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Correspondence between Multiwavelet Shrinkage/Multiple Wavelet Frame Shrinkage and Nonlinear Diffusion

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Abstract

There are numerous methodologies for signal and image denoising. Wavelet, wavelet frame shrinkage, and nonlinear diffusion are effective ways for signal and image denoising. Also, multiwavelet transforms and multiple wavelet frame transforms have been used for signal and image denoising. Multiwavelets have important property that they can possess the orthogonality, short support, good performance at the boundaries, and symmetry simultaneously. The advantage of multiwavelet transform for signal and image denoising was illustrated by Bui et al. in 1998. They showed that the evaluation of thresholding on a multiwavelet basis has produced good results. Further, Strela et al. have showed that the decimated multiwavelet denoising provides superior results than decimated conventional (scalar) wavelet denoising. Mrazek, Weickert, and Steidl in 2003 examined the association between one-dimensional nonlinear diffusion and undecimated Haar wavelet shrinkage. They proved that nonlinear diffusion could be presented by using wavelet shrinkage. High-order nonlinear diffusion in terms of one-dimensional frame shrinkage and two-dimensional frame shrinkage were presented in 2012 by Jiang, and in 2013 by Dong, Jiang, and Shen, respectively. They obtained that the correspondence between both approaches leads to a different form of diffusion equation that mixes benefits from both approaches.

The objective of this dissertation is to study the correspondence between one-dimensional multiwavelet shrinkage and high-order nonlinear diffusion, and to study high-order nonlinear diffusion in terms of one-dimensional multiple frame shrinkage also well. Further, this dissertation formulates nonlinear diffusion in terms of 2D multiwavelet shrinkage and 2D multiple wavelet frame shrinkage. From the experiment results, it can be inferred that nonlinear diffusion in terms of multiwavelet shrinkage/multiple frame shrinkage gives better results than a scalar case.

On the whole, this dissertation expands nonlinear diffusion in terms of wavelet shrinkage and nonlinear diffusion in terms of frame shrinkage from the scalar wavelets and frames to the multiwavelets and multiple frames.

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Contents

List of Figures

List of Tables

Abbreviations

CL(2): Chui-Lian multifilter.
CWT: Continuous wavelet transform.
D_4: Daubechies wavelet filter with 4 wavelet and scaling function coefficients.
1D: One dimension.
2D: Two dimension.
DGHM: Donovan-Geronimo-Hardin-Massopust multifilter.
DMWT: Discrete multiwavelet transform.
DWT: Discrete wavelet transform.
FIR: Finite impulse response filter.
MAX: Maximum possible pixel value of the image.
MRA: Multiresolution analysis.
MSE: Mean squared error.
MWT : Multiwavelet transform.
PSNR: Peak signal to noise ratio.
PM: Perona-Malik diffusion.
SNR: Signal to noise ratio.
UFT: Undecimated frame transform.
UWT: Undecimated wavelet transform.

Chapter 1

Introduction

Wavelets shrinkage, frame shrinkage, and nonlinear diffusion filtering have been effectively utilized as a part of signal and image denoising [9,25,45,46,55,58,61,62,63,66,67]. The advantage of wavelet transform was first illustrated by Donoho and Johnstone [13,14,15]. They determined that the evaluation of thresholding in a wavelet basis has performed best in the worst possible case for sets of piecewise regular images. They implied two thresholding functions: hard-threshold

$$\eta_T^H(x) = \begin{cases} x, |x| \geq T \\ 0, |x| < T \end{cases} \tag{1.1}$$

and soft-threshold

$$\eta_T^S(x) = \begin{cases} sgn(x)(|x| - T), |x| \geq T \\ 0, |x| < T \end{cases} . \tag{1.2}$$

The undecimated transforms are used to minimize artifacts in the denoised data. The establishment of this idea was by Coifman and Donoho [9]. They showed that undecimated single wavelet is better than decimated single wavelet denoising. Furthermore, an undecimated illustration is more advanced than a decimated illustration for image denoising. Numerous undecimated illustrations have been proposed to image denoising [3,58,61,62].

Multiwavelet has drawn much consideration in the signal processing in recent years [8,10,21,22,23,29,30,31,32,33,38,39,44,48,60]. The extension of wavelet was obtained by Donovan, Geronimo, Hardin, and Massopust. They suggested using two scaling functions to approximate a signal [16]. Also, the constructions of multiwavelets and the design of multifilter banks can be found in many papers such as [11,23,38,39,68]. The advantage of multiwavelet transform was illustrated by Bui, Tien, and Chen. They proposed that the evaluation of thresholding in a multiwavelet basis has performed good results [3]. Further, Strela et al. have provided that decimated multiwavelet denoising provides superior results than decimated single wavelet denoising [68]. Thus, Tien, Bui, and Chen developed Coifman and Donoho‘s undecimated single wavelet denoising strategy to multiwavelet and the result demonstrated that undecimated multiwavelet denoising is more appropriate than the singular case for soft thresholding [3],[13]. Multiwavelet has important properties such as orthogonality, short support, good performance at the boundaries, and symmetry, etc. that scalar wavelets fail to possess these properties simultaneously [38,39]. Moreover, multiwavelet achieves better results for image processing in comparison with wavelets in a scalar case.

Association between wavelet shrinkage and diffusion has been studied in [53,54,67]. Also, diffusion in terms of frame shrinkage has been examined in [12,41,71]. Mrazek, Weickert, and Steidl examined the association between one-dimensional nonlinear diffusion and undecimated Haar wavelet shrinkage [54]. They prove that nonlinear diffusion could be presented by wavelet shrinkage. High-order nonlinear diffusion in terms of frame shrinkage is presented in [12,41]. In addition, the equivalent between multiscale wavelet frame shrinkage and nonlinear diffusion in general case is presented by Wang and Huang in [71]. They generalized nonlinear diffusion in terms of frame shrinkage from single scale to multiscale. Equivalent between wavelet/wavelet frame shrinkage and diffusion filtering present diffusion in terms of shrinkage functions with advance achievement. The correspondence between both approaches leads to a different form of diffusion equations. Further, it may assist to move results from one scheme to the other and to mixes benefits from both approaches.

This dissertation is divided into eight chapters starting with a prologue followed by preliminaries in chapter 2, which is concerned with different topics including wavelet, continuous wavelet transforms and multiresolution. Also, decimated and undecimated wavelet transform for 1D and 2D at one scale and multiscale are presented. Further, we discussed wavelet denoising, decimated and undecimated frame transform for 1D and 2D at one scale and multiscale, and frame denoising.

A basic review about multiwavelet, multiresolution approximation and construction of biorthogonal multiwavelet are also provided. In addition, decimated and undecimated multiwavelet transform, multiwavelet shrinkage are presented as well. Also, the discrete multiframe transform and multiple wavelet frame shrinkage are reviewed.

This dissertation will rely on the results of the next two chapters that present nonlinear diffusion in terms of wavelet shrinkage in chapter 3 and nonlinear diffusion in terms of frame shrinkage in chapter 4.

In chapter 5, the equivalence between one-dimensional undecimated multiwavelet shrinkage and second-order nonlinear diffusion equation using Haar, CL(2), and DGHM multifilters are provided. Also, the results are generalized by formulating the equivalence between multiwavelet shrinkage and high-order nonlinear diffusion. Experiment results and comparisons for this scheme are also presented.

In chapter 6, diffusion in terms of one-dimensional multiple frame shrinkage has been formulated. Graphical outcomes and comparisons are also discussed in chapter 6 .

In chapter 7, the second part of the results in this dissertation, the equivalence between two-dimensional multiwavelet shrinkage/multiple frame shrinkage and nonlineare diffusion are provided. In addition, the experiment result are presented in this chapter. This dissertation end with final remarks and conclusions in chapter 8.

Chapter 2

Preliminaries

Basic concepts and preliminary are presented in this chapter. The first section provides a review of wavelets. Section 2.2 discusses continuous wavelet transform. Multiresolution analysis is defined in Section 2.3. Decimated and undecimated wavelet transforms are studied in Sections 2.4 and 2.5, respectively. In addition, Section 2.6 is devoted to wavelet denoising. In Sections 2.7-2.8, we explain an affine frame, quasi-affine frame and frame denoising. The review of multiwavelet and multiwavelet frames are provided with more detail in Sections 2.9 and 2.10, respectively. Nonlinear diffusion equations are discussed in the last Section of this chapter.

2.1 Wavelets

Examining and studying a signal in accordance to the scale is the main idea to wavelet. It is a two channel digital filter bank that restates on the lowpass output if observed from an engineering point of view. At a given scale, lowpass filtering produces an estimated signal, while on the other side, the highpass filtering provides the information that shows the difference between these two consecutive estimates. Bandpass and a group of scaling functions with lowpass filters are associated with a family of wavelet.

The 20th century was the beginning of wavelet. The simplest wavelet, a Haar wavelet, was designed by Alfred Haar. Then Haar wavelet, which is a scale-varying basis function, was discovered by Paul Levy.

The major achievement in the research of wavelets was the multiresolution analysis proposed by Stephane Mallet [41]. Mallet‘s work allowed analyzing signals by using wavelet functions of various resolutions and it helped researchers to construct their own family of wavelets utilizing well defined criteria. The concept of multiresolution analysis was used to construct a set of Daubechies wavelets by Ingrid Daubechies, which contains of outstanding properties such as orthogonality, compact support, continuouity and regularity.

Mother wavelet, which is a unique prototype function, helps to generate the wavelet family. Function ψ is known as the mother wavelet only if it wavers almost to zero

average.
The function $\psi(t) \in L_2(R)$ is called a **wavelet** if it satisfies :

$$PV \int_{-\infty}^{\infty} \psi(t)\ dt = \lim_{A\to\infty} \int_{-A}^{A} \psi(t)\ dt = 0 \tag{2.1}$$

$\psi(t) \to 0$ for $t \to \pm\infty$. Let $b \in \mathbb{R}$ and $a > 0$ be a two real number then:

$$\psi_{a,b} = \frac{1}{a}\psi(\frac{t-b}{a})$$

is called wavelets generated by ψ.

2.2 Continuous wavelet transform (CWT)

Continuous wavelet transform was presented in many papers such as [2,9,15,18,20,24,46]. **Continuous wavelet transform** of $f \in L_2(R)$ is given by:

$$(W_\psi f)(b,a) = \langle f, \psi_{b,a}\rangle = \frac{1}{a}\int_{-\infty}^{\infty} f(t)\psi(\frac{b-t}{a})dt \tag{2.2}$$

and the **inverse of continuous wavelet transform** is defined as

$$f(x) = C_\psi^{-1}\int_0^{\infty}\{\int_{-\infty}^{\infty}(W_\psi f)(b,a)\psi(\frac{x-b}{a})db\}\frac{da}{a^2} \tag{2.3}$$

where C_ψ is given by:

$$0 < C_\psi = \int_0^{\infty}\frac{|\psi(w)|^2}{w}dw < \infty.$$

2.3 Multiresolution analysis

The concept of multiresolution analysis was proposed by Meyer and Mallat, which yields a natural framework for the understanding of wavelet bases. A multiresolution approximation of $L_2(\mathbb{R})$ is denoted as a sequence $\{V_j\}$ of closed subspace of $L_2(\mathbb{R})$ that satisfies some properties[51]:

1. $V_j \subset V_{j+1}\ \forall j \in \mathbb{Z}$
2. $\bigcup_{j=-\infty}^{+\infty} V_j$ *is dense in* $L^2(\mathbb{R})$ *and* $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$
3. $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}\ ,\forall j \in \mathbb{Z}$
4. $f(x) \in V_j \Rightarrow f(x - 2^{-j}k) \in V_j\ ,\ for\ all\ k \in \mathbb{Z}$
5. Also, there exists a real-valued function $\phi \in L_2(\mathbb{R})$ such that $\{\phi(.-k) : k \in \mathbb{Z}\}$ is a Riesz basis of $\mathbb{V}_0$, that is,$\mathbb{V}_0 = \overline{span}\{\phi(.-k) : k \in \mathbb{Z}\}$ and there exist some constant c, $C > 0$, such that

$$c\sum_{k\in\mathbb{Z}} ||c_k||^2 \leq \left|\left|\sum_{k\in\mathbb{Z}} c_k\phi(x-k)\right|\right|^2 \leq C\sum_{k\in\mathbb{Z}}|c_k|^2\ \ for\ all\{c_k\} \in \ell^2(\mathbb{Z})$$

2.4 Decimated wavelet transform

Discrete wavelet transforms has been utilized expressly by numerous studies (see e.g. [2,5,6,9,55,66,70]). Let $p, q, \tilde{p}, \tilde{q}$ be a biorthogonal FIR filter bank, their symbols satisfy:

$$\tilde{P}(z)P(\frac{1}{z}) + \tilde{P}(-z)P(\frac{-1}{z}) = 1$$

$$\tilde{Q}(z)Q(\frac{1}{z}) + \tilde{Q}(-z)Q(-\frac{1}{z}) = 1$$

$$\tilde{Q}(z)P(\frac{1}{z}) + \tilde{Q}(-z)P(-\frac{1}{z}) = 0$$

$$\tilde{P}(z)Q(\frac{1}{z}) + \tilde{P}(-z)Q(-\frac{1}{z}) = 0.$$

Suppose the scaling functions associated with the lowpass filters $p, \tilde{p}$ are $\phi, \tilde{\phi}$, and $\psi, \tilde{\psi}$ are the corresponding biorthogonal wavelet functions with highpass filters $q, \tilde{q}$. Let $\{x_k\}$ be an input sequence x. Then the discrete wavelet transform with the analysis filters $\tilde{p}$, $\tilde{q}$ for one-dimenstion and one-scale is given by:

$$L_n = \frac{1}{\sqrt{2}} \sum_k \tilde{p}_{k-2n} x_k, \quad H_n = \frac{1}{\sqrt{2}} \sum_k \tilde{q}_{k-2n} x_k, \tag{2.4}$$

and inverse discrete wavelet transform with p, q filters is defined by:

$$\tilde{x}_n = \frac{\sqrt{2}}{4} \sum_k p_{n-2k} L_k + \frac{\sqrt{2}}{4} \sum_k q_{n-2k} H_k. \tag{2.5}$$

Discrete wavelet transform is a decimated transform, which downsamples the decomposition by dropping the odd-index term, so Eq.(2.4) can be defined as:

$$\underline{L} = \frac{1}{\sqrt{2}}(\tilde{p}^- * \underline{x}) \downarrow 2 \quad , \underline{H} = \frac{1}{\sqrt{2}}(\tilde{q}^- * \underline{x}) \downarrow 2, \; \tilde{p}_n^- = \tilde{p}_{-n},$$

and upsampling the inverse discrete wavelet transform is given by:

$$\underline{\tilde{x}} = \frac{\sqrt{2}}{4}(p * \{\underline{L} \uparrow 2\}) + \frac{\sqrt{2}}{4}(q * \{\underline{H} \uparrow 2\}).$$

Theorem (2.1)[6]: If the lowpass filters $\tilde{p}, p$ and highpass filters $\tilde{q}, q$ are biorthogonal, then an input x can be recovered from its approximant L and details H defined by:

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \tilde{p}_{k-2n} x_k$$

$$H_n = \frac{1}{2} \sum_{k \in \mathbb{Z}} \tilde{q}_{k-2n} x_k,$$

namely, $\tilde{x}$ defined by

$$\tilde{x}_n = \sum_{k\in\mathbb{Z}} p_{n-2k} L_k + \sum_{k\in\mathbb{Z}} q_{n-2k} H_k,\ \ n \in \mathbb{Z}.$$

is exactly x.

Proof: We want to show that $\tilde{x} = x$. The decimated transform can be defined as:

$$L(2w) = \frac{1}{2}(\overline{\tilde{p}(w)}x(w) + \overline{\tilde{p}(w+\pi)}x(w+\pi)) \tag{2.6}$$

$$H(2w) = \frac{1}{2}(\overline{\tilde{q}(w)}x(w) + \overline{\tilde{q}(w+\pi)}x(w+\pi)) \tag{2.7}$$

$$\tilde{x}(w) = 2p(w)L(2w) + 2q(w)H(2w) \tag{2.8}$$

Plugging Eq.(2.6)and Eq.(2.7) into Eq.(2.8), we get

$$\tilde{x}(w) = 2p(w)\frac{1}{2}[\overline{\tilde{p}(w)}x(w) + \overline{\tilde{p}(w+\pi)}x(w+\pi)] + 2q(w)[\frac{1}{2}(\overline{\tilde{q}(w)}x(w) + \overline{\tilde{q}(w+\pi)}x(w+\pi)]$$

$$\tilde{x}(w) = [p(w)\overline{\tilde{p}(w)} + q(w)\overline{\tilde{q}(w)}]\ x(w) + [p(w)\ \overline{\tilde{p}(w+\pi)} + q(w)\overline{\tilde{q}(w+\pi)}]x(w+\pi).$$

if

$$p(w)\overline{\tilde{p}(w)} + q(w)\overline{\tilde{q}(w)} = 1,\quad p(w)\ \overline{\tilde{p}(w+\pi)} + q(w)\overline{\tilde{q}(w+\pi)} = 0$$

holds, then $\tilde{x} = x$.

