_1)} < \frac{1}{2} < 1 - 2\sin\left(\frac{\pi}{18}\right).$$

Lemma 7.4.40 *For fixed $\rho \in (\frac{1}{2}, 1)$ and $r > 0$ there exists $\delta \in (0, \frac{1}{2})$, such that $\{\frac{|<C>_{H(r)}|}{S(M_1,N_1)} < \frac{1}{2}\} = \mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$ upto sets of P measure zero.*

Proof By definition, $\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r} = \{\frac{|<C>_{H(r)}|}{S(M_1,N_1)} < \rho - \delta\}$. Take $\delta = \rho - \frac{1}{2}$. ∎

Therefore, by lemma 7.4.40, continuity in $r > 0$ and the non-decreasing property of $P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r})$ for decreasing $r > 0$ granted by corollary 5.3.17 and proposition $A.2.63$, respectively, then by inequality 5.9, it follows that

$$R(M_1, N_1) < r_0^* = r_0^*(M_1, N_1)$$

for the event $\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$, since this event is increasing for decreasing $r \leq r_0^*$, a reversal.

Let $\epsilon \in (0, \frac{1}{2})$ be given and let $r_1^* > 0$ and $r_2^* > 0$ guaranteed by corollary 5.3.17 be such that $P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r_1^*}) = 1 - \epsilon$ and $P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r_2^*}) = \epsilon$, respectively. Then, again by corollary 5.3.17, it follows that

$$R(M_1, N_1) < r_1^* < r_0^* = r_0^*(M_1, N_1) < r_2^*$$

since $P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,R(M_1,N_1)}) = 1$. By symmetry, it follows that

$$R(M_1, N_1) < r_1^* < r_0^* = r_0^*(M_1, N_1) < r_2^* < 2r_0^* - R(M_1, N_1).$$

Note that by corollary 7.4.39 and by symmetry that

$$P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}) = 0$$

when $r \geq 2r_0^* - R(M_1, N_1)$. Therefore, if $\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$ occurs with probability 0, then the event $\{\frac{M_1^2}{S(M_1,N_1)} < \frac{1}{2}\}$ occurs with probability 0. Otherwise, $\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$ would occur with positive probability since $\{\frac{M_1^2}{S(M_1,N_1)} < \frac{1}{2}\} \subseteq \{\frac{|<C>_{H(r)}|}{S(M_1,N_1)} < \frac{1}{2}\} = \mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$ upto sets of P-probability measure zero by lemma 7.4.40. Hence, $\{\frac{M_1^2}{S(M_1,N_1)} \geq \frac{1}{2}\}$ occurs with probability 1. As a result,

$$\frac{M_1^2}{M_1^2 + 2M_1N_1 + (N_1-1)^2} \geq \frac{1}{2}$$

with probability 1. Therefore, with probability 1 for M_1, it follows that N_1 has the solution

$$N_1 \geq 1 - M_1^2 + \sqrt{2M_1^2(M_1^2+1)}. \tag{7.2}$$

Lemma 7.4.41 *If $r \geq \frac{1}{2N_1}$, then $P(\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}) = 0$.*

Proof Without loss of generality, suppose area$(\mathcal{B}) = 1$ and further suppose that $\mathcal{B}$ is divided into squares with sides of length $2r = \frac{1}{N_1}$. By hypothesis, $\mathcal{B}$ contains M_1^2 data points and it is to be divided into N_1^2 regions. Clearly then, there are no boundary hexagons separating each of the N_1^2 regions since the sides of $\mathcal{B}$ have length $2rN_1 = 1$ which gives $\mathcal{B}$ an area of 1. Let each square be inscribed with a circle of radius r, which itself may be inscribed within a hexagon. By hypothesis, each of the N_1^2 hexagons in $\mathcal{B}$ contains at least one of the M_1^2 data points. Hence, each of the N_1^2 (occupied) hexagons is connected in a cluster to every other hexagon in $\mathcal{B}$ so that $P(\mathcal{A}^{*}_{S(M_1,N_1),\rho-\delta,r}) = 1$. Since $P(\mathcal{A}^{*}_{S(M_1,N_1),\rho-\delta,r}) = 1$ for $r = \frac{1}{2N_1}$, then $P(\mathcal{A}^{*}_{S(M_1,N_1),\rho-\delta,r}) = 1$ for $r \geq \frac{1}{2N_1}$ by proposition $A.2.63$. ∎