2.4.1 One-dimensional decimated wavelet transform

Let $\{x_k = L_k^0\}$ be an input sequence x, then the decimated wavelet transform consists of DWT algorithms and IDWT algorithm, respectively:

$$L_n = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \tilde{p}_{k-2n} x_k$$

$$H_n = \frac{1}{\sqrt{2}} \sum_{k\in\mathbb{Z}} \tilde{q}_{k-2n} x_k,$$

$$\tilde{x}_n = \frac{\sqrt{2}}{4} \sum_{k\in\mathbb{Z}} p_{n-2k} L_k + \frac{\sqrt{2}}{4} \sum_{k\in\mathbb{Z}} q_{n-2k} H_k,\ \ n \in \mathbb{Z}. \tag{2.9}$$

If $p, q, \tilde{p}, \tilde{q}$ are satisfied these conditions:

$$p(w)\overline{\tilde{p}(w)} + q(w)\overline{\tilde{q}(w)} = 1, \tag{2.10}$$

$$p(w)\overline{\tilde{p}(w+\pi)} + q(w)\overline{\tilde{q}(w+\pi)} = 0 \tag{2.11}$$

then $\tilde{x}_n$ recovers the original signal x_k. The decimated wavelet transform above could be applied again. The J-level decimated discrete wavelet transforms is defined by:

$$\underline{L}^j = \frac{1}{\sqrt{2}}(\tilde{p}^- * \underline{L}^{j-1}) \downarrow 2$$

$$\underline{H}^j = \frac{1}{\sqrt{2}}(\tilde{q}^- * \underline{L}^{j-1}) \downarrow 2,$$

for a filter $\tilde{p} = \{\tilde{p}_k\}$, $\tilde{p}^- = \{\tilde{p}_k^-\}$ denotes its time-reverse given by $\tilde{p}_k^- = \tilde{p}_{-k}$. The J-level inverse decimated discrete wavelet transforms (IDWT) is given by:

$$\underline{L}^{j-1} = \frac{\sqrt{2}}{4}(\ p * [\underline{L}^j \uparrow 2]\ ) + \frac{\sqrt{2}}{4}(\ q * [\underline{H}^j \uparrow 2]\ ),\ \ J > 1\ , j = J, J-1, ..., 1.$$

2.4.2 2D-decimated wavelet transform

Let $\tilde{p} = \{\tilde{p}_k\}, \tilde{q} = \{\tilde{q}_k\}, p = \{p_k\}, q = \{q_k\}$ be a biorthogonal filter banks. Suppose the sequence $\{L^0_{k_1,k_2} = x_{k_1,k_2} : k_1, k_2 \in \mathbb{Z}\}$ is the input x, then the 2D-decimated wavelet transform consists of decomposition algorithm :

$$L_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{k_1-2n_1}\tilde{p}_{k_2-2n_2}x_{k_1,k_2}$$

$$H^1_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-2n_1}\tilde{p}_{k_2-2n_2}x_{k_1,k_2}$$

$$H^2_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{k_1-2n_1}\tilde{q}_{k_2-2n_2}x_{k_1,k_2}$$

$$H^3_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-2n_1}\tilde{q}_{k_2-2n_2}x_{k_1,k_2}, \tag{2.12}$$

and the reconstruction algorithm:

$$\tilde{x}_{n_1,n_2} = \frac{1}{8}\sum_{k_1,k_2} p_{n_1-2k_1}p_{n_2-2k_2}L_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-2k_1}p_{n_2-2k_2}H^1_{k_1,k_2}$$

$$+\frac{1}{8}\sum_{k_1,k_2} p_{n_1-2k_1}q_{n_2-2k_2}H^2_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-2k_1}q_{n_2-2k_2}H^3_{k_1,k_2}. \tag{2.13}$$

Since $\tilde{p} = \{\tilde{p}_k\}, \tilde{q} = \{\tilde{q}_k\}, p = \{p_k\}, q = \{q_k\}$ are biorthogonal filter banks satisfying Eq.(2.10) and Eq.(2.11), then $\tilde{x}_{k_1,k_2} = x_{k_1,k_2}$.
Continuing doing this procedure, for $J > 1$, the J-level decimated wavelet transform is given by:

$$L^j_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{k_1-2n_1}\tilde{p}_{k_2-2n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,1}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-2n_1}\tilde{p}_{k_2-2n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,2}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}^{j-1}_{k_1-2n_1}\tilde{q}_{k_2-2n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,3}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-2n_1}\tilde{q}_{k_2-2n_2}L^{j-1}_{k_1,k_2}, \tag{2.14}$$

$$L^{j-1}_{n_1,n_2} = \frac{1}{8}\sum_{k_1,k_2} p_{n_1-2k_1}p_{n_2-2k_2}L^{j}_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-2k_1}p_{n_2-2k_2}H^{j,1}_{k_1,k_2}$$
$$+\frac{1}{8}\sum_{k_1,k_2} p_{n_1-2k_1}q_{n_2-2k_2}H^{j,2}_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-2k_1}q_{n_2-2k_2}H^{j,3}_{k_1,k_2}. \quad (2.15)$$

2.5 Undecimated wavelet transform

The UDWT has been found for different purposes and under diverse names. For example, the redundant wavelet transform, shift invariant DWT [4,20,25,55,52]. There are many advantages of undecimated wavelet transform such as repetitive, shift invariant, direct, and providing a superior estimate to CWT rather than the orthonormal (ON) discrete wavelet transform (DWT). The undecimated discrete wavelet transform (UDWT) is an adjusted rendition of the decimated wavelet transform (DWT). In the UDWT, the filters are up-sampled at each level of decomposition instead of down-sampled. Up-sampling the discrete wavelet transform is done by adding zero between every two consecutive terms. Then the undecimated DWT and inverse undecimated DWT are defined respectively by:

$$\underline{L} = \frac{1}{\sqrt{2}}(\tilde{p}^{-} * \underline{x}) \quad , \underline{H} = \frac{1}{\sqrt{2}}(\tilde{q}^{-} * \underline{x}),$$
$$\tilde{\underline{x}} = \frac{\sqrt{2}}{4}(p * \underline{L}) + \frac{\sqrt{2}}{4}(q * \underline{H}). \quad (2.16)$$

The J-level undecimated decomposition algorithm is given by:

$$\underline{L}^{j} = \frac{1}{\sqrt{2}}(\hat{\tilde{p}}^{-}_{j-1} * \underline{L}^{j-1}) \quad , \underline{H}^{j} = \frac{1}{\sqrt{2}}(\hat{\tilde{q}}^{-}_{j-1} * \underline{L}^{j-1}); \;\; J > 1, j = J, J-1, ..., 1.$$

2.5.1 One-dimensional undecimated wavelet transform

The undecimated wavelet transform using filter bank $\{p, q, \tilde{p}, \tilde{q}\}$ of an one-dimensional signal $\{x_k = L^0_k\}$ consists of decomposition algorithm:

$$L_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \tilde{p}_{k-n}x_k$$

$$H_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} \tilde{q}_{k-n}x_k$$

and reconstruction algorithm:

$$\tilde{x}_n = \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} p_{n-k}L_k + \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} q_{n-k}H_k. \quad (2.17)$$

The filter bank needs to verify the reconstruction condition:

$$p(w)\overline{\tilde{p}(w)} + q(w)\overline{\tilde{q}(w)} = 1.$$

to recovers the original siginal i.e. $x_k = \tilde{x}_k$.
The multi-level undecimated wavelet decomposition algorithm is obtained by:

$$\underline{L}^j = \frac{1}{\sqrt{2}}(\hat{\tilde{p}}^{-}_{j-1} * \underline{L}^{j-1})$$

$$\underline{H}^j = \frac{1}{\sqrt{2}}(\hat{\tilde{q}}^{-}_{j-1} * \underline{L}^{j-1}) \; for \; j = 1, 2, ..., J.$$

and the undecimated wavelet reconstruction algorithm for $j = J, J-1, ..., 1$ is given by:

$$\underline{L}^{j-1} = \frac{\sqrt{2}}{4}(\hat{p}_{j-1} * \underline{L}^j) + \frac{\sqrt{2}}{4}(\hat{q}_{j-1} * \underline{H}^j) \tag{2.18}$$

where

$$(\hat{p}_j)_k = \begin{cases} p_n & if \quad k = n2^j \\ 0 & if \quad k \neq n2^j \end{cases}$$

2.5.2 2D-undecimated wavelet transform

Suppose $L^0_k = x_k$ denotes the input x, that is x= $\{x_{k_1,k_2} \;\; , k_1, k_2 \in \mathbb{Z}\}$. The shift invariant wavelet transform for two-dimensional and one-level consists of decomposition algorithm with analysis filters $\tilde{p}, \tilde{q}$:

$$L_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{k_1-n_1}\tilde{p}_{k_2-n_2}x_{k_1,k_2}$$

$$H^1_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-n_1}\tilde{p}_{k_2-n_2}x_{k_1,k_2}$$

$$H^2_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{k_1-n_1}\tilde{q}_{k_2-n_2}x_{k_1,k_2}$$

$$H^3_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{k_1-n_1}\tilde{q}_{k_2-n_2}x_{k_1,k_2}, \tag{2.19}$$

and reconstruction algorithm with the synthesis filters p, q :

$$\tilde{x}_{n_1,n_2} = \frac{1}{8}\sum_{k_1,k_2} p_{n_1-k_1}p_{n_2-k_2}L_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-k_1}p_{n_2-k_2}H^1_{k_1,k_2}$$

$$+\frac{1}{8}\sum_{k_1,k_2} p_{n_1-k_1}q_{n_2-k_2}H^2_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2} q_{n_1-k_1}q_{n_2-k_2}H^3_{k_1,k_2}. \tag{2.20}$$

Continuing doing this procedure, we get the J-level shift invariant DWT:

$$L^j_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \hat{\tilde{p}}^{j-1}_{k_1-n_1}\hat{\tilde{p}}^{j-1}_{k_2-n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,1}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}}\hat{\tilde{q}}^{j-1}_{k_1-n_1}\hat{\tilde{p}}^{j-1}_{k_2-n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,2}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}}\hat{\tilde{p}}^{j-1}_{k_1-n_1}\hat{\tilde{q}}^{j-1}_{k_2-n_2}L^{j-1}_{k_1,k_2}$$

$$H^{j,3}_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}}\hat{\tilde{q}}^{j-1}_{k_1-n_1}\hat{\tilde{q}}^{j-1}_{k_2-n_2}L^{j-1}_{k_1,k_2}, \tag{2.21}$$

and J-level shift invariant IDWT is given by:

$$L^{j-1}_{n_1,n_2} = \frac{1}{8}\sum_{k_1,k_2}\hat{p}^{j-1}_{n_1-k_1}\hat{p}^{j-1}_{n_2-k_2}L^{j}_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2}\hat{q}^{j-1}_{n_1-k_1}\hat{p}^{j-1}_{n_2-k_2}H^{j,1}_{k_1,k_2}$$

$$+\frac{1}{8}\sum_{k_1,k_2}\hat{p}^{j-1}_{n_1-k_1}\hat{q}^{j-1}_{n_2-k_2}H^{j,2}_{k_1,k_2} + \frac{1}{8}\sum_{k_1,k_2}\hat{q}^{j-1}_{n_1-k_1}\hat{q}^{j-1}_{n_2-k_2}H^{j,3}_{k_1,k_2}. \tag{2.22}$$

2.6 Wavelet denoising

The methodology of removing the noise is defined as signal denoising or denoising simply. There are numerous methodologies for the assignment of denoising, which can be generalized into two different kinds: denoising in the original signal domain, and denoising in the transform domain. The image denoising issue has been generally considered, and as of not long ago wavelet procedures has been created.

Wavelet shrinkage is a famous denoising strategy in picture transforming due to its effectiveness and simplicity,. The wavelet delicate thresholding system presented by Donoho et al, was mulled over and stretched out in a few papers [9,13,14]. They have presented a general edge T

$$T = \sqrt{2\sigma^2 logN}$$

σ is the varince and N is the total number of pixels. The threshold choice assumes an essential part in the wavelet shrinkage system. In the event that the threshold is excessively little, much commotion is still in the denoised picture. Unexpectedly, if the threshold is excessively substantial, some critical little subtle elements, for example, surfaces, will be evacuated. There are different sorts of shrinkage techniques, for example, SureShrink, BayesShrink, and OracleShrink [5].

Three-stage procedure required to be followed in order to remove noise from the highpass coefficients by the wavelet shrinkage. First, calculate the wavelet coefficient. Then modify the details coefficient by shrinking process. The most popular threshold strategies are hard-thresholding methods, which deletes all the coefficients having values less then T and keeps the others as it is. The general hard-shrinkage rule is set by

$$\hat{x}^H_T = \begin{cases} x, |x| \geq T \\ 0, |x| < T \end{cases} \tag{2.23}$$

and soft thresholding rule which is set by:

$$\hat{x}_T^S = \begin{cases} sgn(x)(|x| - T), |x| \geq T \\ 0, |x| < T. \end{cases} \tag{2.24}$$

Where x is the wavelet coefficient, T is the threshold, and $\hat{x}$ is modified coefficient. After applying the shrinkage method, we get the adjusted wavelet coefficients. Finally, restore the denoised version associated with original signal through the shrunken wavelet coefficient by computing the inverse discrete wavelet transform. One-dimension undecimated denoising algorithm is defined by:

$$u_k = \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} p_n L_{k-n} + \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} q_n s_\theta(H_{k-n}).$$

If the filters bank satisfy

$$\overline{p(w)}\tilde{p}(w) + \overline{q(w)}\tilde{q}(w) = 1,$$

then u_k can be written as

$$u_k = u_k^0 + \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} q_n [s_\theta(H_{k-n}) - H_{k-n}] \tag{2.25}$$

Two-dimension undecimated denoising algorithm is given by:

$$u_{k_1,k_2} = \frac{1}{8} \sum_{k_1,k_2} p_{n_1} p_{n_2} L_{k_1-n_1,k_2-n_2} + \frac{1}{8} \sum_{k_1,k_2} q_{n_1} p_{n_2} s_\theta(H^1_{k_1-n_1,k_2-n_2})$$

$$+ \frac{1}{8} \sum_{k_1,k_2} p_{n_1} q_{n_2} s_\theta(H^2_{k_1-n_1,k_2-n_2}) + \frac{1}{8} \sum_{k_1,k_2} q_{n_1} q_{n_2} s_\theta(H^3_{k_1-n_1,k_2-n_2}),$$

the shrinkage function s_θ is depending upon a parameter θ. The resulting signal u_{k_1,k_2} after one-step wavelet shrinkage can be written as:

$$u_{k_1,k_2} = u^0_{k_1,k_2} + \frac{1}{8} \sum_{k_1,k_2} q_{n_1} p_{n_2} [s_\theta(H^1_{k_1-n_1,k_2-n_2}) - H^1_{k_1-n_1,k_2-n_2}]$$

$$+ \frac{1}{8} \sum_{k_1,k_2} p_{n_1} q_{n_2} [s_\theta(H^2_{k_1-n_1,k_2-n_2}) - H^2_{k_1-n_1,k_2-n_2}]$$

$$+ \frac{1}{8} \sum_{k_1,k_2} q_{n_1} q_{n_2} [s_\theta(H^3_{k_1-n_1,k_2-n_2}) - H^3_{k_1-n_1,k_2-n_2}] \tag{2.26}$$

2.7 Frame: affine frame and quasi-affine frame

A gathering of components $X = \{x_j : j \in \mathbb{Z}\} \subset L_2(\mathbb{R}^d)$ with $d \in \mathbb{N}$ is known as a **frame** if

$$A||f||^2_{L_2(\mathbb{R}^d)} \leq \sum_{j\in\mathbb{Z}} |\langle f, x_j\rangle|^2 \leq B||f||^2_{L_2(\mathbb{R}^d)}, \quad \forall f \in L_2(\mathbb{R}^d) \tag{2.27}$$

where A, B are constants and ,$0 < A \leq B < \infty$. The maximum of all such numbers A and the minimum of all such numbers B are known as the **frame bounds** of the frame. A frame is said to be **tight** when A = B i.e.

$$\sum_{j\in\mathbb{Z}} |\langle f, x_j\rangle|^2 = A||f||^2_{L_2(\mathbb{R}^d)}, \quad \forall f \in L_2(\mathbb{R}^d) \tag{2.28}$$

For any provided frame X of $L_2(\mathbb{R}^d)$ there is another frame present $\tilde{X} = \{\tilde{x}_j : j \in \mathbb{Z}\}$ of $L_2(\mathbb{R}^d)$ such that:

$$f = \sum_j \langle f, x_j\rangle \, \tilde{x}_j \ , \forall f \in L_2(\mathbb{R}^d)$$

$\tilde{X}$ is called **dual frame** of X. And the pair, or $(X, \tilde{X})$, are called **bi-frame**. If X is a tight frame, then

$$f = \sum_j \langle f, x_j\rangle \, x_j \ , \forall f \in L_2(\mathbb{R}^d).$$

Affine system $X(\Psi)$and **Quasi-affine system** $X_{qu}(\Psi)$ generated by Ψ are given, respectively, by:

$$X(\Psi) = \{\psi_{l,n,k} = 2^{\frac{nd}{2}}\psi_l(2^n. - k) \ : 1 \leq l \leq L, n \in \mathbb{Z}, k \in \mathbb{Z}^d\}, \tag{2.29}$$

$$X_{qu}(\Psi) = \{\psi_{l,n,k} : 1 \leq l \leq L, n \in \mathbb{Z}, k \in \mathbb{Z}^d\}, \tag{2.30}$$

where $\Psi := \{\psi_1, ..., \psi_L\} \subset L_2(\mathbb{R})$ and $\psi_{l,n,k}$ is given by:

$$\psi_{l,n,k} := \begin{cases} 2^{\frac{nd}{2}}\psi_l(2^n. - k), n \geq 0 \\ 2^{nd}\psi_l(2^n. - 2^{n-J}k), n < 0. \end{cases}$$

$X(\Psi)$ is known as a **(tight) frame system** when $X(\Psi)$ forms (tight) frame of $L_2(\mathbb{R}^d)$ is called a **(tight) framelet**.