As a result of the preceding lemma, a conservative estimate for r_0^* is given by the minimal solution to

$$2r_0^* - R(M_1,N_1) \geq \frac{1}{2N_1}$$

where for M_1, the value of N_1 satisfies $N_1 \geq 1 - M_1^2 + \sqrt{2M_1^2(M_1^2+1)}$ and a minimal solution is found when $N_1 = 1 - M_1^2 + \sqrt{2M_1^2(M_1^2+1)}$ such that

$$2r_0^* - R(M_1,N_1) = \frac{1}{2N_1}. \tag{7.3}$$

As such, for $\epsilon \in (0, \frac{1}{2})$, since $(r_1^*, r_2^*) \subset (\ R(M_1,N_1), 2r_0^* - R(M_1,N_1)\ )$, then by equation 7.3

$$\begin{aligned} r_2^* - r_1^* &\approx 2r_0^* - 2R(M_1,N_1) \\ &= \frac{1}{2N_1} - R(M_1,N_1) \end{aligned} \tag{7.4}$$

is an estimate of the length of the sharp threshold interval $r_2^* - r_1^*$ about r_0^*.

Theorem 7.4.42 *Let $\Delta^*(M_1, N_1)$ denote the sharp threshold interval length for the event of classifying M_1^2 random data points into N_1^2 classes. Then,*

$$\Delta^*(M_1, N_1) = O(N_1^{-1}).$$

Proof Follows directly from equation 7.4, equation 7.1 and theorem 7.3.38. ∎

Using the value of r_0^* given by equation 7.3 and by using the estimate for the length of the sharp threshold interval about r_0^* given by equation 7.4, an estimate for the value of r_1^* can be obtained. Thus, when $r \leq r_1^*$, the event $\mathcal{A}^{*c}_{S(M_1,N_1),\rho-\delta,r}$ occurs with probability at least $1 - \epsilon$. Now, assuming that the means of each class have been calculated using the current members of the respective class, the car purchase of new customers associated to any new data points can be "predicted" with probability greater than $1 - \epsilon$ merely by measuring the distance from each data point to the mean of each class to determine if the distance is less than $2r_1^*$.

The process of measuring distances from new data points to the mean of each class is deterministic. As such, if at least N_1^2 classes are required to classify each possible data point encountered, then classification of a new data point only depends upon whether or not the corresponding class is formed. Now, by corollary 7.4.39, there is a non-zero probability that less than N_1^2 classes form when $r = r_1^* > R(M_1, N_1)$. If p_{N_1} represents the probability that less than N_1^2 classes form when $r = r_1^*$, then the mean number of misclassified data points is computed as $p_{N_1} M_1^2$ since the M_1^2 data points are uniformly distributed throughout $\mathcal{B}$.

Proposition 7.4.43 *When $r = r_1^*$, there is a non-zero probability that less than N_1^2 classes form.*

Proof Follows directly from theorem 7.3.38, corollary 7.4.39 and corollary 5.3.17. ∎

Corollary 7.4.44 *When $r = r_1^*$, the mean number of misclassified data points is $p_{N_1} M_1^2$, where $p_{N_1} > 0$ is the probability that less than N_1^2 classes form.*

Proof By proposition 7.4.43 there is a $p_{N_1} > 0$ probability that any new data point cannot be classified correctly when $r = r_1^*$. Since the data points are uniformly distributed, then the mean number of misclassified points is $p_{N_1} M_1^2$. ∎

Chapter 8

Summary and Future Research

8.1 Summary

It was assumed that n nodes in a set $\mathcal{X}_n$ were uniformly distributed throughout the continuum of a bounded region $\mathcal{B}$ of area 1 in the 2-dimensional plane with individual nodes communicating provided that their Euclidean distance $d(x, y) \leq r$ for $x, y \in \mathcal{X}_n$ and some predefined radius, r. For $\rho \in (\frac{1}{2}, 1)$, the set such that at least half of all n nodes communicate in a connected cluster was defined as $\mathcal{A}_{n,\rho,r} = \{< C >_r \subseteq \mathcal{X}_n : \rho_n(C) \geq \rho\}$, where $\rho_n(C) = \frac{N}{n}$. The probability distribution $P(\mathcal{A}_{n,\rho,r})$ was shown to be continuous as a function of r so that there exists $r_0 > 0$ such that $P(\mathcal{A}_{n,\rho,r_0}) = \frac{1}{2}$. If $\epsilon > 0$ is given and $r(n, \rho, \epsilon) = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon\}$, then the length of the interval about r_0 such that $P(\mathcal{A}_{n,\rho,r})$ increases from ϵ to $1 - \epsilon$ was defined to be $\Delta(n, \rho, \epsilon) = r(n, \rho, 1 - \epsilon) - r(n, \rho, \epsilon)$. The length $\Delta(n, \rho, \epsilon)$ was to be estimated. Using techniques from [7], it was shown that $\Delta(n, \rho, \epsilon) = O(r_0 \log^{\frac{1}{4}}(n))$. Yet, this estimate depends upon the unknown quantity, r_0. As such, a theorem was stated and later proven which gives a lower bound on $P(\mathcal{A}_{n,\rho,r})$ for $r \geq r_0$ and an upper bound on $P(\mathcal{A}_{n,\rho,r})$ for $r \leq r_0$. By continuity of $P(\mathcal{A}_{n,\rho,r})$ as a function of r, then given $\epsilon > 0$ such that $P(\mathcal{A}_{n,\rho,r_1}) = \epsilon$ and $P(\mathcal{A}_{n,\rho,r_2}) = 1 - \epsilon$, it must be the case that $r_1 < r_0 < r_2$ and $\Delta(n, \rho, \epsilon) = r_2 - r_1$. In order to show the upper and lower bound on $P(\mathcal{A}^*_{n,\rho,r})$, similar results to those proven in the continuum case were restated and proven in the presence of a hexagonal partition of $\mathcal{B}$ for the analogues, $\mathcal{A}^*_{n,\rho,r}$ and $P(\mathcal{A}^*_{n,\rho,r})$. It was shown that $r_0 \leq r^*_0$. Coupled with the upper and lower bounds on $P(\mathcal{A}^*_{n,\rho,r})$, the upper and lower bounds on $P(\mathcal{A}_{n,\rho,r})$ were established. As such, an estimate of the length $\Delta(n, \rho, r)$ of the interval about r_0 as $r_2 - r_1$ was established.