2.7.1 One dimensional frame transform

Let $p(w) = \frac{1}{2}\sum_{k\in\mathbb{Z}} p_k e^{-ikw}$ be the filter for a one dimensional sequence $\{p_k\}_{k\in\mathbb{Z}}$. Assume the lowpass filters $\{p, \tilde{p}\}$, and the highpass filters $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ are a biorthogonal FIR filter bank satisfying:

$$p(w)\overline{\tilde{p}(w)} + \sum_{l=1}^{L} q^{(l)}(w)\overline{\tilde{q}^{(l)}(w)} = 1, \tag{2.31}$$

$$p(w)\overline{\tilde{p}(w+\pi)}+\sum_{l=1}^{L}q^{(l)}(w)\overline{\tilde{q}^{(l)}(w+\pi)}=0, \qquad (2.32)$$

and $x_k = L_k^0$ is the initial data. Then the **decimated discrete frame transform** with the analysis filters $\{\tilde{p}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ is given by:

$$L_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\tilde{p}_{k-2n}x_k$$

$$H_n^{(l)} = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\tilde{q}_{k-2n}^{(l)}x_k,$$

and the **inverse decimated discrete frame transform** with the synthesis filters $\{p, q^{(1)}, ..., q^{(L)}\}$ is defined by:

$$\tilde{x}_n = \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}p_{n-2k}L_k + \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}q_{n-2k}^{(l)}H_k^{(l)},\ n\in\mathbb{Z}. \qquad (2.33)$$

The J-level,$J > 1$ decimated frame transform is given by:

$$\underline{L}^j = \frac{1}{\sqrt{2}}(\tilde{p}^- * \underline{L}^{j-1})\downarrow 2$$

$$\underline{H}^{(l),j} = \frac{1}{\sqrt{2}}(\tilde{q}^{-(l)} * \underline{L}^{j-1})\downarrow 2$$

$$\underline{L}^{j-1} = \frac{\sqrt{2}}{4}(\ p * [\underline{L}^j \uparrow 2]\ ) + \frac{\sqrt{2}}{4}(\ q^{(l)} * [\underline{H}^{(l),j}\uparrow 2]\ ) \qquad (2.34)$$

Assume $\{p, \tilde{p}\}$, and $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ are finite impulse response filters satisfying Eq.(2.31), then **undecimated frame transform** of one-dimensional signal x= $\{L_k^0 = x_k\}$ consists of decomposition algorithm:

$$L_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\tilde{p}_{k-n}x_k$$

$$H_n^{(l)} = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}\tilde{q}_{k-n}^{(l)}x_k,$$

and reconstruction algorithm:

$$\tilde{x}_n = \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}p_{n-k}L_k + \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}q_{n-k}^{(l)}H_k^{(l)}. \qquad (2.35)$$

Repeating this process,we get the multi-level undecimated frame transform. For $j = 1, 2, ..., J$, the J-level undecimated discrete frame transform is obtained by:

$$\underline{L}^j = \frac{1}{\sqrt{2}}(\hat{\tilde{p}}_{j-1}^- * \underline{L}^{j-1})$$

$$\underline{H}^{(l),j} = \frac{1}{\sqrt{2}}(\hat{\tilde{q}}^{-(l)}_{j-1} * \underline{L}^{j-1}),$$

and inverse undecimated discrete frame transform is defined by

$$\underline{L}^{j-1} = \frac{\sqrt{2}}{4}(\hat{p}^{-}_{j-1} * \underline{L}^{j}) + \frac{\sqrt{2}}{4}(\hat{q}^{-(l)}_{j-1} * \underline{H}^{(l),j}), \quad (\hat{p}_j)_k = \begin{cases} p_n & if \quad k = n2^j \\ 0 & if \quad k \neq n2^j. \end{cases}$$

2.7.2 Two dimensional frame transform

Suppose f is a two-dimensional image , and the set of two-dimensional images represents by $I_2 = \mathbb{R}^{N_1 \times N_2}$. **Tensor products** of univariate frames is one method to construct frame for two dimensional $L_2(\mathbb{R}^2)$. Let $\psi_{\mathbf{i}}(x,y) = \psi_{i_1}\psi_{i_2}, \ 0 \leq i_1, i_2 \leq r, \quad (k_1,k_2) \in \mathbb{R}^2$, and $\psi_0 := \phi$ be the two-dimensional refinable function and framelets with $\mathbf{i} = (i_1, i_2)$ and $\mathbf{k} = (k_1, k_2)$. Let $\{q_l : l = 0, 1, ..., r, \ q_0 = p\}$ and $\{\tilde{q}_l : l = 0, 1, ..., r, \ \tilde{q}_0 = \tilde{p}\}$ be a pair set of univariate masks $\{q_l\}$ and $\{\tilde{q}_l\}$ satisfying the Mixed extension principle (MEP):

$$\hat{p}(\xi)\overline{\hat{\tilde{p}}(\xi)} + \sum_{l=1}^{L} \hat{q}^{(l)}(\xi)\overline{\hat{\tilde{q}}^{(l)}(\xi)} = 1 \tag{2.36}$$

$$\hat{p}(\xi)\overline{\hat{\tilde{p}}(\xi+\nu)} + \sum_{l=1}^{L} \hat{q}^{(l)}(\xi)\overline{\hat{\tilde{q}}^{(l)}(\xi+\nu)} = 0 \tag{2.37}$$

for all $\nu \in \{0,\pi\}^2 \setminus \{0\}$ and $\xi \in [-\pi,\pi]^2$. Let the two-dimensional masks be defined by:

$$q_{\mathbf{i},\mathbf{k}} = q_{i_1,k_1} q_{i_2,k_2}, \ 0 \leq i_1, i_2 \leq r, \ (k_1,k_2) \in \mathbb{Z}^2, \tag{2.38}$$

and let $\Psi_2, \ \tilde{\Psi}_2$ be denoted by:

$$\Psi_2 := \{\psi_{\mathbf{i}}, 0 \leq i_1, i_2 \leq r, \mathbf{i} \neq (0,0)\}$$
$$\tilde{\Psi}_2 := \{\tilde{\psi}_{\mathbf{i}}, 0 \leq i_1, i_2 \leq r, \mathbf{i} \neq (0,0)\}.$$

The decomposition of 2D-fast lev-level WFT with $\{p, q^{(1)}, ..., q^{(L)}\}$ is defined by:

$$Wu = \{W_{\iota,l}u = q_{\iota,l} * u : 0 \leq \iota \leq Lev - 1, 0 \leq l \leq L\}, \quad u \in I_2 \tag{2.39}$$

where $*$ denotes the convolution. $q_{\iota,l}$ is denoted by:

$$q_{\iota,l} = \tilde{q}_{\iota,l} * \tilde{q}_{\iota-1,0} * ... * \tilde{q}_{0,0} \ with \ \ \tilde{q}_{\iota,l}[k] = \begin{cases} q_l[2^{-\iota}], & k \in 2^{\iota}\mathbb{Z}^2 \\ 0 & k \notin 2^{\iota}\mathbb{Z}^2 \end{cases} \tag{2.40}$$

If $Lev = 1$ then $W_{0,l} = W_l$ and $q_{0,l} = q_l$
The wavelet frame reconstruction is the adjoint operator of $\tilde{W}$. The inverse wavelet frame transform is defined as $\tilde{W}^T$ for a bi-frame system, and as W^T for a tight frame system.
The perfect reconstruction algorithm by the mixed extension principle (MEP) is defined by:

$$u = \tilde{W}^T W u, \ for \ all \ \ u \in I_2, \tag{2.41}$$

Two-dimensional decimated frame transform

Suppose that $\{p, \tilde{p}\}$, and $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ are finite impulse response filters satisfying Eq.(2.31) and Eq.(2.32) with the initial data $\{x_k = L_k^0,\ k \in \mathbb{Z}^2\}$. Then the one-level decimated wavelet frame transform in a **compact form** is given by the following processes:

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}^2} \tilde{p}_{k-2n} x_k$$

$$H_n^{(l)} = \frac{1}{2} \sum_{k \in \mathbb{Z}^2} \tilde{q}_{k-2n}^{(l)} x_k,\ n \in \mathbb{Z}^2,\ 1 \leq l \leq L,$$

$$\tilde{x}_n = \frac{1}{8} \sum_{k \in \mathbb{Z}^2} p_{n-2k} L_k + \frac{1}{8} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^2} q_{n-2k}^{(l)} H_k^{(l)},\ n \in \mathbb{Z}^2 \tag{2.42}$$

The J-level decimated frame transform can be obtained by the following processes:

$$\underline{L}^j = \frac{1}{2}(\tilde{p}^- * \underline{L}^{j-1}) \downarrow 2$$

$$\underline{H}^{j,(l)} = \frac{1}{2}(\tilde{q}^{-(l)} * \underline{L}^{j-1}) \downarrow 2$$

$$\underline{L}^{j-1} = \frac{1}{8}(\ p * [\underline{L}^j \uparrow 2]\ ) + \frac{1}{8}(\ q^{(l)} * [\underline{H}^{j,(l)} \uparrow 2]\ ).$$

Two-dimensional undecimated frame transform

Suppose x=$\{x_k = L_k^0\,, k \in \mathbb{Z}^2\}$ denotes the input x for 2-dimensional signal and $\{p, \tilde{p}\}$, $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ are finite impulse response filters satisfying Eq.(2.31). Thus,the shift invariant frame transform consist of decomposition algorithm :

$$L_n = \frac{1}{2} \sum_{k \in \mathbb{Z}^2} p_{k-n} x_k$$

$$H_n^{(l)} = \frac{1}{2} \sum_{k \in \mathbb{Z}^2} q_{k-n}^{(l)} x_k,\ n \in \mathbb{Z}^2,\ 1 \leq l \leq L,$$

and reconstruction algorithm:

$$\tilde{x}_n = \frac{1}{8} \sum_{k \in \mathbb{Z}^2} \tilde{p}_{n-k} L_k + \frac{1}{8} \sum_{l=1}^{L} \sum_{k \in \mathbb{Z}^2} \tilde{q}_{n-k}^{(l)} H_k^{(l)},\ n \in \mathbb{Z}^2 \tag{2.43}$$

$\tilde{x}_n$ recovers the original signal x_k if the filters $\{p, \tilde{p}\}$, $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ satisfy Eq.(2.31). The J-level shift invariant frame transform for $J > 1, \quad j = J, J-1, ..., 1,$ is defined as:

$$\underline{L}^j = \frac{1}{2}(\hat{\tilde{p}}_{j-1}^- * \underline{L}^{j-1})$$

$$\underline{H}^{j,(l)} = \frac{1}{2}(\hat{\tilde{q}}_{j-1}^{-(l)} * \underline{L}^{j-1})$$

$$\underline{L}^{j-1} = \frac{1}{8}(\hat{p}_{j-1} * \underline{L}^j) + \frac{1}{8}(\hat{q}_{j-1}^{(l)} * \underline{H}^{j,(l)}). \tag{2.44}$$

2.8 Frame denoising

Suppose an input signal is x, ζ is a noisy signal and y is a noisy data. Then an image signal corrupted with additive noise is given by:

$$y = x + \zeta.$$

Few steps should be followed to get image frame denoising algorithm. First applying frame transform to get frame coefficient. Second modifying the frame coefficient by using shrinking process (hard threshold or soft-threshold). Finally applying inverse frame transform on modify coefficient to get denoised image. Let consider 2D undecimated discrete frame transform with initial input x_k:

$$L_n = \frac{1}{2}\sum_{k\in\mathbb{Z}^2} p_k x_{n+k}$$

$$H_n^{(l)} = \frac{1}{2}\sum_{k\in\mathbb{Z}^2} q_k^{(l)} x_{n+k},\ n \in \mathbb{Z}^2,\ 1 \le l \le L,$$

thus the denoising algorithm is given by:

$$u_k = \frac{1}{8}\sum_{k\in\mathbb{Z}^2} \tilde{p}_n L_{k-n} + \frac{1}{8}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^2} \tilde{q}_n^{(l)} s_{\theta_l}^{l}(H_{k-n}^{(l)}),\ n \in \mathbb{Z}^2 \tag{2.45}$$

u_k is called the denoising signal of the original signal x_k with noise. If the filters satisfy Eq.(2.31), then Eq.(2.45) can be written as:

$$u_k = x_k + \frac{1}{8}\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^2} \tilde{q}_n^{(l)} [s_{\theta_l}^{l}(H_{k-n}^{(l)}) - H_{k-n}^{(l)}],\ n \in \mathbb{Z}^2 \tag{2.46}$$

2.9 Multiwavelets

The major contrast between scalar wavelets and multiwavelets is definitely the change from scalar wavelets to multiwavelets, the symbols and the coefficients are matrices. Multiwavelet require a scaling functions r (multiscaling functions) and a set of wavelet functions r(multiwavelet functions). The scaling functions r form Riesz basis for V_0, and the wavelet functions r(multiwavelet functions) form an orthonormal basis of $L^2(\mathbb{R})$, which reduces to the scalar wavelet case when r = 1.

Multiwavelet has important properties such as orthogonality, symmetry, short support, higher order of vanishing moments and good performance at the boundaries, etc. that scalar wavelets fail to possess these properties simultaneously. Multiwavelet present higher achievement for image processing in comparison with wavelets in scalar case.

Likewise to wavelet, construction of multiwavelet is connected with a multiresolution analysis using r scaling functions in the system.

The extension of wavelet was submitted by J.S. Geronimo, D.P Hardin, P.R. Massopust. They suggested using two scaling functions to approximate a signal [16]. Multiwavelet has drawn much consideration in current years [7,8,10,21,22,23,29,30,31,32,33,34 ,35,36,37,38,39,40,44,48,49,50,57,60,64,65]. In addition the constructions of multiwavelets and the design of multifilter banks can be found in many paper such as [11,23,38,39,68].

Definition (2.1): A refinable function vector is defined as:

$$\Phi(x) = \begin{pmatrix} \phi_1(x) \\ . \\ . \\ . \\ \phi_r(x) \end{pmatrix}, \qquad \phi_n : \mathbb{R} \to \mathbb{C}$$

which satisfies

$$\Phi(x) = 2\sum_k P_k \Phi(2x-k) \tag{2.47}$$

where the multiplicity of Φ is r and the dilation factor is the integer 2. P_k are the recursion coefficients of $r \times r$ matrices.

Definition (2.2): The symbol of a refinable function vector is a trigonometric matrix polynomial:

$$P(w) = \sum_{k=k_0}^{k_1} P_k e^{-ikw}$$

The refinment equation (2.47) can be written in term of Fourier transform as :

$$\Phi(w) = P(\frac{w}{2})\hat{\Phi}(\frac{w}{2})$$

Definition (2.3): A vector-valued function $\Psi = (\psi_1, .., \psi_r)^T$ is called a multiwavelet if the collections of the integer translates and the dilations of factor 2 of $\psi_1, .., \psi_r$ make an orthonormal basis of $L^2(R)$.

$$\Psi(x) = 2\sum_{k\in Z} Q_k \Phi(2x-k) \tag{2.48}$$

2.9.1 Multiresolution analysis using multiwavelet (MRA)

An MRA of multiwavelet is denoted as a sequence $\{V_j\}$ of closed subspace in $L^2(\mathbb{R})$ that satisfy some properties:

$$\begin{aligned}
&V_j \subset V_{j+1}\\
&f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}\\
&\bigcap_{j\in\mathbb{Z}} V_j = 0\\
&\bigcup_{j\in\mathbb{Z}} V_j =: L^2(\mathbb{R})\\
&There\ exists\ a\ set\ of\ functions\ \phi_j(x) \in V_0\ s.t.\{\phi_j(x-k), 1 \le j \le r,\ k \in \mathbb{Z}\}\\
&\quad is\ a\ Riesz\ basis\ for\ V_0.
\end{aligned}$$

The multiscaling function $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$ (r is the multiplicity) are derived as in Eq.(2.47).
Similarly multiwavelet function $\Psi = (\psi_1, \psi_2, \ldots, \psi_r)^T$ can be derived as in Eq.(2.48). where P_k, Q_k are FIR filters. In the filters, every coefficient is an $r \times r$ matrix. In the Fourier transform domain, multiscaling and multiwavelet functions can be defined as:

$$\hat{\Phi}(w) = P(\frac{w}{2})\hat{\Phi}(\frac{w}{2})$$
$$\hat{\Psi}(w) = Q(\frac{w}{2})\hat{\Phi}(\frac{w}{2})$$

There are some typical examples of design and implementation of multiwavelet such as DGHM multiwavelet by Geronimo, Hardin and Massopust, the CL multiwavelet by Chui and Lian, design of multifilter banks and orthonormal multiwavelet, and multiwavelet with optimum time-frequency resolution by Jiang [8,16,17,18,38,39].