8.2 Future Research

To reiterate, it is assumed that n nodes are uniformly distributed throughout the continuum of some finite region, $\mathcal{B}$. As in [44], the network is self-organizing and each of the nodes operates in one of X, Y or Z mode. Given fixed communications radii r_y and r_z on Y and Z nodes respectively, it will be shown that there is a critical radius $r_{0,x} = r_{0,x}(n, \rho, r_y, r_z)$ on X nodes such that at least half of all nodes are connected with probability $\frac{1}{2}$. This will be accomplished by

1) Modifying the continuum model (chapter 4) to account for multiple node radii.

2) Modifying the hexagonal partition model (chapter 5) so that all nodes in the same hexagon are configured the same. As such, a binary word over the hexagons will constitute a configuration such that X, Y and Z nodes are represented with unoccupied hexagons being treated as though they were failed nodes.

With these adjustments, the analysis can continue as before. Lastly, using a genetic algorithm, a representative node configuration will be found such that coverage area, node overlap errors, isolated node errors, master-node ratio and network energy are minimized.

Appendix A

Supporting Results

A.1 Continuum Model

A.1.1 Graph

Proposition A.1.45 *If $r < r'$, then $G(\mathcal{X}_n; r) \subseteq G(\mathcal{X}_n; r')$.*

Proof Suppose $r < r'$. If $< x, y >_r \in G(\mathcal{X}_n; r)$, then $d(x, y) \leq r < r'$ so that $< x, y >_r \in G(\mathcal{X}_n; r')$. Hence, $G(\mathcal{X}_n; r) \subseteq G(\mathcal{X}_n; r')$. $\blacksquare$

A.1.2 Increasing Property

Lemma A.1.46 $|\mathcal{A}_{n,\rho,r}| \leq 1$.

Proof If $\mathcal{A}_{n,\rho,r} = \emptyset$, then there is nothing to prove. Thus, suppose that $\mathcal{A}_{n,\rho,r}$ occurs and $< C >_r$ $\in \mathcal{A}_{n,\rho,r}$. Since $\rho_n(C) \geq \rho > \frac{1}{2}$, then all other connected components are of order strictly less than half of all nodes. Therefore, $|\mathcal{A}_{n,\rho,r}| = 1$. $\blacksquare$

Proposition A.1.47 *$\mathcal{A}_{n,\rho,r}$ is an increasing property in r.*

Proof Suppose $< C >_r \in \mathcal{A}_{n,\rho,r}$ and fix arbitrary $r' > r$. Then, $d(x, y) \leq r < r'$ for all $x, y \in < C >_r$. Thus, $< C >_r \subseteq < C >_{r'}$ implies $N = | < C >_r | \leq | < C >_{r'} |$. Hence, $< C >_r \in \mathcal{A}_{n,\rho,r}$ implies $< C >_{r'} \in \mathcal{A}_{n,\rho,r}$. Since $r' > r$ is arbitrary, then $\mathcal{A}_{n,\rho,r}$ is an increasing property in r. $\blacksquare$