2.9.2 Construction of biorthogonal multiwavelets

Suppose that $P(w) = \sum_{k\in Z} P_k e^{-ikw}$, $\tilde{P}(w) = \sum_{k\in Z} \tilde{P}_k e^{-ikw}$ are $r \times r$ matrices satisfying

$$P(w)\tilde{P}(w)^* + P(w+\pi)\tilde{P}(w+\pi)^* = I_r. \tag{2.49}$$

Also, suppose $\Phi = (\phi_1, ..., \phi_r)^T$ and $\tilde{\Phi} = (\tilde{\phi}_1, ..., \tilde{\phi}_r)^T \in L^2(\mathbb{R})^r$ are given as:

$$\Phi(x) = 2\sum_{k\in Z} P_k \Phi(2x-k), \quad \tilde{\Phi}(x) = 2\sum_{k\in Z} \tilde{P}_k \tilde{\Phi}(2x-k)$$

Suppose $V_0(\Phi), V_0(\tilde{\Phi})$ are two subspace defined as:
$V_0(\Phi) := \overline{span}\{\phi_i(.-k) : 1 \le i \le r, k \in \mathbb{Z}\}$, $V_0(\tilde{\Phi}) := \overline{span}\{\tilde{\phi}_i(.-k) : 1 \le i \le r, k \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$. The scaling function is defined by $\Phi = (\phi_1, ..., \phi_r)^T$.
Suppose the two MRAs of multiplicity r with $\Phi, \tilde{\Phi}$ are given by $V_j(\Phi)$ and $V_j(\tilde{\Phi})$. Then $\Psi = (\psi_1, ..., \psi_r)^T$ and $\tilde{\Psi} = (\tilde{\psi}_1, ..., \tilde{\psi}_r)^T$ can be written as:

$$\Psi(x) = 2\sum_{k\in\mathbb{Z}} Q_k \Phi(2x-k), \quad \tilde{\Psi}(x) = 2\sum_{k\in\mathbb{Z}} \tilde{Q}_k \tilde{\Phi}(2x-k)$$

In the Fourier domain:

$$\hat{\Psi}(w) = Q(\frac{w}{2})\hat{\Phi}(\frac{w}{2}), \quad \hat{\tilde{\Psi}}(w) = \tilde{Q}(\frac{w}{2})\hat{\tilde{\Phi}}(\frac{w}{2})$$

The Riesz bases of $V_1(\Phi)$ and the Riesz bases of $V_1(\tilde{\Phi})$ are given by $\{\phi_i(.-k), \psi_i(.-k) : 1 \le i \le r, k \in \mathbb{Z}\}$ and $\{\tilde{\phi}_i(.-k), \tilde{\psi}_i(.-k) : 1 \le i \le r, k \in \mathbb{Z}\}$ with

$$\begin{aligned} \left\langle \phi_l, \tilde{\phi}_{l'}(.-k) \right\rangle &= \left\langle \psi_l, \tilde{\psi}_{l'}(.-k) \right\rangle = \delta(k)\delta(l-l') \\ \left\langle \phi_l, \tilde{\psi}_{l'}(.-k) \right\rangle &= \left\langle \tilde{\phi}_l, \psi_{l'}(.-k) \right\rangle = 0 \end{aligned} \tag{2.50}$$

If $\{2^{\frac{j}{2}}\psi_l(2^j.-k) : 1 \le l \le r, k \in \mathbb{Z}\}$ and $\{2^{\frac{j}{2}}\tilde{\psi}_l(2^j.-k) : 1 \le l \le r, k \in \mathbb{Z}$ create a pair of dual Riesz bases of $L^2(\mathbb{R})$, then we said that $\Psi, \tilde{\Psi}$ design a set of biorthogonal multiwavelets.
Therfore, P ,Q ,$\tilde{P}$,$\tilde{Q}$ satisfy the perfect reconstruction conditions:

$$\begin{aligned} P(w)\tilde{P}(w)^* + P(w+\pi)\tilde{P}(w+\pi)^* &= I_r \\ Q(w)\tilde{Q}(w)^* + Q(w+\pi)\tilde{Q}(w+\pi)^* &= I_r \\ P(w)\tilde{Q}(w)^* + P(w+\pi)\tilde{Q}(w+\pi)^* &= 0_r \\ Q(w)\tilde{P}(w)^* + Q(w+\pi)\tilde{P}(w+\pi)^* &= 0_r \end{aligned} \tag{2.51}$$

Then the multiwavelet filter bank or multifilter bank are formed by P ,Q ,$\tilde{P}$,$\tilde{Q}$.

2.9.3 Discrete multiwavelet transform

Suppose that the symbols of a biorthogonal FIR multifilter bank P ,Q ,$\tilde{P}$,$\tilde{Q}$ satisfy:

$$\begin{aligned} P(w)\tilde{P}(w)^* + P(w+\pi)\tilde{P}(w+\pi)^* &= I_2 \\ Q(w)\tilde{Q}(w)^* + Q(w+\pi)\tilde{Q}(w+\pi)^* &= I_2 \\ P(w)\tilde{Q}(w)^* + P(w+\pi)\tilde{Q}(w+\pi)^* &= 0_2 \\ Q(w)\tilde{P}(w)^* + Q(w+\pi)\tilde{P}(w+\pi)^* &= 0_2 \end{aligned} \tag{2.52}$$

let $\Phi = (\phi_1, \phi_2)^T$ and $\tilde{\Phi} = (\tilde{\phi}_1, \tilde{\phi}_2)^T$ be the scaling functions, and P and $\tilde{P}$ be matrices lowpass filters

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = 2\sum_k P_k \begin{pmatrix} \phi_1(2x-k) \\ \phi_2(2x-x) \end{pmatrix}$$

$$\begin{pmatrix} \tilde{\phi}_1(x) \\ \tilde{\phi}_2(x) \end{pmatrix} = 2\sum_k \tilde{P}_k \begin{pmatrix} \tilde{\phi}_1(2x-k) \\ \tilde{\phi}_2(2x-x) \end{pmatrix}$$

where $P(w) = \sum_k P_k e^{-ikw}$ and $\tilde{P}(w) = \sum_k \tilde{P}_k e^{-ikw}$
and $\Psi = (\psi_1, \psi_2)^T$ and $\tilde{\Psi} = (\tilde{\psi}_1, \tilde{\psi}_2)^T$ be the corresponding biorthogonal multiwavelet defined by matrices highpass filters $Q, \tilde{Q}$

$$\begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} = 2\sum_k Q_k \begin{pmatrix} \phi_1(2x-k) \\ \phi_2(2x-x) \end{pmatrix}$$

$$\begin{pmatrix}\tilde{\psi}_1(x)\\ \tilde{\psi}_2(x)\end{pmatrix} = 2\sum_k \tilde{Q}_k \begin{pmatrix}\tilde{\phi}_1(2x-k)\\ \tilde{\phi}_2(2x-x)\end{pmatrix}$$

where $Q(w) = \sum_k Q_k e^{-ikw}$ and $\tilde{Q}(w) = \sum_k \tilde{Q}_k e^{-ikw}$
The discrete multiwavelet transform algorithms can be obtain by:
Let $f \in V_0$ then

$$f(x) = \sum_{k\in\mathbb{Z}} (x_{1,k}\phi_1(x-k) + x_{2,k}\phi_2(x-k))$$

$$f(x) = \sum_{k\in\mathbb{Z}} [\ L_{1,k}\phi_1(\frac{x}{2}-k) + L_{2,k}\phi_2(\frac{x}{2}-k)\ ] + \sum_{k\in\mathbb{Z}} [\ H_{1,k}\psi_1(\frac{x}{2}-k) + H_{2,k}\psi_2(\frac{x}{2}-k)\ ]$$

Then the decomposition multiwavelet algorithms are defined as:

$$\begin{pmatrix}L_{1,k}\\ L_{2,k}\end{pmatrix} = \sum_n P_n \begin{pmatrix}x_{1,n+2k}\\ x_{2,n+2k}\end{pmatrix},$$
$$\begin{pmatrix}H_{1,k}\\ H_{2,k}\end{pmatrix} = \sum_n Q_n \begin{pmatrix}x_{1,n+2k}\\ x_{2,n+2k}\end{pmatrix}, \quad n,k \in \mathbb{Z} \tag{2.53}$$

and the reconstruction algorithm is given by:

$$\begin{pmatrix}\tilde{x}_{1,k}\\ \tilde{x}_{2,k}\end{pmatrix} = \sum_n \tilde{P}_n^T \begin{pmatrix}L_{1,k-2n}\\ L_{2,k-2n}\end{pmatrix} + \sum_n \tilde{Q}_n^T \begin{pmatrix}H_{1,k-2n}\\ H_{2,k-2n}\end{pmatrix} \tag{2.54}$$

Theorem (2.2): If P, Q are biorthogonal, then an input $(x_{i,k})^T, i = 1,2$ can be recovered from its approximate $(L_{i,k})^T$ and details $(H_{i,k})^T$, namely: $(\tilde{x}_{i,k})^T$ is exactly $(x_{i,k})^T$.
Proof: Suppose that

$$\begin{pmatrix}L_1(2w)\\ L_2(2w)\end{pmatrix} = \overline{P(w)} \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix} + \overline{P(w+\pi)} \begin{pmatrix}x_1(w+\pi)\\ x_2(w+\pi)\end{pmatrix}$$

$$\begin{pmatrix}H_1(2w)\\ H_2(2w)\end{pmatrix} = \overline{Q(w)} \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix} + \overline{Q(w+\pi)} \begin{pmatrix}x_1(w+\pi)\\ x_2(w+\pi)\end{pmatrix}$$

and

$$\begin{pmatrix}\tilde{x}_1(w)\\ \tilde{x}_2(w)\end{pmatrix} = \tilde{P}^T(w) \begin{pmatrix}L_1(2w)\\ L_2(2w)\end{pmatrix} + \tilde{Q}^T(w) \begin{pmatrix}H_1(2w)\\ H_2(2w)\end{pmatrix}$$

Plugging $(L_1(2w), L_2(2w))^T$ and $(H_1(2w), H_2(2w))^T$ into $(\tilde{x}_1(w), \tilde{x}_2(w))^T$,we get

$$\begin{pmatrix}\tilde{x}_1(w)\\ \tilde{x}_2(w)\end{pmatrix} = \tilde{P}^*(w)\ [P(w) \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix} + P(w+\pi) \begin{pmatrix}x_1(w+\pi)\\ x_2(w+\pi)\end{pmatrix}] + \tilde{Q}^*(w)\ [Q(w) \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix} + Q(w+\pi) \begin{pmatrix}x_1(w+\pi)\\ x_2(w+\pi)\end{pmatrix}]$$

$$\begin{pmatrix}\tilde{x}_1(w)\\ \tilde{x}_2(w)\end{pmatrix} = [\tilde{P}^*(w)\ P(w) + \tilde{Q}^*(w)\ Q(w)] \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix}$$
$$+ [\tilde{P}^*(w)\ P(w+\pi) + \tilde{Q}^*(w)Q(w+\pi)] \begin{pmatrix}x_1(w+\pi)\\ x_2(w+\pi)\end{pmatrix}$$

If

$$\tilde{P}^*(w)\ P(w) + \tilde{Q}^*(w)\ Q(w) = I_2$$
$$\tilde{P}^*(w)\ P(w+\pi) + \tilde{Q}^*(w)Q(w+\pi) = 0_2$$

holds, then

$$\begin{pmatrix}\tilde{x}_1(w)\\ \tilde{x}_2(w)\end{pmatrix} = \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix}. \tag{2.55}$$

2.9.4 Undecimated discrete multiwavelet transform

The undecimated multiwavelet transform using the multifilter bank $P, Q, \tilde{P}, \tilde{Q}$ of the signal $\begin{pmatrix}x_{1,k}\\ x_{2,k}\end{pmatrix}$ consists of undecimated discrete multiwavelet algorithms:

$$\begin{pmatrix}L_{1,k}\\ L_{2,k}\end{pmatrix} = \sum_n P_n \begin{pmatrix}x_{1,n+k}\\ x_{2,n+k}\end{pmatrix},$$
$$\begin{pmatrix}H_{1,k}\\ H_{2,k}\end{pmatrix} = \sum_n Q_n \begin{pmatrix}x_{1,n+k}\\ x_{2,n+k}\end{pmatrix}, \quad j,k \in \mathbb{Z} \tag{2.56}$$

and the reconstruction algorithm:

$$\begin{pmatrix}\tilde{x}_{1,k}\\ \tilde{x}_{2,k}\end{pmatrix} = \sum_n \tilde{P}_n^T \begin{pmatrix}L_{1,k-n}\\ L_{2,k-n}\end{pmatrix} + \sum_n \tilde{Q}_n^T \begin{pmatrix}H_{1,k-n}\\ H_{2,k-n}\end{pmatrix} \tag{2.57}$$

Theorem (2.3): If $P, Q, \tilde{P}, \tilde{Q}$ are undecimated bi-orthogonal multifilter bank, then an input $\begin{pmatrix}x_{1,k}\\ x_{2,k}\end{pmatrix}$ can be recovered from its approximate $\begin{pmatrix}L_{1,k}\\ L_{2,k}\end{pmatrix}$ and details $\begin{pmatrix}H_{1,k}\\ H_{2,k}\end{pmatrix}$, namely: $\begin{pmatrix}\tilde{x}_{1,k}\\ \tilde{x}_{2,k}\end{pmatrix}$ is exactly $\begin{pmatrix}x_{1,k}\\ x_{2,k}\end{pmatrix}$.

Proof: Suppose that

$$\begin{pmatrix}L_1(w)\\ L_2(w)\end{pmatrix} = \overline{P(w)} \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix}$$
$$\begin{pmatrix}H_1(w)\\ H_2(w)\end{pmatrix} = \overline{Q(w)} \begin{pmatrix}x_1(w)\\ x_2(w)\end{pmatrix}$$

and

$$\begin{pmatrix}\tilde{x}_1(w)\\ \tilde{x}_2(w)\end{pmatrix} = \tilde{P}^T(w) \begin{pmatrix}L_1(w)\\ L_2(w)\end{pmatrix} + \tilde{Q}^T(w) \begin{pmatrix}H_1(w)\\ H_2(w)\end{pmatrix}$$

Plugging $(L_1(w), L_2(w))^T$ and $(H_1(w), H_2(w))^T$ into $(\tilde{x}_1(w), \tilde{x}_2(w))^T$,we get

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = \tilde{P}^*(w)\ [P(w) \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}] + \tilde{Q}^*(w)\ [Q(w) \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}]$$

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = [\tilde{P}^*(w)\ P(w) + \tilde{Q}^*(w)\ Q(w)] \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}$$

If

$$\tilde{P}^*(w)\ P(w) + \tilde{Q}^*(w)\ Q(w) = I_2$$

holds, then

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}. \tag{2.58}$$

2.9.5 Undecimated multiwavelets transform based denoising

Let $\underline{c}_k^0 = (c_{i,k})^T,\ i = 1,2$ be an initial data, and P, Q be FIR multiwavelet filter bank. Assume they are satisfy:

$$P^*(w)\ P(w) + Q^*(w)\ Q(w) = I_2 \tag{2.59}$$

Then undecimated multiwavelet transform based denoising consists of decomposition algorithm:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \sum_{k\in\mathbb{Z}} P_k \begin{pmatrix} c_{1,k+n} \\ c_{2,k+n} \end{pmatrix}$$
$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \sum_{k\in\mathbb{Z}} Q_k \begin{pmatrix} c_{1,k+n} \\ c_{2,k+n} \end{pmatrix} \tag{2.60}$$

and denoising algorithm:

$$\underline{u}_k = \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \sum_{n\in\mathbb{Z}} P_n^T \begin{pmatrix} L_{1,k-n} \\ L_{2,k-n} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \tag{2.61}$$

2.10 Multiple frames

The set $\Psi_i^{(l)} = (\psi_1^{(l)}, ..., \psi_r^{(l)})^T$ generates a **multiple wavelet frame** in $L_2(\mathbb{R})$ if

$$A\,||f||^2 \le \sum_{l=1}^{L} \sum_{i=1}^{r} \sum_{j,k\in\mathbb{Z}} \left| < f, \psi_{i,j,k}^{(l)} > \right|^2 \le B\,||f||^2 \quad \forall f \in L_2(\mathbb{R})$$

where the maximum of all positive constant A and the minimum of all positive constant B are known as the multiple wavelet frame bounds and $\psi_{i,j,k}^{(l)} := 2^{\frac{j}{2}}\psi_i^{(l)}(2^j x - k), i = 1, ..., r.$

When $A = B = 1$, the set $\Psi_i^{(l)}$ generates a **tight multiple wavelet frame** in $L_2(\mathbb{R})$:

$$\sum_{l=1}^{L}\sum_{i=1}^{r}\sum_{j,k\in\mathbb{Z}}\left|< f, \psi_{i,j,k}^{(l)} >\right|^2 = ||f||^2$$

The pair $\{\psi_1^{(l)}, \ldots, \psi_r^{(l)}\}$ and $\{\tilde{\psi}_1^{(l)}, \ldots, \tilde{\psi}_r^{(l)}\}$ generate **dual multiple wavelet frames** in $L_2(\mathbb{R})$ if both $\{\psi_1^{(l)}, \ldots, \psi_r^{(l)}\}$ and $\{\tilde{\psi}_1^{(l)}, \ldots, \tilde{\psi}_r^{(l)}\}$ generate multiple wavelet frames in $L_2(\mathbb{R})$ and satisfy

$$< f, g >= \sum_{l=1}^{L}\sum_{i=1}^{r}\sum_{j,k\in\mathbb{Z}} < f, \psi_{i,j,k}^{(l)} >< \tilde{\psi}_{i,j,k}^{(l)}, g >, \quad \forall f, g \in L_2(\mathbb{R})$$