Proposition A.1.48 *$\mathcal{A}_{n,\rho,r}$ is a decreasing property in n.*

Proof Suppose $< C >_r \in \mathcal{A}_{n',\rho,r}$. If $n' < n$, then $| < C >_r |/n' > | < C >_r |/n \geq \rho$ so that $< C >_r \in \mathcal{A}_{n',\rho,r}$. Hence, $\mathcal{A}_{n,\rho,r} \subseteq \mathcal{A}_{n',\rho,r}$. Since $n' < n$, then $\mathcal{A}_{n,\rho,r}$ is decreasing in n. $\blacksquare$

A.1.3 Probability Measure

Proposition A.1.49 *The event $\mathcal{A}_{n,\rho,r}$ is P-measurable.*

Proof For $x, y \in \mathcal{X}_n$ and $S \subseteq \mathcal{X}_n$, define the state on $< x, y >_r$ to be 1 if and only if $< x, y >_r \in G(S; r)$ and -1 otherwise. Then, S mutually determines an element $\omega_S \in \Omega = \{-1, 1\}^{\mathcal{X}_n}$ so that S is P-measureable. Since $\mathcal{A}_{n,\rho,r}$ is the event that there exists $\omega_S \in \Omega$ mutually determined by $S \subseteq \mathcal{X}_n$ such that $(\max_{y \in S} | < C_y >_r |)/n \geq \rho$, then $\mathcal{A}_{n,\rho,r}$ is P-measureable. ∎

Proposition A.1.50 *$P(\mathcal{A}_{n,\rho,r})$ is a non-decreasing function of r.*

Proof Suppose $r_1 \leq r_2$. Since $\mathcal{A}_{n,\rho,r}$ is an increasing property in r by proposition $A.1.47$, then $\mathcal{A}_{n,\rho,r_1} \subseteq \mathcal{A}_{n,\rho,r_2}$ so that $P(\mathcal{A}_{n,\rho,r_1}) \leq P(\mathcal{A}_{n,\rho,r_2})$ by properties of probability measures. Thus, $P(\mathcal{A}_{n,\rho,r})$ is non-decreasing in r. ∎

Proposition A.1.51 *$P(\mathcal{A}_{n,\rho,r})$ is a non-increasing function of n.*

Proof Suppose $n' < n$. Since $\mathcal{A}_{n,\rho,r}$ is a decreasing property in n by proposition $A.1.48$, then $\mathcal{A}_{n,\rho,r} \subseteq \mathcal{A}_{n',\rho,r}$ so that $P(\mathcal{A}_{n,\rho,r}) \leq P(\mathcal{A}_{n',\rho,r})$ by properties of probability measures. Thus, $P(\mathcal{A}_{n,\rho,r})$ is non-increasing in n. ∎

A.1.4 Connection Radius

Proposition A.1.52 *$r(n, \rho, \epsilon)$ is a non-decreasing function of ϵ.*

Proof Suppose $\epsilon_1, \epsilon_2 \in (0, \frac{1}{2})$ such that $\epsilon_1 \leq \epsilon_2$. Define $r_1 = r(n, \rho, \epsilon_1)$ and $r_2 = r(n, \rho, \epsilon_2)$ and suppose $r_1 > r_2$. Since $P(\mathcal{A}_{n,\rho,r})$ is non-decreasing in r by proposition $A.1.50$, then $P(\mathcal{A}_{n,\rho,r_1}) \geq P(\mathcal{A}_{n,\rho,r_2}) \geq \epsilon_2 \geq \epsilon_1$. Hence, $r_2 \in \{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_1\}$ and $r_2 < r_1 = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_1\}$. Contradiction. Thus, $r_1 \leq r_2$ so that $r(n, \rho, \epsilon)$ is non-decreasing in ϵ. ∎

Lemma A.1.53 *If $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$, then $\mathcal{X}_n = \{x \in \mathcal{X}_n : d(x, y) \leq R\}$ for all fixed $y \in \mathcal{X}_n$.*

Proof Clearly, $\{x \in \mathcal{X}_n : d(x, y) \leq R\} \subseteq \mathcal{X}_n$. Conversely, fix any $y \in \mathcal{X}_n$. For every $x \in \mathcal{X}_n$, it is true that $d(x, y) \leq 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\} = R$. Hence, $\mathcal{X}_n \subseteq \{x \in \mathcal{X}_n : d(x, y) \leq R\}$ for all fixed $y \in \mathcal{X}_n$. Thus, $\mathcal{X}_n = \{x \in \mathcal{X}_n : d(x, y) \leq R\}$ for all fixed $y \in \mathcal{X}_n$. ∎

Corollary A.1.54 *If $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$, then $< C_y >_R \in \mathcal{A}_{n,\rho,R}$ for all $y \in \mathcal{X}_n$ and $n \geq 1$.*