2.10.1 Discrete multiple wavelet frame transform

Suppose that $\{P, \tilde{P}\}$ and $\{Q^{(1)}, ..., Q^{(L)}, \tilde{Q}^{(1)}, ..., \tilde{Q}^{(L)}\}$ are biorthogonal FIR multiple wavelet frame filter banks satisfying:

$$\tilde{P}^*(w)P(w) + \sum_{l=1}^{L}\tilde{Q}^{(l)*}(w)Q^{(l)}(w) = I$$

$$\tilde{P}^*(w)P(w+\pi) + \sum_{l=1}^{L}\tilde{Q}^{(l)*}(w)Q^{(l)}(w+\pi) = 0.$$

Suppose the scaling functions $(\phi_i)^T, (\tilde{\phi}_i)^T,\ i = 1, 2$ with lowpass multifilters $P, \tilde{P}$ are given as:

$$\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\end{pmatrix} = 2\sum_k P_k \begin{pmatrix}\phi_1(2x-k)\\ \phi_2(2x-k)\end{pmatrix}$$

$$\begin{pmatrix}\tilde{\phi}_1(x)\\ \tilde{\phi}_2(x)\end{pmatrix} = 2\sum_k \tilde{P}_k \begin{pmatrix}\tilde{\phi}_1(2x-k)\\ \tilde{\phi}_2(2x-k)\end{pmatrix}$$

,and let $(\psi_i^{(l)})^T, (\tilde{\psi}_i^{(l)})^T$ be the corresponding biorthogonal multiple wavelet frame defined by matrices highpass filter $Q^{(l)}, \tilde{Q}^{(l)}$

$$\begin{pmatrix}\psi_1^{(l)}(x)\\ \psi_2^{(l)}(x)\end{pmatrix} = 2\sum_k\sum_{l=1}^{L} Q_k^{(l)} \begin{pmatrix}\phi_1(2x-k)\\ \phi_2(2x-k)\end{pmatrix}$$

$$\begin{pmatrix}\tilde{\psi}_1^{(l)}(x)\\ \tilde{\psi}_2^{(l)}(x)\end{pmatrix} = 2\sum_k\sum_{l=1}^{L} \tilde{Q}_k^{(l)} \begin{pmatrix}\tilde{\phi}_1(2x-k)\\ \tilde{\phi}_2(2x-k)\end{pmatrix}$$

Then the discrete multiple wavelet frame transform algorithms consist of the decomposition algorithm:

$$\begin{pmatrix} L_{1,k} \\ L_{2,k} \end{pmatrix} = \sum_n P_n \begin{pmatrix} x_{1,n+2k} \\ x_{2,n+2k} \end{pmatrix},$$
$$\begin{pmatrix} H_{1,k}^{(l)} \\ H_{2,k}^{(l)} \end{pmatrix} = \sum_n \sum_{l=1}^{L} Q_n^{(l)} \begin{pmatrix} x_{1,n+2k} \\ x_{2,n+2k} \end{pmatrix}, \quad k \in \mathbb{Z} \tag{2.62}$$

and the reconstruction algorithm:

$$\begin{pmatrix} \tilde{x}_{1,k} \\ \tilde{x}_{2,k} \end{pmatrix} = \sum_n \ \tilde{P}_n^T \begin{pmatrix} L_{1,k-2n} \\ L_{2,k-2n} \end{pmatrix} + \sum_n \sum_{l=1}^{L} \tilde{Q_n^{(l)}}^T \begin{pmatrix} H_{1,k-2n}^{(l)} \\ H_{2,k-2n}^{(l)} \end{pmatrix} \tag{2.63}$$

Theorem (2.4) :Suppose that $\{P, Q^{(1)}, ..., Q^{(L)}\}$ and $\{\tilde{P}, \tilde{Q}^{(1)}, ..., \tilde{Q}^{(L)}\}$ are biorthogonal multiple frame filter banks, then an input $(x_{i,k})^T, i = 1, 2$ can be recovered from its approximate $(L_{i,k})^T$ and details $(H_{i,k}^{(l)})^T$, namely: $(\tilde{x}_{i,k})^T$ is exactly $(x_{i,k})^T$.
Proof: Suppose that

$$\begin{pmatrix} L_1(2w) \\ L_2(2w) \end{pmatrix} = \overline{P(w)} \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix} + \overline{P(w+\pi)} \begin{pmatrix} x_1(w+\pi) \\ x_2(w+\pi) \end{pmatrix}$$

$$\begin{pmatrix} H_1^{(l)}(2w) \\ H_2^{(l)}(2w) \end{pmatrix} = \sum_{l=1}^{L} [\overline{Q^{(l)}(w)} \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix} + \overline{Q^{(l)}(w+\pi)} \begin{pmatrix} x_1(w+\pi) \\ x_2(w+\pi) \end{pmatrix}]$$

and

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = \tilde{P}^T(w) \begin{pmatrix} L_1(2w) \\ L_2(2w) \end{pmatrix} + \sum_{l=1}^{L} \tilde{Q}^{T^{(l)}}(w) \begin{pmatrix} H_1^{(l)}(2w) \\ H_2^{(l)}(2w) \end{pmatrix}$$

Plugging $(L_1(2w), L_2(2w))^T$ and $(H_1^{(l)}(2w), H_2^{(l)}(2w))^T$ into $(\tilde{x}_1(w), \tilde{x}_2(w))^T$,we get

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = \tilde{P}^*(w) \ [P(w) \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix} + P(w+\pi) \begin{pmatrix} x_1(w+\pi) \\ x_2(w+\pi) \end{pmatrix}]$$
$$+ \sum_{l=1}^{L} \tilde{Q}^{*(l)}(w) \ [Q^{(l)}(w) \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix} + Q^{(l)}(w+\pi) \begin{pmatrix} x_1(w+\pi) \\ x_2(w+\pi) \end{pmatrix}]$$

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = [\tilde{P}^*(w) \ P(w) + \sum_{l=1}^{L} \tilde{Q}^{*(l)}(w) \ Q^{(l)}(w)] \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}$$
$$+ [\tilde{P}^*(w) \ P(w+\pi) + \sum_{l=1}^{L} \tilde{Q}^{*(l)}(w) Q^{(l)}(w+\pi)] \begin{pmatrix} x_1(w+\pi) \\ x_2(w+\pi) \end{pmatrix}$$

If

$$\tilde{P}^*(w)\ P(w) + \sum_{l=1}^{L} \tilde{Q}^{*(l)}(w)\ Q^{(l)}(w) = I$$

$$\tilde{P}^*(w)\ P(w+\pi) + \sum_{l=1}^{L} \tilde{Q}^{*(l)}(w) Q^{(l)}(w+\pi) = 0$$

holds, then

$$\begin{pmatrix} \tilde{x}_1(w) \\ \tilde{x}_2(w) \end{pmatrix} = \begin{pmatrix} x_1(w) \\ x_2(w) \end{pmatrix}. \tag{2.64}$$

2.10.2 Undecimated multiple wavelet frame transform based denoising

Suppose $\underline{c}_k^0 = \begin{pmatrix} c_{1,k} \\ c_{2,k} \end{pmatrix}$ is an initial input, $\{P, Q^{(1)}, ..., Q^{(L)}\}$ and $\{\tilde{P}, \tilde{Q}^{(1)}, ..., \tilde{Q}^{(L)}\}$ are FIR multiple wavelet frame filter banks. Assume they are satisfy:

$$\tilde{P}^*(w)\ P(w) + \sum_{l=1}^{L} \tilde{Q}^{(l)*}(w)\ Q^{(l)}(w) = I. \tag{2.65}$$

Then undecimated multiple frame decmposition algorithm is given by:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \sum_{k\in\mathbb{Z}} P_k \begin{pmatrix} c_{1,k+n} \\ c_{2,k+n} \end{pmatrix}$$

$$\begin{pmatrix} H_{1,n}^{(l)} \\ H_{2,n}^{(l)} \end{pmatrix} = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{L} Q_k^{(l)} \begin{pmatrix} c_{1,k+n} \\ c_{2,k+n} \end{pmatrix}, \tag{2.66}$$

and the denoising algorithm:

$$\underline{u}_k = \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \sum_{n\in\mathbb{Z}} \tilde{P}_n^T \begin{pmatrix} L_{1,k-n} \\ L_{2,k-n} \end{pmatrix} + \sum_{n\in\mathbb{Z}} \sum_{l=1}^{L} \begin{pmatrix} \tilde{q}_{n,11}^{(l)} S_{\theta_{11}^l}^{(l)}(H_{1,(k-n)}^{(l)}) + \tilde{q}_{n,21}^{(l)} S_{\theta_{21}^l}^{(l)}(H_{2,(k-n)}^{(l)}) \\ \tilde{q}_{n,12}^{(l)} S_{\theta_{12}^l}^{(l)}(H_{1,(k-n)}^{(l)}) + \tilde{q}_{n,22}^{(l)} S_{\theta_{22}^l}^{(l)}(H_{2,(k-n)}^{(l)}) \end{pmatrix}. \tag{2.67}$$

2.11 Nonlinear diffusion equations

2.11.1 Second-order nonlinear diffusion

The second-order nonlinear diffusion equation for 1-D signal f is given by [41]:

$$u_t = \frac{\partial}{\partial x}(g(u_x^2)u_x) \tag{2.68}$$

with a noise. Where $u(x,0) = f(x)$ acts as the inital condition, and u_x represents $\frac{\partial}{\partial x}u(x,t)$ and g is the diffusivity.

Let the time and spatial step size be represented by τ and h respectively. Suppose $u_k^0 = f(kh),\ k \in \mathbb{Z}$. Thus the solution of $u(x,t)$ at $(kh, j\tau)$ is given by $u_k^j,\ j \geq 1$. The approximation of $\frac{\partial}{\partial x}u(x,t)$ at $(kh, j\tau)$ is given by:

$$u_x = \frac{u_{k+1}^j - u_k^j}{h},$$

and the approximation of $\frac{\partial}{\partial t}u(x,t)$ at $(kh, j\tau)$ is given by:

$$u_t = \frac{u_k^{j+1} - u_k^j}{\tau}.$$

Thus Eq.(2.68) can be discretized as:

$$u_k^{j+1} = u_k^j + \frac{\tau}{h^2}g[(\frac{u_{k+1}^j - u_k^j}{h})^2](u_{k+1}^j - u_k^j) - \frac{\tau}{h^2}g[(\frac{u_k^j - u_{k-1}^j}{h})^2](u_k^j - u_{k-1}^j). \quad (2.69)$$

2.11.2 Formula of high-order nonlinear diffusion

Let

$$u_t = \frac{\partial}{\partial x}(g_1(u_x^2)u_x) - \frac{\partial^2}{\partial x^2}(g_2(u_{xx}^2)u_{xx}), \quad (2.70)$$

be the formula of fourth-order nonlinear diffusion with $u(x,0) = f(x)$ acts as initial condition. Suppose that u_k^j represent the value of $u(x,t)$ at $(kh, j\tau)$. Then the first and second partial derivatives with respect to x are given, respectively, by:

$$u_x(kh, j\tau) \approx \frac{u_{k+1}^j - u_{k-1}^j}{2h}$$
$$u_{xx}(kh, j\tau) \approx \frac{u_{k-1}^j - 2u_k^j + u_{k+1}^j}{h^2},$$

where $u(x) \in \mathbb{R}$ and $h > 0$. Then the discretization of fourth-order nonlinear diffusion can be given by:

$$\begin{aligned} u_k^{j+1} &= u_k^j + \frac{\tau}{4h^2}g_1\frac{(u_{k+2}^j - u_k^j)^2}{4h^2}(u_{k+2}^j - u_k^j) - \frac{\tau}{4h^2}g_1\frac{(u_k^j - u_{k-2}^j)^2}{4h^2})(u_k^j - u_{k-2}^j) \\ &\quad -\frac{\tau}{h^4}g_2\frac{(u_{k-2}^j - 2u_{k-1}^j + u_k^j)^2}{h^4}(u_{k-2}^j - 2u_{k-1}^j + u_k^j) \\ &\quad +\frac{2\tau}{h^4}g_2\frac{(u_{k-1}^j - 2u_k^j + u_{k+1}^j)^2}{h^4})(u_{k-1}^j - 2u_k^j + u_{k+1}^j) \\ &\quad -\frac{\tau}{h^4}g_2\frac{(u_k^j - 2u_{k+1}^j + u_{k+2}^j)^2}{h^4}(u_k^j - 2u_{k+1}^j + u_{k+2}^j), \end{aligned} \quad (2.71)$$

Assume that the wavelet frame filter banks satisfy Eq.(2.31), and the vanishing moment order of highpass filter $q^{(l)},\ and\ \tilde{q}^{(l)}, l = 1, ..., L$ are $\alpha_l\ and\ \beta_l$, respectively. Then high-order nonlinear diffusion equation with $u(x,0) = f(x)$ is defined as:

$$u_t = \sum_{l=1}^{L}(-1)^{1+\alpha_l}\frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}}(g_l((\frac{\partial^{\beta_l}u}{\partial x^{\beta_l}})^2)\frac{\partial^{\beta_l}u}{\partial x^{\beta_l}}) \quad (2.72)$$

Lemma(2.1)[41]: Let $q(w)$ be a finite impulse response highpass filter with vanishing moment order Jthen F(x) smooth if

$$\frac{1}{C_J}\frac{1}{\varepsilon^J}\sum_{k\in\mathbb{Z}} q_k F(x+k\varepsilon) = F^{(J)}(x) + o(1)$$

$$\frac{1}{C_J}\frac{(-1)^J}{\varepsilon^J}\sum_{k\in\mathbb{Z}} q_k F(x-k\varepsilon) = F^{(J)}(x) + o(1)$$

where C_J is defined as:

$$C_J = \frac{1}{J!}\sum_{k\in\mathbb{Z}} k^j q_k$$

Equation (2.72) can be discretized as:
From lemma(2.1) with $\varepsilon = h$, the approximation of $\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$ *and* $\frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}} G(x,t)$ is defined by:

$$\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}(kh, j\tau) \approx \frac{1}{C_{\beta_l}}\frac{1}{h^{\beta_l}}\sum_{n\in\mathbb{Z}} q_n^{(l)} u(kh+nh, j\tau) \approx \frac{1}{C_{\beta_l}}\frac{1}{h^{\beta_l}}\sum_{n\in\mathbb{Z}} q_n^{(l)} u_{n+k}^j$$

$$\frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}} G(kh, j\tau) \approx \frac{(-1)^{\alpha_l}}{\tilde{C}_{\alpha_l}}\frac{1}{h^{\alpha_l}}\sum_{m\in\mathbb{Z}} \tilde{q}_m^{(l)} G(kh-mh, j\tau),$$

where $G(x,t) := g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2)\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$. Then Eq.(2.72) can be written as:

$$u_k^{j+1} = u_k^j - \tau\sum_{l=1}^{L}\frac{1}{\tilde{C}_{\alpha_l}}\frac{1}{h^{\alpha_l}}\sum_{m\in\mathbb{Z}} \tilde{q}_m^{(l)} g_l((\frac{1}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l),j})^2)(\frac{1}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l),j}) \quad (2.73)$$

2.11.3 Nonlinear diffusion equation in two-dimension

The two-scale symbol of the FIR highpass filter q is defined by $\hat{q}(w) = \sum_{k\in\mathbb{Z}^2} q_k e^{-ikw}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2, and\ w \in \mathbb{R}^2$, denote

$$\alpha! = \alpha_1!\alpha_2!,\ |\alpha| = \alpha_1 + \alpha_2,\ \ \frac{\partial^\alpha}{\partial w^\alpha} = \frac{\partial^{\alpha_1+\alpha_2}}{\partial w_2^{\alpha_2}\partial w_1^{\alpha_1}}.$$

The vanishing moments of order $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ for the FIR highpass filter q and $\hat{q}(w)$ is defined by:

$$\sum_{k\in\mathbb{Z}^2} k^\beta q_k = i^{|\beta|}\frac{\partial^\beta}{\partial w^\beta}\hat{q}(w)|_{w=0} = 0,$$

Assume that the wavelet frame filter banks are tight i,e. $p = \tilde{p}$ and $q^{(l)} = \tilde{q}^{(l)}$ $1 \le l \le L$, and $q^{(l)}$ have vanishing moments of orders β_l. Then the nonlinear diffusion equation is defined as:

$$u_t = \sum_{l=1}^{L}(-1)^{1+|\beta_l|}\frac{\partial^{\beta_l}}{\partial x^{\beta_l}}(g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2)\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}) \quad (2.74)$$

with $u(x,0)=f(x)$ acts as initial condition, $x \in \mathbb{R}^2,\ t \geq 0$, and $g_l : \mathbb{R} \mapsto \mathbb{R}^+$.
Lemma (2.2) [12]: If a finite impulse response highpass filter q has vanishing moment order $\alpha \in \mathbb{Z}_+^2$, then F(x) on $\mathbb{R}^2$ is smooth if

$$\frac{1}{\varepsilon^{|\alpha|}} \sum_{k\in\mathbb{Z}^2} q_k F(x+\varepsilon k) = C_\alpha \frac{\partial^\alpha}{\partial x^\alpha} F(x) + O(\varepsilon),\ \ as\ \varepsilon \longrightarrow 0, \tag{2.75}$$

where C_α is the constant defined by

$$C_\alpha = \frac{1}{\alpha!} \sum_{k\in\mathbb{Z}^2} k^\alpha q_k = \frac{i^{|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial w^\alpha} \hat{q}(w)|_{w=0}.$$

Therefore, the approximation of $\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$ *and* $\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G(x,t)$ where $G(x,t) := g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2)\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$. can be obtained from FIR filter $q^{(l)}$. Then from lemma (2.2) with $\varepsilon = h$ and $\varepsilon = -h$, the approximation of $\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$ *and* $\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G_l(x,t)$ is defined by:

$$\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}(kh, j\tau) \approx \frac{1}{C_{\beta_l}^{(l)}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q_n^{(l)} u(kh+nh, j\tau) \approx \frac{1}{C_{\beta_l}^{(l)}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q_n^{(l)} u_{n+k}^j$$

$$\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G_l(kh, j\tau) \approx \frac{(-1)^{|\beta_l|}}{C_{\beta_l}^{(l)}} \frac{1}{h^{|\beta_l|}} \sum_{m\in\mathbb{Z}^2} q_{k-m}^{(l)} G_l(mh, j\tau)$$

Therefore Eq.(2.74) can be discretized as:

$$\tilde{u}_k^j = \tilde{u}_k^{j-1} - \tau \sum_{l=1}^{L} \frac{1}{C_{\beta_l}^{(l)}} \frac{1}{h^{|\beta_l|}} \sum_m q_{k-m}^{(l)} g_l((\frac{1}{C_{\beta_l}^{(l)} h^{|\beta_l|}} H_m^{(l),j-1})^2)$$
$$(\frac{1}{C_{\beta_l}^{(l)} h^{|\beta_l|}} H_m^{(l),j-1})\ k \in \mathbb{Z}^2.$$

Chapter 3

Correspondence between wavelet shrinkage and nonlinear diffusion

Wavelet shrinkage and nonlinear diffusion filtering are powerful methods with same target. Since they have the same aim, it would be advantageous to recognize if there are association between both methods. This correspondence may assist to move results from one scheme to the other. Also, it helps to design hybrid method that mixes benefits from both approaches.