Proof Fix an arbitrary $y \in \mathcal{X}_n$. By lemma $A.1.53$, if $< C_y >_R = \{x \in \mathcal{X}_n : d(x,y) \leq R\}$, then $< C_y >_R = \mathcal{X}_n$ so that $| < C_y >_R | = |\mathcal{X}_n| = n$. Therefore, since $y \in \mathcal{X}_n$ is arbitrary, then $< C_y >_R \in \mathcal{A}_{n,\rho,R}$ for all $y \in \mathcal{X}_n$ and $n \geq 1$. ∎

Corollary A.1.55 *If $R = 2 * \max\{d(x,y) : x, y \in \mathcal{X}_n\}$, then $P(\mathcal{A}_{n,\rho,R}) = 1$ for all $n \geq 1$.*

Proof By lemma $A.1.53$ and corollary $A.1.54$, it is true that $\mathcal{X}_n \in \mathcal{A}_{n,\rho,R}$ for all $n \geq 1$ and $\rho \in (\frac{1}{2}, 1)$. Thus, $\mathcal{A}_{n,\rho,R} \neq \emptyset$ for all $n \geq 1$ and $\rho \in (\frac{1}{2}, 1)$. Hence, $P(\mathcal{A}_{n,\rho,R}) = 1$ for all $n \geq 1$. ∎

Lemma A.1.56 *If $R = 2 * \max\{d(x,y) : x, y \in \mathcal{X}_n\}$, then $0 < r(n,\rho,\epsilon) \leq R$ for all $\epsilon \in (0, \frac{1}{2})$.*

Proof By lemma $A.1.53$, it is true that $\mathcal{X}_n = \{x \in \mathcal{X}_n : d(x,y) \leq R\}$ for all fixed $y \in \mathcal{X}_n$. Therefore, $P(\mathcal{A}_{n,\rho,R}) = 1 \geq \epsilon$ for all $\epsilon \in (0, \frac{1}{2})$. Suppose that $\epsilon_0 \in (0, \frac{1}{2})$ exists such that $r_0 = r(n,\rho,\epsilon_0) > R$. Thus, $\mathcal{A}_{n,\rho,R} \subseteq \mathcal{A}_{n,\rho,r_0}$ so that

$$\begin{aligned} 1 &= P(\mathcal{A}_{n,\rho,R}) \\ &\leq P(\mathcal{A}_{n,\rho,R}) \\ &\leq P(\mathcal{A}_{n,\rho,r_0}) \end{aligned}$$

since $P(\mathcal{A}_{n,\rho,r})$ is non-increasing in n by proposition $A.1.54$, non-decreasing in r by proposition $A.1.50$ and by properties of probability measures. Hence, $P(\mathcal{A}_{n,\rho,r_0}) = 1$. But, then $R \in \{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_0\}$ and $R < r_0 = \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_0\}$. Contradiction. Thus, $0 < r_0 \leq R$. Therefore, since ϵ_0 is arbitrary, then $0 < r(n,\rho,\epsilon) \leq R$ for all $\epsilon \in (0, \frac{1}{2})$. ∎

Proposition A.1.57 *Suppose $\{\epsilon_k \in (0, \frac{1}{2})\}_{k \geq 1}$ is any convergent sequence such that $\epsilon_k \to \epsilon_0$. Define $r_k = r(n,\rho,\epsilon_k)$ and $r_0 = r(n,\rho,\epsilon_0)$. For arbitrary $\xi > 0$, if $\{k \geq 1 : |P(\mathcal{A}_{n,\rho,r_k}) - P(\mathcal{A}_{n,\rho,r_0})| \geq \xi\}$ is a set of measure zero, then $r_k \to r_0$ as $k \to \infty$.*

Proof If $\xi > 0$ is arbitrary and $\{k \geq 1 : |P(\mathcal{A}_{n,\rho,r_k}) - P(\mathcal{A}_{n,\rho,r_0})| \geq \xi\}$ is a set of measure zero, then

$$P(\mathcal{A}_{n,\rho,r_k}) = P(\mathcal{A}_{n,\rho,r_0}) \geq \epsilon_0$$

for all $k \geq 1$. Hence, $r_k \in \{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_0\}$ for all $k \geq 1$. Thus,

$$\lim_{k \to \infty} r_k = \lim_{k \to \infty} r(n,\rho,\epsilon_k)$$

$$\begin{aligned} &= \lim_{k\to\infty} \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_k\} && \text{(A.1)} \\ &= \inf\{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_0\} && \text{(A.2)} \\ &= r(n, \rho, \epsilon_0) \\ &= r_0 \end{aligned}$$

where equation $A.1$ and equation $A.2$ follow since $r_k \in \{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_k\} \cap \{r > 0 : P(\mathcal{A}_{n,\rho,r}) \geq \epsilon_0\}$ for all $k \geq 1$ and $\epsilon_k \to \epsilon_0$ as $k \to \infty$. ∎