In this chapter, we present an overview of the existing methods in the literature that illustrate association between wavelet shrinkage and nonlinear diffusion.

We start this chapter by presenting the equivalence between wavelet shrinkage and second-order nonlinear diffusion equation in Section 3.1. Further, 2D-nonlinear diffusion in terms of wavelet shrinkage is presented in Section 3.2. We conclude this chapter by illustrating association between wavelet shrinkage and nonlinear diffusion equation in general case.

3.1 Association between wavelet shrinkage and second-order nonlinear diffusion equation

3.1.1 Haar wavelet shrinkage

Suppose the symbols $p(w)$ and $q(w)$ of the sequences $\{p_k, q_k\}_{k\in\mathbb{Z}}$ are given by:

$$p(w) = \frac{1}{2}\sum_{k\in\mathbb{Z}} p_k e^{-ikw}, \quad q(w) = \frac{1}{2}\sum_{k\in\mathbb{Z}} q_k e^{-ikw},$$

and $\{u_k^0 = c_k\}_k$ be a signal. Then the shift invariant wavelet decomposition algorithm is defined as:

$$L_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} p_k u_{k+n}^0$$

$$H_n = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} q_k u_{k+n}^0,$$

and denoising algorithm is given by:

$$u_k = \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} p_n L_{k-n} + \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} q_n s_\theta(H_{k-n}).$$

The wavelet shrinkage process could be applied iteratively:

$$L_n^j = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} p_k u_{k+n}^j$$

$$H_n^j = \frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}} q_k u_{k+n}^j,$$

$$u_k^{j+1} = \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} \tilde{p}_n L_{k-n}^j + \frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}} \tilde{q}_n s_\theta(H_{k-n}^j)$$

Let $\{p_0,\ p_1\} = \{1, 1\},\ \{q_0, q_1\} = \{1, -1\},\ p_k = q_k = 0,\ \ k \neq 0, 1.$ be the Haar filter coefficients. Then the undecimated discrete wavelet transform algorithm for one-scale is defined by:

$$L_n = \frac{1}{\sqrt{2}}[c_n + c_{n+1}]$$

$$H_n = \frac{1}{\sqrt{2}}[c_n - c_{n+1}],$$

and denoising algorithm for one-scale is given by:

$$\begin{aligned} u_k =& \frac{1}{4}[c_{k-1} + 2c_k + c_{k+1}] + \frac{\sqrt{2}}{4}s_\theta[\frac{1}{\sqrt{2}}(c_k - c_{k+1})] \\ &- \frac{\sqrt{2}}{4}s_\theta[\frac{1}{\sqrt{2}}(c_k - c_{k-1})]. \end{aligned} \tag{3.1}$$

Iterated undecimated wavelet transform based denoising with Haar filter are given by:

$$L_n^j = \frac{1}{\sqrt{2}}[u_n^j + u_{n+1}^j]$$

$$H_n^j = \frac{1}{\sqrt{2}}[u_n^j - u_{n+1}^j],$$

and

$$\begin{aligned} u_k^{j+1} =& \frac{1}{4}[u_{k-1}^j + 2u_k^j + u_{k+1}^j] + \frac{\sqrt{2}}{4}s_\theta[\frac{1}{\sqrt{2}}(u_k^j - u_{k+1}^j)] \\ &+ \frac{\sqrt{2}}{4}s_\theta[\frac{1}{\sqrt{2}}(u_k^j - u_{k-1}^j)]. \end{aligned} \tag{3.2}$$

3.1.2 Second-order nonlinear diffusion in terms of shrinkage

The second-order nonlinear diffusion equation for 1-D signal f with a noise is given by:

$$u_t = \frac{\partial}{\partial x}(g(u_x^2)u_x) \tag{3.3}$$

$u(x,0) = f(x)$ acts as the initial condition. The approximation of $\frac{\partial u}{\partial x}$ at $(kh, j\tau)$ is given by:

$$u_x = \frac{u_{k+1}^j - u_k^j}{h},$$

and the approximation of $\frac{\partial u}{\partial t}$ at $(kh, j\tau)$ is defined as:

$$u_t = \frac{u_k^{j+1} - u_k^j}{\tau}.$$

Then Eq.(3.3) can be written as:

$$u_k^{j+1} = u_k^j + \frac{\tau}{h^2} g[(\frac{u_{k+1}^j - u_k^j}{h})^2](u_{k+1}^j - u_k^j) - \frac{\tau}{h^2} g[(\frac{u_k^j - u_{k-1}^j}{h})^2](u_k^j - u_{k-1}^j). \tag{3.4}$$

when $j = 0$, we get the signal after 1-step diffusing with $u_k^0 = f(kh) = c_k,\ k \in \mathbb{Z}$:

$$u_k^1 = \frac{1}{4}[c_{k-1} + 2c_k + c_{k+1}] + \sqrt{2}[\frac{\tau}{h^2} g \frac{(\sqrt{2})^2 (c_{k+1} - c_k)^2}{\sqrt{2}^2 h^2} - \frac{1}{4}] \frac{1}{\sqrt{2}} (c_{k+1} - c_k)$$

$$\sqrt{2}[\frac{1}{4} - \frac{\tau}{h^2} g \frac{(\sqrt{2})^2 (c_k - c_{k-1})^2}{\sqrt{2}^2 h^2}] \frac{1}{\sqrt{2}} (c_k - c_{k-1}) \tag{3.5}$$

Also, the signal after $(j+1)$-step diffusing is given by:

$$u_k^{j+1} = \frac{1}{4}[u_{k-1}^j + 2u_k^j + u_{k+1}^j] + \sqrt{2}[\frac{\tau}{h^2} g \frac{(\sqrt{2})^2 (u_{k+1}^j - u_k^j)^2}{(\sqrt{2})^2 h^2} - \frac{1}{4}] \frac{1}{\sqrt{2}} (u_{k+1}^j - u_k^j)$$

$$+\sqrt{2}[\frac{1}{4} - \frac{\tau}{h^2} g \frac{(\sqrt{2})^2 (u_k^j - u_{k-1}^j)^2}{(\sqrt{2})^2 h^2}] \frac{1}{\sqrt{2}} (u_k^j - u_{k-1}^j) \tag{3.6}$$

Theorem (3.1)[54].
Let u_k in Eq.(3.1) be the denoising algorithm after 1-step wavelet shrinking with $u_k^0 = f(kh) = c_k$ as the inital input and u_k^1 in Eq.(3.5) be the resulting signal after 1-step diffusing with $u_k^0 = f(kh), k \in \mathbb{Z}$. If

$$s_\theta(x) = x(1 - \frac{4\tau}{h^2} g(\frac{2x^2}{h^2})), \tag{3.7}$$

then $u_k = u_k^1$.
Corollary (3.1)[54]. With diffusivity function $g(x)$ and shrinkage function $s_\theta(x)$ satisfying Eq.(3.7), iterated wavelet shrinking with Haar filter bank and nonlinear diffusing with Eq.(3.3) result in the same signal.

3.2 Correspondence between 2D wavelet shrinkage and nonlinear diffusion

3.2.1 2D wavelet shrinkage

Suppose a two dimensions signal is given by $c = (c_{i,j})$ then a one-level undecimated wavelet transform based denoising is given by :

$$v = L_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{i-n_1}\tilde{p}_{j-n_2}c_{i,j}$$

$$w_x = H^1_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{i-n_1}\tilde{p}_{j-n_2}c_{i,j}$$

$$w_y = H^2_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{p}_{i-n_1}\tilde{q}_{j-n_2}c_{i,j}$$

$$w_{xy} = H^3_{n_1,n_2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} \tilde{q}_{i-n_1}\tilde{q}_{j-n_2}c_{i,j},$$

We have 4-neighbourhoods in the pixel (i,j):

$$\begin{cases} \alpha : \{i, i+1\} \times \{j, j+1\} \\ \beta : \{i, i+1\} \times \{j-1, j\} \\ \gamma : \{i-1, i\} \times \{j, j+1\} \\ \delta : \{i-1, i\} \times \{j-1, j\} \end{cases}$$

Using Haar wavelet transform coefficients $p = (1, 1), \quad and \quad q = (1, -1)$ we have:

$$v^\alpha = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\alpha_y = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\alpha_x = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\alpha_{xy} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v^\beta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ w^\beta_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix}, \ w^\beta_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \ w^\beta_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$v^\gamma = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\gamma_y = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\gamma_x = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ w^\gamma_{xy} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$v^\delta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ w^\delta_y = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \ w^\delta_x = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \ w^\delta_{xy} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix}$$

The undecimated discrete wavelet transform in the neighbourhood α is given by :

$$v^\alpha = \frac{1}{2}c_{i,j} + \frac{1}{2}c_{i,j+1} + \frac{1}{2}c_{i+1,j} + \frac{1}{2}c_{i+1,j+1}$$

$$w_y^\alpha = -\frac{1}{2}c_{i,j} + \frac{1}{2}c_{i,j+1} - \frac{1}{2}c_{i+1,j} + \frac{1}{2}c_{i+1,j+1}$$
$$w_x^\alpha = \frac{1}{2}c_{i,j} + \frac{1}{2}c_{i,j+1} - \frac{1}{2}c_{i+1,j} - \frac{1}{2}c_{i+1,j+1}$$
$$w_{xy}^\alpha = -\frac{1}{2}c_{i,j} + \frac{1}{2}c_{i,j+1} + \frac{1}{2}f_{i+1,j} - \frac{1}{2}c_{i+1,j+1}$$

and denoising algorithm is defined as:

$$u_{i,j}^\alpha = \frac{1}{8}(v^\alpha + s(w_x^\alpha) + s(w_y^\alpha) + s(w_{xy}^\alpha))$$

following the same steps for β, γ, δ , then the one level and 2-D Haar wavelet shrinkage:

$$u_{i,j} = \frac{1}{8}(v^\alpha + s(w_x^\alpha) + s(w_y^\alpha) + s(w_{xy}^\alpha)) + \frac{1}{8}(v^\beta + s(w_x^\beta) - s(w_y^\beta) - s(w_{xy}^\beta))$$
$$\frac{1}{8}(v^\gamma - s(w_x^\gamma) + s(w_y^\gamma) - s(w_{xy}^\alpha)) + \frac{1}{8}(v^\delta - s(w_x^\delta) - s(w_y^\delta) + s(w_{xy}^\delta)) \tag{3.8}$$

3.2.2 Nonlinear diffusion equation in terms of shrinkage

The diffusion process is given by [53]:

$$u_t = div(g(|\nabla u|^2)\nabla u) \tag{3.9}$$

$f(x,y) = u(x,y,0)$ acts as the initial condition, where $g(|\nabla u|^2)$ is a nonnegative function. By using explicit finite difference schemes:

$$div(g(|\nabla u|^2)\nabla u) = \sum_{p=1}^{2} \partial_{r_p}(g(|\nabla u|^2)\partial_{r_p} u)$$

Then the nonlinear diffusion by using explicit finite difference schemes is represented by:

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} = \sum_{(m,n)\in N(i,j)} g_{m,n}^k \frac{u_{m,n}^k - u_{i,j}^k}{|(m,n)-(i,j)|^2}$$
$$u_{i,j}^{k+1} = u_{i,j}^k + \tau \sum_{(m,n)\in N(i,j)} g_{m,n}^k \frac{u_{m,n}^k - u_{i,j}^k}{|(m,n)-(i,j)|^2} \tag{3.10}$$

Where $g_{m,n}^k$ estimates $g(|\nabla u(x,y,t)|^2)$ at position $(\frac{i+m}{2}, \frac{j+n}{2})$, $N(i,j)$ consists of neighbours of pixel (i,j), and $\frac{u_{m,n}^k - u_{i,j}^k}{|(m,n)-(i,j)|}$ represents the space between (i,j)and(m,n). We examine diagonal, vertical and horizontal. Each of them gives various discrete schemes for Eq.(3.10).

First: $r_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $r_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ are the basis vectors in diagonal direction. Then the nonlinear diffusion becomes:

$$u_{i,j}^{k+1} = u_{i,j}^k + \tau \sum_{(m,n)\in D(i,j)} g_{m,n}^k \frac{u_{m,n}^k - u_{i,j}^k}{2} \tag{3.11}$$

with $k = 0$, then $u^o_{i,j} = c_{i,j}$. Thus, Eq.(3.11) can be written as:

$$u_{i,j} = c_{i,j} + \tau \sum_{(m,n)\in D(i,j)} g_{m,n} \frac{c_{m,n} - c_{i,j}}{2} \tag{3.12}$$

Then the Eq.(3.12) after applying the value of $c_{i,j}$can be written as:

$$\begin{aligned} u_{i,j} = & \frac{1}{8}(v^\alpha + w^\alpha_x(1 - 4\tau g^\alpha) + w^\alpha_y(1 - 4\tau g^\alpha) + w^\alpha_{xy}) \\ & + \frac{1}{8}(v^\beta + w^\beta_x(1 - 4\tau g^\beta) - w^\beta_y(1 - 4\tau g^\beta) - w^\beta_{xy}) \\ & + \frac{1}{8}(v^\gamma - w^\gamma_x(1 - 4\tau g^\gamma) + w^\gamma_y(1 - 4\tau g^\gamma) - w^\gamma_{xy}) \\ & + \frac{1}{8}(v^\delta - w^\delta_x(1 - 4\tau g^\delta) - w^\delta_y(1 - 4\tau g^\delta) + w^\delta_{xy}) \end{aligned} \tag{3.13}$$

Then comparing a single iteration of nonlinear diffusion Eq.(3.13) with one-level wavelet shrinkage in Eq.(3.8), we end with the correspondence between Eq.(3.13),and Eq.(3.8), under the conditions:

$$\begin{aligned} s(w^\omega_x) &= w^\omega_x(1 - 4\tau g^\omega) \\ s(w^\omega_y) &= w^\omega_y(1 - 4\tau g^\omega) \\ s(w^\omega_{xy}) &= w^\omega_{xy} \end{aligned} \tag{3.14}$$

Theorem (3.2).
Let $u_{i,j}$ in Eq.(3.8) be the denoising algorithm after one-step wavelet shrinking with 2D Haar filter bank, and $u_{i,j}$ in Eq.(3.13) be the signal after one-step diffusing with diagonal connectivity. If

$$s(w^\omega_x) = w^\omega_x(1 - 4\tau g^\omega)$$

$$s(w^\omega_y) = w^\omega_y(1 - 4\tau g^\omega)$$

$$s(w^\omega_{xy}) = w^\omega_{xy},$$

then $u_{i,j}$=$u_{i,j}$
Second: vertical and horizontal connectivity . In this case $r_1 = (1, 0)$, $\quad r_2 = (0, 1)$, then:

$$u_{i,j} = c_{i,j} + \tau \sum_{(m,n)\in \nu(i,j)} g_{m,n}(c_{m,n} - c_{i,j}) \tag{3.15}$$

where

$$\nu(i, j) = \{(i + 1, j), (i, j + 1), (i - 1, j), (i, j - 1)\}$$

and the diffusivity g can be approximated from [53]:

$$g_A = \frac{1}{2}(g^\alpha + g^\beta), \quad g_E = \frac{1}{2}(g^\gamma + g^\delta)$$

$$g_C = \frac{1}{2}(g^\alpha + g^\gamma), \quad g_G = \frac{1}{2}(g^\beta + g^\delta)$$

$c_{m,n} - c_{i,j}$ in term of wavelet coefficients of the pixels (m,n) is defined in two ways:

$$c_{m,n} - c_{i,j} := \begin{cases} d_A := -w_x^\alpha - w_{xy}^\alpha \ = -w_x^\beta + w_{xy}^\beta for \ (m,n) = (i+1,j) \\ d_C := -w_y^\alpha - w_{xy}^\alpha = -w_y^\gamma + w_{xy}^\gamma \quad for \ (m,n) = (i, j+1) \\ d_E := w_x^\gamma + w_{xy}^\gamma = w_x^\delta - w_{xy}^\delta \ \ for \ (m,n) = (i-1,j) \\ dG := w_y^\beta + w_{xy}^\beta = w_y^\delta - w_{xy}^\delta \ for (m,n) = (i, j-1) \end{cases}$$