A.2 Hexagonal Partition Model

A.2.1 Graph

Proposition A.2.58 *If $r < r'$, then $G(\mathcal{X}_n; H(r)) \subseteq G(\mathcal{X}_n; H(r'))$.*

Proof Suppose $r < r'$. Choose a coordinate system so that $\hat{0} = (0,0)$ is defined such that $d(x,\hat{0}) = \frac{d(x,y)}{2} = d(\hat{0}, y)$ and x and y lie on the same coordinate axis. Orient $H(r)$ and $H(r')$ so that $x \in h^r_{i_x,j_x} \cap h^{r'}_{i_x,i_y}$ and $y \in h^r_{i_y,j_y} \cap h^{r'}_{i_y,j_y}$ are centered along the same coordinate axis. If $< x, y >_{H(r)} \in G(\mathcal{X}_n; H(r))$, then $h(h^r_{i_x,j_x}, h^r_{i_y,j_y}) \leq 1$. If $h(h^{r'}_{i_x,j_x}, h^{r'}_{i_y,j_y}) > 1$, then $h(h^r_{i_x,j_x}, h^r_{i_y,j_y}) = h(h^r_{i_x,j_x} \cap h^{r'}_{i_x,j_x}, h^r_{i_y,j_y} \cap h^{r'}_{i_y,j_y}) > 1$. Contradiction. Thus, $< x, y >_{H(r)} \in G(\mathcal{X}_n; H(r'))$ so that $G(\mathcal{X}_n; H(r)) \subseteq G(\mathcal{X}_n; H(r'))$. ∎

A.2.2 Increasing Property

Lemma A.2.59 $|\mathcal{A}^*_{n,\rho,r}| \leq 1$.

Proof If $\mathcal{A}^*_{n,\rho,r} = \emptyset$, then there is nothing to prove. Thus, suppose that $\mathcal{A}^*_{n,\rho,r}$ occurs and $< C >_{H(r)} \in \mathcal{A}^*_{n,\rho,r}$. Since $\rho^*_n(C) \geq \rho > \frac{1}{2}$, then all other connected components are of order strictly less than half of all nodes. Therefore, $|\mathcal{A}^*_{n,\rho,r}| = 1$. ∎

Proposition A.2.60 *$\mathcal{A}^*_{n,\rho,r}$ is an increasing property in r.*

Proof Suppose $< C >_{H(r)} \in \mathcal{A}^*_{n,\rho,r}$. Fix arbitrary $r' > r$ and let $x, y \in < C >_{H(r)}$. By proposition $A.2.58$, it is true that $< x, y >_{H(r)} \in G(\mathcal{X}_n; H(r)) \subseteq G(\mathcal{X}_n; H(r'))$. Thus, $< C >_{H(r)} \subseteq < C >_{H(r')}$. Since $N \leq | < C >_{H(r)} | \leq | < C >_{H(r')} |$, then $< C >_{H(r)} \in \mathcal{A}^*_{n,\rho,r}$ implies $< C >_{H(r')} \in \mathcal{A}^*_{n,\rho,r}$. Since $r' > r$ is arbitrary, then $\mathcal{A}^*_{n,\rho,r}$ is an increasing property in r. ∎

Proposition A.2.61 *$\mathcal{A}^*_{n,\rho,r}$ is a decreasing property in n.*

Proof Suppose $< C >_{H(r)} \in \mathcal{A}^*_{n',\rho,r}$. If $n' < n$, then $| < C >_{H(r)} |/n' > | < C >_{H(r)} |/n \geq \rho$ so that $< C >_{H(r)} \in \mathcal{A}^*_{n',\rho,r}$. Hence, $\mathcal{A}^*_{n,\rho,r} \subseteq \mathcal{A}^*_{n',\rho,r}$. Since $n' < n$, then $\mathcal{A}^*_{n,\rho,r}$ is decreasing in n. ∎

A.2.3 Probability Measure

Proposition A.2.62 *The event $\mathcal{A}^*_{n,\rho,r}$ is P-measurable.*

Proof For $x, y \in \mathcal{X}_n$ and $S \subseteq \mathcal{X}_n$, define the state on $< x, y >_{H(r)}$ to be 1 if and only if $< x, y >_{H(r)}$ $\in G(S; H(r))$ and -1 otherwise. Then, S mutually determines an element $\omega_S \in \Omega = \{-1, 1\}^{\mathcal{X}_n}$ so that S is P-measureable. Since $\mathcal{A}^*_{n,\rho,r}$ is the event that there exists ω_S mutually determined by S such that $(\max_{y \in S} | < C_y >_{H(r)} |)/n \geq \rho$, then $\mathcal{A}^*_{n,\rho,r}$ is P-measurable. ∎