The nonlinear diffusion then becomes:

$$\begin{aligned} u_{i,j} = \frac{1}{8}(v^\alpha + v^\beta + v^\gamma + v^\delta) + \frac{1}{8}(w_x^\alpha + w_y^\alpha + w_{xy}^\alpha + w_x^\beta - w_y^\beta - w_{xy}^\beta \\ -w_x^\gamma + w_y^\gamma - w_{xy}^\gamma - w_x^\delta - w_y^\delta + w_{xy}^\delta) - \tau\frac{g^\alpha + g^\beta}{2} d_A \\ -\tau\frac{g^\alpha + g^\gamma}{2} d_C - \tau\frac{g^\gamma + g^\delta}{2} d_E - \tau\frac{g^\beta + g^\delta}{2} d_G \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} g^\alpha d_A + g^\beta d_A &= g^\alpha(-w_x^\alpha - w_{xy}^\alpha) + g^\beta(-w_x^\beta + w_{xy}^\beta) \\ g^\alpha d_C + g^\gamma d_C &= g^\alpha(-w_y^\alpha - w_{xy}^\alpha) + g^\gamma(-w_y^\gamma + w_{xy}^\gamma) \\ g^\gamma d_E + g^\delta d_E &= g^\gamma(w_x^\gamma + w_{xy}^\gamma) + g^\delta(w_x^\delta - w_{xy}^\delta) \\ g^\beta d_G + g^\delta d_G &= g^\beta(w_y^\beta + w_{xy}^\beta) + g^\delta(w_y^\delta - w_{xy}^\delta) \end{aligned}$$

applying $g^\alpha d_A + g^\beta d_A$, $g^\alpha d_C + g^\gamma d_C$, $g^\gamma d_E + g^\delta d_E$, and $g^\beta d_G + g^\delta d_G$ into Eq(3.19), we get

$$\begin{aligned} u_{i,j} = \frac{1}{8}(v^\alpha + w_x^\alpha(1 - 4\tau g^\alpha) + w_y^\alpha(1 - 4\tau g^\alpha) + w_{xy}^\alpha(1 - 8\tau g^\alpha)) \\ +\frac{1}{8}(v^\beta + w_x^\beta(1 - 4\tau g^\beta) - w_y^\beta(1 - 4\tau g^\beta) - w_{xy}^\beta(1 - 8\tau g^\beta)) \\ +\frac{1}{8}(v^\gamma - w_x^\gamma(1 - 4\tau g^\gamma) + w_y^\gamma(1 - 4\tau g^\gamma) - w_{xy}^\gamma(1 - 8\tau g^\gamma)) \\ +\frac{1}{8}(v^\delta - w_x^\delta(1 - 4\tau g^\delta) - w_y^\delta(1 - 4\tau g^\delta) + w_{xy}^\delta(1 - 8\tau g^\delta)) \end{aligned} \tag{3.17}$$

Theorem (3.3).
Let $u_{i,j}$ in Eq.(3.8) be the denoising algorithm after one-step wavelet shrinking with 2D Haar filter bank, and let $u_{i,j}$ in Eq.(3.17) be the signal after one-step diffusing with vertical and horizontal connectivity. If

$$s(w_x^\omega) = w_x^\omega(1 - 4\tau g^\omega)$$

$$s(w_y^\omega) = w_y^\omega(1 - 4\tau g^\omega)$$

$$s(w_{xy}^\omega) = w_{xy}^\omega(1 - 8\tau g^\omega),$$

then $u_{i,j}$=$u_{i,j}$

3.3 Correspondence between wavelet shrinkage and nonlinear diffusion equation in general case

3.3.1 Wavelet shrinkage in general case

Undecimated discrete wavelet transform is defined by:

$$L_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} p_k u^0_{k+n}$$

$$H_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} q_k u^0_{k+n},$$

and shrunk data is given by:

$$u_k = \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{p}_n L_{k-n} + \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{q}_n s_\theta(H_{k-n}).$$

with a suitable shrinking function s_θ, u_k can be written as:

$$u_k = c_k + \frac{\sqrt{2}}{4} \sum_{n \in \mathbb{Z}} \tilde{q}_n (s_\theta(H_{k-n}) - H_{k-n}) \tag{3.18}$$

iterated undecimated wavelet transform based denoising is given by:

$$L^j_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} p_k u^j_{k+n}$$

$$H^j_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} q_k u^j_{k+n},$$

$$u^{j+1}_k = \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{p}_n L^j_{k-n} + \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{q}_n s_\theta(H^j_{k-n})$$

with a suitable s_θ, u^{j+1}_k is the denoising algorithm of the original signal with noise. Then u^{j+1}_k can be defined as

$$u^{j+1}_k = u^j_k + \frac{\sqrt{2}}{4} \sum_{n \in \mathbb{Z}} \tilde{q}_n (s_\theta(H^j_{k-n}) - H^j_{k-n}) \tag{3.19}$$

3.3.2 Shrinkage for high-order nonlinear diffusion

Assume that wavelet filter banks $\{p, \tilde{p}\}$ *and* $\{q, \tilde{q}\}$ satisfy Eq.(2.31), and suppose the vanishing moment order of q, and $\tilde{q}$ is s, namely q, and $\tilde{q}$ are tight wavelet filter banks.

Then the formula of high-order nonlinear diffusion for $u = (x,t)$ with f as initial condition $u(x,0) = f(x)$ is formula like:

$$u_t = (-1)^{s+1}\frac{\partial^s}{\partial x^s}(g((\frac{\partial^s u}{\partial x^s})^2)\frac{\partial^s u}{\partial x^s}), \quad s \geq 2 \tag{3.20}$$

Equation (3.20) can be discretized as:
From lemma(2.1) with $\varepsilon = h$, the approximation of $\frac{\partial^s}{\partial x^s}$ *and* $\frac{\partial^s}{\partial x^s}G(x,t)$ are written as:

$$\frac{\partial^s u}{\partial x^s}(kh, j\tau) \approx \frac{1}{C_s}\frac{1}{h^s}\sum_{n\in\mathbb{Z}} q_n u(kh+nh, j\tau) \approx \frac{1}{C_s}\frac{1}{h^s}\sum_{n\in\mathbb{Z}} q_n u^j_{n+k}$$

$$\frac{\partial^s}{\partial x^s}G(kh, j\tau) \approx \frac{(-1)^s}{C_s}\frac{1}{h^s}\sum_{m\in\mathbb{Z}} q_m G(kh-mh, j\tau)$$

where $G(x,t) := g((\frac{\partial^s u}{\partial x^s})^2)\frac{\partial^s u}{\partial x^s}$. Suppose $\frac{u^{j+1}_k - u^j_k}{\tau}$ is the approximation of $\frac{\partial^s u}{\partial t^s}$ at $(kh, j\tau)$, then Eq.(3.20) can be discretized as:

$$u^{j+1}_k = u^j_k - \tau\frac{1}{C_s}\frac{1}{h^s}\sum_{m\in\mathbb{Z}} q_m g((\frac{\sqrt{2}}{C_s h^s}H^j_{k-m})^2)(\frac{\sqrt{2}}{C_s h^s}H^j_{k-m}) \tag{3.21}$$

By setting $j = 0$ *with* $c_k = u^0_k$, then Eq.(3.21) is defined as:

$$u^1_k = c_k - \tau\frac{1}{C_s h^s}\sum_{m\in\mathbb{Z}} q_m g((\frac{\sqrt{2}}{C_s h^s}H_{k-m})^2)(\frac{\sqrt{2}}{C_s h^s}H_{k-m}) \tag{3.22}$$

Theorem (3.4) [53].
Let u_k in Eq.(3.18) be the denoised signal after one-step undecimated tight wavelet shrinking of $u^0_k = f(kh)$ with shrinkage function s_θ. Let u^1_k in Eq.(3.22) be the signal after one-step diffusing for diffusion equation (3.20) with $u^0_k = f(kh), k \in \mathbb{Z}$. If

$$s_\theta(x) = x(1 - \frac{4\tau}{C_s^2 h^{2s}}g(\frac{2x^2}{(C_s)^2 h^{2s}})), \tag{3.23}$$

then $u_k = u^1_k$ for all k.

Corollary (3.2). With diffusivity function $g(x)$ and shrinkage function $s_\theta(x)$ satisfying Eq.(3.23), iterated tight wavelet shrinking and high-order nonlinear diffusing Eq.(3.20) result in the same signal.

Chapter 4

Equivalence between wavelet frame shrinkage and nonlinear diffusion

We rely on the results of this chapter and the previous chapter that present nonlinear diffusion equation in term of frame shrinkage in chapter 4 and in term of wavelet shrinkage in chapter 3.

In Sections 4.1-4.6, we provide an overview of a few current approach in the lectures that present the equivalence between wavelet frame shrinkage and nonlinear diffusion.

4.1 Equivalence between tight frame shrinkage and fourth-order nonlinear diffusion

4.1.1 Frame shrinkage

Let $\{p, \tilde{p}\}$ *and* $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ be a finite impulse response(FIR) filter banks. Suppose $\{c_k = u_k^o\}_k$ is the inital data, then the outputs L_n *and* $H_n^{(l)}$ are defined by:

$$L_n = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} p_k c_{k+n}$$

$$H_n^{(l)} = \frac{1}{\sqrt{2}} \sum_{k \in \mathbb{Z}} q_k^{(l)} c_{k+n},\ n \in (\mathbb{Z}),\ l = 1, ..., L$$

and the shrunk data is given by:

$$u_k = \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{p}_n L_{k-n} + \frac{\sqrt{2}}{4} \sum_{k \in \mathbb{Z}} \tilde{q}_n^{(l)} s_{\theta_l}^l (H_{k-n}^{(l)}) \tag{4.1}$$

If the frame filter bank satisfy:

$$\overline{p(w)} \tilde{p}(w) + \sum_{l=1}^{L} \overline{q^{(l)}(w)} \tilde{q}^{(l)}(w) = 1.$$

and $s^l_{\theta_l}$ are identity i.e $s^l_{\theta_l}(x) = x$, then $u_k = c_k$

The nonzero coefficients of Ron-Shen's filter bank are defined as:

$$(p_{-1}, p_0, p_1) = (\frac{1}{2}, 1, \frac{1}{2})$$

$$(q^{(1)}_{-1}, q^{(1)}_0, q^{(1)}_1) = (\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2})$$

$$(q^{(2)}_{-1}, q^{(2)}_0, q^{(2)}_1) = (-\frac{1}{2}, 1, -\frac{1}{2})$$

By applying the coefficients, then the outputs L_n *and* $H^{(l)}_n$ are given by:

$$L_n = \frac{1}{2\sqrt{2}}[c_{n-1} + 2c_n + c_{n+1}]$$

$$H^{(1)}_n = \frac{1}{2}[c_{n-1} - c_{n+1}]$$

$$H^{(2)}_n = \frac{1}{2\sqrt{2}}[2c_n - c_{n-1} - c_{n+1}]$$

and the shrunk data is given by:

$$u_k = \frac{1}{16}[c_{k-2} + 4c_{k-1} + 6c_k + 4c_{k+1} + c_{k+2}] + \frac{1}{4}s^1_\theta[\frac{c_k - c_{k+2}}{2}] - \frac{1}{4}s^1_\theta[\frac{c_{k-2} - c_k}{2}]$$
$$+\frac{\sqrt{2}}{4}s^2_\sigma(\frac{\sqrt{2}}{4}[2c_k - c_{k-1} - c_{k+1}]) - \frac{\sqrt{2}}{8}s^2_\sigma(\frac{\sqrt{2}}{4}[2c_{k-1} - c_{k-2} - c_k])$$
$$-\frac{\sqrt{2}}{8}s^2_\sigma(\frac{\sqrt{2}}{4}[2c_{k+1} - c_k - c_{k+2}]) \tag{4.2}$$

4.1.2 Fourth-order nonlinear diffusion in terms of shrinkage

The formula of fourth-order nonlinear diffusion for $u = (x, t)$ with f as initial condition $u(x, 0) = f(x)$ can be written as:

$$u_t = \frac{\partial}{\partial x}(g_1(u^2_x)u_x) - \frac{\partial^2}{\partial x^2}(g_2(u^2_{xx})u_{xx}) \tag{4.3}$$

with noise. The discretization of fourth-order nonlinear diffusion can be given as:

$$u^{j+1}_k = u^j_k + \frac{\tau}{4h^2}g_1\frac{(u^j_{k+2} - u^j_k)^2}{4h^2}(u^j_{k+2} - u^j_k) - \frac{\tau}{4h^2}g_1\frac{(u^j_k - u^j_{k-2})^2}{4h^2})(u^j_k - u^j_{k-2})$$
$$-\frac{\tau}{h^4}g_2\frac{(u^j_{k-2} - 2u^j_{k-1} + u^j_k)^2}{h^4}(u^j_{k-2} - 2u^j_{k-1} + u^j_k)$$
$$+\frac{2\tau}{h^4}g_2\frac{(u^j_{k-1} - 2u^j_k + u^j_{k+1})^2}{h^4})(u^j_{k-1} - 2u^j_k + u^j_{k+1})$$

$$-\frac{\tau}{h^4}g_2\frac{(u_k^j-2u_{k+1}^j+u_{k+2}^j)^2}{h^4}(u_k^j-2u_{k+1}^j+u_{k+2}^j), \tag{4.4}$$

when j=0, Eq.(4.4) can be written as:

$$u_k^1=\frac{1}{16}[c_{k-2}+4c_{k-1}+6c_k+4c_{k+1}+c_{k+2}]+(\frac{\tau}{4h^2}g_1\frac{(c_{k+2}-c_k)^2}{4h^2}-\frac{1}{8})[c_{k+2}-c_k]$$
$$+(\frac{1}{8}-\frac{\tau}{4h^2}g_1\frac{(c_k-c_{k-2})^2}{4h^2})[c_k-c_{k-2}]-\frac{\tau}{h^4}g_2\frac{(c_k-2c_{k-1}+c_k)^2}{h^4})$$
$$[c_k-2c_{k-1}+c_k]+(\frac{2\tau}{h^4}g_2\frac{(c_{k-1}-2c_k+c_{k+1})^2}{h^4}-\frac{1}{8})[c_{k-1}-2c_k+c_{k+1}]$$
$$+(\frac{1}{16}-\frac{\tau}{h^4}g_2\frac{(c_k-2c_{k+1}+c_{k+2})^2}{h^4})[c_k-2c_{k+1}+c_{k+2}] \tag{4.5}$$

Theorem (4.1)[41].
let u_k in Eq.(4.2) be the resulting signal after one-step Ron-shen‘s frame shrinking with $c_k=f(kh), k\in\mathbb{Z}$ and u_k^1 in Eq.(4.5) be the signal after one-step diffusing with $u_k^0=f(kh), k\in\mathbb{Z}$. If

$$s_\theta^1(x)=x(1-\frac{2\tau}{h^2}g_1(\frac{x^2}{h^2})),\quad s_\sigma^2(x)=x(1-\frac{16\tau}{h^4}g_2(\frac{8x^2}{h^4})), \tag{4.6}$$

then $u_k=u_k^1$.