Proposition A.2.63 *$P(\mathcal{A}^*_{n,\rho,r})$ is a non-decreasing function of r.*

Proof Suppose $r_1^* \leq r_2^*$. Since $\mathcal{A}^*_{n,\rho,r}$ is an increasing property in r by proposition $A.2.60$, then $\mathcal{A}^*_{n,\rho,r_1^*} \subseteq \mathcal{A}^*_{n,\rho,r_2^*}$ so that $P(\mathcal{A}^*_{n,\rho,r_1^*}) \leq P(\mathcal{A}^*_{n,\rho,r_2^*})$ by properties of probability measures. Thus, $P(\mathcal{A}^*_{n,\rho,r})$ is non-decreasing in r. ∎

Proposition A.2.64 *$P(\mathcal{A}^*_{n,\rho,r})$ is a non-increasing function of n.*

Proof Suppose $n' < n$. Since $\mathcal{A}^*_{n,\rho,r}$ is a decreasing property in n by proposition $A.2.61$, then $\mathcal{A}^*_{n,\rho,r} \subseteq \mathcal{A}^*_{n',\rho,r}$ so that $P(\mathcal{A}^*_{n,\rho,r}) \leq P(\mathcal{A}^*_{n',\rho,r})$ by properties of probability measures. Thus, $P(\mathcal{A}^*_{n,\rho,r})$ is non-increasing in n. ∎

A.2.4 Connection Radius

Proposition A.2.65 *$r^*(n, \rho, \epsilon)$ is a non-decreasing function of ϵ.*

Proof Suppose $\epsilon_1, \epsilon_2 \in (0, \frac{1}{2})$ such that $\epsilon_1 \leq \epsilon_2$. Define $r_1^* = r^*(n, \rho, \epsilon_1)$ and $r_2^* = r^*(n, \rho, \epsilon_2)$ and suppose $r_1^* > r_2^*$. Since $P(\mathcal{A}^*_{n,\rho,r}$ is non-decreasing in r by proposition $A.2.63$, then $P(\mathcal{A}^*_{n,\rho,r_1^*}) \geq P(\mathcal{A}^*_{n,\rho,r_2^*}) \geq \epsilon_2 \geq \epsilon_1$. Hence, $r_2^* \in \{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_1\}$ and $r_2^* < r_1^* = \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_1\}$. Contradiction. Thus, $r_1^* \leq r_2^*$ so that $r^*(n, \rho, \epsilon)$ is non-decreasing in ϵ. ∎

Lemma A.2.66 *If $R = 2 * \max\{d(x, y) : x, y \in \mathcal{X}_n\}$, then $\mathcal{X}_n = \{x \in \mathcal{X}_n : h(h^R_{i_x,j_x}, h^R_{i_y,j_y}) = 0\}$ for all fixed $y \in \mathcal{X}_n$.*

Proof By lemma $A.1.56$, $\mathcal{X}_n = \{x \in \mathcal{X}_n : d(x, y) \leq R\}$. Let C_R be a circle of radius R such that $\mathcal{X}_n \subset C_R \subset \mathcal{B}$. Choose an orientation of H^R such that $h^R_{i_x,j_x} \in H^R$ inscribes C_R for some fixed $x \in \mathcal{X}_n$. Then, for all $y \in \mathcal{X}_n$, it is true that $y \in h^R_{i_x,j_x}$. Hence, $h^R_{i_x,j_x} = h^R_{i_y,j_y}$ for all $y \in \mathcal{X}_n$ so that $h(h^R_{i_x,j_x}, h^R_{i_y,j_y}) = 0$. ∎

Corollary A.2.67 *If $R = 2 * \max\{d(x,y) : x, y \in \mathcal{X}_n\}$, then $< C_y >_{H^R} \in \mathcal{A}^*_{n,\rho,R}$ for all $y \in \mathcal{X}_n$ and $n \geq 1$.*

Proof Fix an arbitrary $y \in \mathcal{X}_n$. By lemma $A.2.66$, if $< C_y >_{H^R} = \{x \in \mathcal{X}_n : h(h^R_{i_x,j_x}, h^R_{i_y,j_y}) = 0\}$, then $< C_y >_{H^R} = \mathcal{X}_n$ so that $| < C_y >_{H^R} | = |\mathcal{X}_n| = n$. Therefore, since $y \in \mathcal{X}_n$ is arbitrary, then $< C_y >_{H^R} \in \mathcal{A}^*_{n,\rho,R}$ for all $y \in \mathcal{X}_n$ and $n \geq 1$. ∎

Corollary A.2.68 *If $R = 2 * \max\{d(x,y) : x, y \in \mathcal{X}_n\}$, then $P(\mathcal{A}^*_{n,\rho,R}) = 1$ for all $n \geq 1$.*

Proof By lemma $A.2.66$ and corollary $A.2.67$, $\mathcal{X}_n \in \mathcal{A}^*_{n,\rho,R}$ for all $n \geq 1$ and $\rho \in (\frac{1}{2}, 1)$. Thus, $\mathcal{A}^*_{n,\rho,R} \neq \emptyset$ for all $n \geq 1$ and $\rho \in (\frac{1}{2}, 1)$. Hence, $P(\mathcal{A}^*_{n,\rho,R}) = 1$ for all $n \geq 1$ and $\rho \in (\frac{1}{2}, 1)$. ∎

Lemma A.2.69 *If $R = 2 * \max\{d(x,y) : x, y \in \mathcal{X}_n\}$, then $0 < r^*(n, \rho, \epsilon) \leq R$ for all $\epsilon \in (0, \frac{1}{2})$.*

Proof By lemma $A.2.66$, $\mathcal{X}_n = \{x \in \mathcal{X}_n : h(h^R_{i_x,j_x}, h^R_{i_y,j_y}) = 0\}$ for all fixed $y \in \mathcal{X}_n$. Therefore, $P(\mathcal{A}^*_{n,\rho,R}) = 1 \geq \epsilon$ for all $\epsilon \in (0, \frac{1}{2})$. Suppose that $\epsilon_0 \in (0, \frac{1}{2})$ exists such that $r^*_0 = r^*(n, \rho, \epsilon_0) > R$. Thus, $\mathcal{A}^*_{n,\rho,R} \subseteq \mathcal{A}^*_{n,\rho,r^*_0}$ so that

$$\begin{aligned} 1 &= P(\mathcal{A}^*_{n,\rho,R}) \\ &\leq P(\mathcal{A}^*_{n,\rho,R}) \\ &\leq P(\mathcal{A}^*_{n,\rho,r^*_0}) \end{aligned}$$

since $P(\mathcal{A}^*_{n,\rho,r})$ is non-increasing in n by proposition $A.2.64$, non-decreasing in r by proposition $A.2.63$ and by properties of probability measures. Hence, $P(\mathcal{A}^*_{n,\rho,r^*_0}) = 1$. But, then $R \in \{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_0\}$ and $R < r^*_0 = \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_0\}$. Contradiction. Thus, $0 < r^*_0 \leq R$. Therefore, since ϵ_0 is arbitrary, then $0 < r^*(n, \rho, \epsilon) \leq R$ for all $\epsilon \in (0, \frac{1}{2})$. ∎

Proposition A.2.70 *Suppose $\{\epsilon_k \in (0, \frac{1}{2})\}_{k \geq 1}$ is any convergent sequence such that $\epsilon_k \to \epsilon_0$. Define $r^*_k = r^*(n, \rho, \epsilon_k)$ and $r^*_0 = r^*(n, \rho, \epsilon_0)$. For arbitrary $\xi > 0$, if $\{k \geq 1 : |P(\mathcal{A}^*_{n,\rho,r^*_k}) - P(\mathcal{A}^*_{n,\rho,r^*_0})| \geq \xi\} = \emptyset$, then $r^*_k \to r^*_0$ as $k \to \infty$.*

Proof If $\xi > 0$ is arbitrary and $\{k \geq 1 : |P(\mathcal{A}^*_{n,\rho,r^*_k}) - P(\mathcal{A}^*_{n,\rho,r^*_0})| \geq \xi\} = \emptyset$, then

$$P(\mathcal{A}^*_{n,\rho,r^*_k}) = P(\mathcal{A}^*_{n,\rho,r^*_0}) \geq \epsilon_0$$

for all $k \geq 1$. Hence, $r^*_k \in \{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_0\}$ for all $k \geq 1$. Thus,

$$\lim_{k \to \infty} r^*_k = \lim_{k \to \infty} r^*(n, \rho, \epsilon_k)$$

$$
\begin{aligned}
&= \lim_{k\to\infty} \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_k\} && \text{(A.3)}\\
&= \inf\{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_0\} && \text{(A.4)}\\
&= r^*(n, \rho, \epsilon_0)\\
&= r^*_0
\end{aligned}
$$

where equation $A.3$ and equation $A.4$ follow since $r^*_k \in \{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_k\}, \{r > 0 : P(\mathcal{A}^*_{n,\rho,r}) \geq \epsilon_0\}$ for all $k \geq 1$ and $\epsilon_k \to \epsilon_0$ as $k \to \infty$. ∎

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