4.2 Equivalence between one-dimensional frame shrinkage and high-order nonlinear diffusion

4.2.1 Frame shrinkage in general case

Suppose that the denoising algorithm after one-step frame shrinking is given by Eq.(4.1), and the lowpass and highpass filters are undecimated bi-frame filter banks. Then

$$u_k=c_k+\frac{\sqrt{2}}{4}\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}}\tilde{q}_n^{(l)}(s_{\theta_l}^l(x)-x)|_{x=H_{k-n}^{(l)}} \tag{4.7}$$

The UFT-based denoising can be applied iteratively:

$$L_n^j=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}p_k u_{k+n}^j$$

$$H_n^{(l),j}=\frac{1}{\sqrt{2}}\sum_{k\in\mathbb{Z}}q_k^{(l)}u_{k+n}^j,\ 1\le l\le L$$

$$u_k^{j+1}=\frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}\tilde{p}_n L_{k-n}^j+\frac{\sqrt{2}}{4}\sum_{k\in\mathbb{Z}}\tilde{q}_n^{(l)}s_\theta(H_{k-n}^{(l),j}) \tag{4.8}$$

Assume the filter banks satisfy this condition:

$$\overline{p(w)}\tilde{p}(w)+\sum_{l=1}^{L}\overline{q^{(l)}(w)}\tilde{q}^{(l)}(w)=1,$$

then u_k^j can be written as:

$$u_k^j=\frac{\sqrt{2}}{4}\sum_{n\in\mathbb{Z}}\tilde{p}_n L_{k-n}^j+\frac{\sqrt{2}}{4}\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}}\tilde{q}_n^{(l)}(H_{k-n}^{(l),j}),$$

and Eq.(4.8) becomes:

$$u_k^{j+1}=u_k^j+\frac{\sqrt{2}}{4}\sum_{l=1}^{L}\sum_{n\in\mathbb{Z}}\tilde{q}_n^{(l)}(s_{\theta_l}^l(H_{k-n}^{(l),j})-H_{k-n}^{(l),j}). \tag{4.9}$$

4.2.2 High-order nonlinear diffusion

The formula of nonlinear diffusion for $u=(x,t)$ with f as initial condition $u(x,0)=f(x)$ is defined by:

$$u_t=\sum_{l=1}^{L}(-1)^{1+\alpha_l}\frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}}(g_l((\frac{\partial^{\beta_l}u}{\partial x^{\beta_l}})^2)\frac{\partial^{\beta_l}u}{\partial x^{\beta_l}}) \tag{4.10}$$

where vanishing moment order of $q^{(l)},\ and\ \tilde{q}^{(l)}$ are $\alpha_l\ and\ \beta_l$, respectively.
Then from subsecion 2.11.2, Eq.(4.10) can be discretized as:

$$u_k^{j+1}=u_k^j-\tau\sum_{l=1}^{L}\frac{1}{\tilde{C}_{\alpha_l}}\frac{1}{h^{\alpha_l}}\sum_{m\in\mathbb{Z}}\tilde{q}_m^{(l)}g_l((\frac{\sqrt{2}}{C_{\beta_l}h^{\beta_l}}H_{k-m}^{(l),j})^2)(\frac{\sqrt{2}}{C_{\beta_l}h^{\beta_l}}H_{k-m}^{(l),j}) \tag{4.11}$$

let $j=0$ in Eq.(4.11) with $c_k=u_k^0$, we have:

$$u_k^1=c_k-\tau\sum_{l=1}^{L}\frac{1}{\tilde{C}_{\alpha_l}h^{\alpha_l}}\sum_{m\in\mathbb{Z}}\tilde{q}_m^{(l)}g_l((\frac{\sqrt{2}}{C_{\beta_l}h^{\beta_l}}H_{k-m}^{(l)})^2)(\frac{\sqrt{2}}{C_{\beta_l}h^{\beta_l}}H_{k-m}^{(l)}) \tag{4.12}$$

Theorem (4.2)[41].

let u_k in Eq(4.7) be the denoising algorithm after one-step undecimated bi-frame shrinking with $c_k=f(kh),k\in\mathbb{Z}$ and shrinkage function $s_{\theta_l}^l$. Let u_k^1 in Eq.(4.12) be the signal after one-step diffusing with $u_k^0=f(kh),k\in\mathbb{Z}$ as initial input. If

$$s_{\theta_l}^l(x)=x(1-\frac{4\tau}{\tilde{C}_{\alpha_l}C_{\beta_l}h^{\alpha_l+\beta_l}}g_l(\frac{2x^2}{(C_{\beta_l})^2h^{2\beta_l}}))|_{x=H_{k-m}^{(l)}},\quad 1\le l\le L, \tag{4.13}$$

then $u_k = u_k^1$,for all k.
If the frame filters bank is tight. i.e the vanishing moment order of $q^{(l)},\ and\ \tilde{q}^{(l)}$ are α_l,the nonlinear diffusion equation is given by:

$$u_t = \sum_{l=1}^{L} (-1)^{1+\beta_l} \frac{\partial^{\beta_l}}{\partial x^{\beta_l}} (g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2) \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}), \tag{4.14}$$

with $u(x,0) = f(x)$ acts as the initial condition. Eq.(4.14) can be discretized as:

$$u_k^{j+1} = u_k^j + \tau \sum_{l=1}^{L} (-1)^{1+\beta_l} \frac{(-1)^{\beta_l}}{C_{\beta_l}} \frac{1}{h^{\beta_l}} \sum_{m\in\mathbb{Z}} q_m^{(l)} g_l((\frac{\sqrt{2}}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l)})^2)(\frac{\sqrt{2}}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l)})$$

and u_k^1 after one-step diffusing is defined by:

$$u_k^1 = c_k - \tau \sum_{l=1}^{L} \frac{1}{C_{\beta_l} h^{\beta_l}} \sum_{m\in\mathbb{Z}} q_m^{(l)} g_l((\frac{\sqrt{2}}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l)})^2)(\frac{\sqrt{2}}{C_{\beta_l} h^{\beta_l}} H_{k-m}^{(l)}) \tag{4.15}$$

Theorem (4.3)[41].
let u_k be the denoising algorithm Eq.(4.7) after one-step undecimated tight frame shrinking with $c_k = f(kh), k \in \mathbb{Z}$ and shrinkage function $s_{\theta_l}^l$. Let u_k^1 in Eq.(4.15) be the signal after one-step diffusing for diffusing equation (4.14) with $u_k^0 = f(kh), k \in \mathbb{Z}$ as initial data. If

$$s_{\theta_l}^l(x) = x(1 - \frac{4\tau}{(C_{\beta_l})^2 h^{2\beta_l}} g_l(\frac{2x^2}{(C_{\beta_l})^2 h^{2\beta_l}}))|_{x=H_{k-m}^{(l)}}, \quad 1 \le l \le L. \tag{4.16}$$

then $u_k = u_k^1$ for all k.

4.3 Equivalence between 2D frame shrinkage and high-order nonlinear diffusion

4.3.1 2D frame shrinkage

Suppose that highpass and lowpass frame filter banks are satisfying:

$$\overline{p(w)}\tilde{p}(w) + \sum_{l=1}^{L} \overline{q^{(l)}(w)}\tilde{q}^{(l)}(w) = 1.$$

Let the initial input is $\{u_k^0\}$ $k \in \mathbb{Z}^2$, then the one-level undecimated wavelet frame transform based denoising are given by:

$$L_n = \sum_{k\in\mathbb{Z}^2} p_k u_{k+n}^0$$

$$H_n^{(l)} = \sum_{k\in\mathbb{Z}^2} q_k^{(l)} u_{k+n}^0,\ n \in \mathbb{Z}^2,\ 1 \le l \le L$$

$$u_k^1 = \sum_{n \in \mathbb{Z}^2} \tilde{p}_{k-n} L_n + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} s_l(H_n^{(l)}) \tag{4.17}$$

Eq.(4.17) can be written as:

$$u_k^1 = u_k^0 + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} [s_l(\xi) - \xi]|_{\xi = H_n^{(l)}}, \ k \in \mathbb{Z}^2 \tag{4.18}$$

iterated frame shrinking process above are given by:

$$L_n^{j-1} = \sum_{k \in \mathbb{Z}^2} p_k u_{k+n}^{j-1}$$

$$H_n^{(l),j-1} = \sum_{k \in \mathbb{Z}^2} q_k^{(l)} u_{k+n}^{j-1}, \ n \in \mathbb{Z}^2, \ 1 \le l \le L$$

$$u_k^j = u_k^{j-1} + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} [s_l(H_n^{(l),j-1}) - H_n^{(l),j-1}], j = 1, 2, ... \tag{4.19}$$

4.3.2 Nonlinear diffusion equation in terms of 2D frame shrinkage

The two-scale symbol of the FIR highpass filter q is defined by $\hat{q}(w) = \sum_{k \in \mathbb{Z}^2} q_k e^{-ikw}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2, and\ w \in \mathbb{R}^2$, denote

$$\alpha! = \alpha_1! \alpha_2!, \ |\alpha| = \alpha_1 + \alpha_2, \quad \frac{\partial^\alpha}{\partial w^\alpha} = \frac{\partial^{\alpha_1 + \alpha_2}}{\partial w_2^{\alpha_2} \partial w_1^{\alpha_1}}.$$

The vanishing moments of order $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ for the FIR highpass filter q and $\hat{q}(w)$ is defined by:

$$\sum_{k \in \mathbb{Z}^2} k^\beta q_k = i^{|\beta|} \frac{\partial^\beta}{\partial w^\beta} \hat{q}(w)|_{w=0} = 0,$$

with $|\beta| < |\alpha|$ and $|\beta| = |\alpha|$ but $\beta \neq \alpha$ for all $\beta \in \mathbb{Z}_+^2$. Assume the pair of frame filter banks $\{p, q^{(1)}, ..., q^{(L)}\}\ and\ \{\tilde{p}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ are tight i,e. $p = \tilde{p}$ and $q^{(l)} = \tilde{q}^{(l)}, \ 1 \le l \le L$, and $q^{(l)}$ have vanishing moments of orders β_l. Then the nonlinear diffusion equation is defined as:

$$u_t = \sum_{l=1}^{L} (-1)^{1+|\beta_l|} \frac{\partial^{\beta_l}}{\partial x^{\beta_l}} (g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2) \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}) \tag{4.20}$$

with f as initial condition $u(x, 0) = f(x)$, $x \in \mathbb{R}^2, \ t \ge 0$, and $g_l : \mathbb{R} \mapsto \mathbb{R}^+$.

The approximation of $\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}\ and\ \frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G(x, t)$ where $G(x, t) := g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2) \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$. can be obtained from FIR filter $q^{(l)}$. Then from lemma (2.2) with $\varepsilon = h$ and $\varepsilon = -h$, the approximation of $\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}$ *and* $\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G_l(x,t)$ is defined by:

$$\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}(kh, j\tau) \approx \frac{1}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q^{(l)}_n u(kh+nh, j\tau) \approx \frac{1}{C_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q^{(l)}_n u^j_{n+k}$$

$$\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G_l(kh, j\tau) \approx \frac{(-1)^{|\beta_l|}}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{m\in\mathbb{Z}^2} q^{(l)}_{k-m} G_l(mh, j\tau)$$

Then Eq.(4.20) can be discretized as:

$$\tilde{u}^1_k = \tilde{u}^0_k - \tau \sum_{l=1}^{L} \frac{1}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_m q^{(l)}_{k-m} g_l((\frac{1}{C^{(l)}_{\beta_l} h^{|\beta_l|}} H^{(l)}_m)^2)(\frac{1}{C^{(l)}_{\beta_l} h^{|\beta_l|}} H^{(l)}_m). \tag{4.21}$$

Continue doing this process, we get the approximated values $\tilde{u}^2_k, \tilde{u}^3_k, \tilde{u}^4_k, \ldots$, of the solution $u(x,t)$ at $(hk, 2\tau), (hk, 3\tau), (hk, 4\tau), \ldots$.
Let $\tilde{u}^{j-1}_k$ be the approximated solution of $u(x,t)$ at $(kh, (j-1)\tau),\ k \in \mathbb{Z}^2$, then

$$\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}(kh, (j-1)\tau) \approx \frac{1}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q^{(l)}_n u(kh+nh, (j-1)\tau)$$

$$\approx \frac{1}{C_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{n\in\mathbb{Z}^2} q^{(l)}_n u^{j-1}_{n+k}$$

$$\frac{\partial^{\beta_l}}{\partial x^{\beta_l}} G_l(kh, (j-1)\tau) \approx \frac{(-1)^{|\beta_l|}}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_{m\in\mathbb{Z}^2} q^{(l)}_{k-m} G_l(mh, (j-1)\tau)$$

Therefore Eq.(4.20) can be discretized as:

$$\tilde{u}^j_k = \tilde{u}^{j-1}_k - \tau \sum_{l=1}^{L} \frac{1}{C^{(l)}_{\beta_l}} \frac{1}{h^{|\beta_l|}} \sum_m q^{(l)}_{k-m} g_l((\frac{1}{C^{(l)}_{\beta_l} h^{|\beta_l|}} H^{(l),j-1}_m)^2)$$

$$(\frac{1}{C^{(l)}_{\beta_l} h^{|\beta_l|}} H^{(l),j-1}_m)\ k \in \mathbb{Z}^2. \tag{4.22}$$

Comparing Eq.(4.19) with Eq.(4.22), we conclude that $\tilde{u}^j_k = u^j_k$ under the condition:

$$s_l(\xi) = \xi\{1 - \frac{\tau}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}} g_l(\frac{\xi^2}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}})\},\ \xi \in \mathbb{R},\ 1 \le l \le L \tag{4.23}$$

Theorem (4.4)[12].
Let u^j be the resulting sequence of j-steps tight wavelet frame shrinkage in Eq.(4.19) with $u^0_k = f(kh), k \in \mathbb{Z}^2$ and q^l having vanishing moment β_l. Then u^j is a discrete approximation of $\{u(kh, \tau j)\}, k \in \mathbb{Z}^2, j = 1, 2, \ldots$ with u(x,t) the solution of Eq.(4.20) provided that the shrinkage function satisfy Eq.(4.23).

The correspondence nonlinear diffusion equation to bi-frame filter banks is given by:

$$u_t = \sum_{l=1}^{L} (-1)^{1+|\alpha_l|} \frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}} (g_l((\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2) \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}) \tag{4.24}$$

with $u(x,0) = f(x),\ x \in \mathbb{R}^2,\ t \geq 0$ acts as initial condition, and $g_l : \mathbb{R} \mapsto \mathbb{R}^+$. Then Eq.(4.24) is discretized as:

$$\tilde{u}_k^j = \tilde{u}_k^{j-1} - \tau \sum_{l=1}^{L} \frac{1}{\tilde{C}_{\beta_l}^{(l)}} \frac{1}{h^{|\beta_l|}} \sum_m \tilde{q}_{k-m}^{(l)} g_l((\frac{1}{C_{\beta_l}^{(l)} h^{|\beta_l|}} H_m^{(l),j-1})^2)$$

$$(\frac{1}{C_{\beta_l}^{(l)} h^{|\beta_l|}} H_m^{(l),j-1})\ k \in \mathbb{Z}^2. \tag{4.25}$$

Thus u_k^j after j-steps frame shrinking is u_k^j after j-steps diffusing if :

$$s_l(\xi) = \xi \{1 - \frac{\tau}{\tilde{C}_{\alpha_l}^{(l)} C_{\beta_l}^{(l)} h^{|\alpha_l|+|\beta_l|}} g_l(\frac{\xi^2}{(C_{\beta_l}^{(l)})^2 h^{2|\beta_l|}})\},\ \xi \in \mathbb{R},\ 1 \leq l \leq L \tag{4.26}$$

Theorem (4.5)[12].
Let u^j be the resulting sequence of j-steps bi-frame shrinkage in Eq.(4.19) with $u_k^0 = f(kh), k \in \mathbb{Z}^2$ and q^l(resp $\tilde{q}^l$) having vanishing moment β_l=(resp.α_l). Then u^j is a discrete approximation of $\{u(kh, \tau j)\}, k \in \mathbb{Z}^2, j = 1, 2, ...$ with u(x,t) the solution of Eq.(4.24) provided that the shrinkage function operator of Eq.(4.19) is chosen as Eq.(4.26).

4.4 Equivalence between channel mixed frame shrinkage and channel mixed high-order nonlinear diffusion in two-dimension

4.4.1 Channel mixed wavelet frame shrinkage

The one-level 2-D channel mixed undecimated wavelet frame transform based denoising are given by the following processes:

$$L_n = \sum_{k \in \mathbb{Z}^2} p_k u_{k+n}^0$$

$$H_n^{(l)} = \sum_{k \in \mathbb{Z}^2} q_k^{(l)} u_{k+n}^0,\ n \in \mathbb{Z}^2,\ 1 \leq l \leq L$$

$$u_k^1 = \sum_{n \in \mathbb{Z}^2} \tilde{p}_{k-n} L_n + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} s_l(H_n^{(1)}, H_n^{(2)}, ..., H_n^{(L)}), \tag{4.27}$$

If the highpass and lowpass filters are bi-frame filter banks, then u_k^1 can be written as

$$u_k^1 = u_k^0 + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} (s_l(H_n^{(1)}, H_n^{(2)}, ..., H_n^{(L)}) - H_n^{(l)}) \tag{4.28}$$

iterated channel mixed frame shrinking process above are given by:

$$L_n^{j-1} = \sum_{k \in \mathbb{Z}^2} p_k u_{k+n}^{j-1}$$

$$H_n^{(l),j-1} = \sum_{k \in \mathbb{Z}^2} q_k^{(l)} u_{k+n}^{j-1}, \ n \in \mathbb{Z}^2,$$

$$u_k^j = u_k^{j-1} + \sum_{l=1}^{L} \sum_{n \in \mathbb{Z}^2} \tilde{q}_{k-n}^{(l)} [s_l(H_n^{(1),j-1}, H_n^{(2),j-1}, ..., H_n^{(L),j-1}) - H_n^{(l),j-1}], \tag{4.29}$$

where $j = 1, 2, ...$ and $1 \leq l \leq L$.

4.4.2 Channel mixed high-order nonlinear diffusion equation

The corresponding channel mixed nonlinear diffusion equation to the bi-frame filter banks $\{q^{(1)}, ..., q^{(L)}, \tilde{q}^{(1)}, ..., \tilde{q}^{(L)}\}$ with vanishing moments of orders β_l and α_l is defined as:

$$u_t = \sum_{l=1}^{L} (-1)^{1+|\alpha_l|} \frac{\partial^{\alpha_l}}{\partial x^{\alpha_l}} [g_l(\frac{\partial^{\beta_1} u}{\partial x^{\beta_1}}, \frac{\partial^{\beta_2} u}{\partial x^{\beta_2}}, ..., \frac{\partial^{\beta_L} u}{\partial x^{\beta_L}}) \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}] \tag{4.30}$$

The discretization of Eq.(4.31) with $k \in \mathbb{Z}^2$ is defined as:

$$\tilde{u}_k^j = \tilde{u}_k^{j-1} - \tau \sum_{l=1}^{L} \frac{1}{C_{\alpha_l}^{(l)} h^{|\alpha_l|}} \sum_m q_{k-m}^{(l)} g_l [\ \frac{H_m^{(1),j-1}}{C_{\beta_1}^{(1)} h^{|\beta_1|}}, \frac{H_m^{(2),j-1}}{C_{\beta_2}^{(2)} h^{|\beta_2|}}, ..., \frac{H_m^{(L),j-1}}{C_{\beta_L}^{(L)} h^{|\beta_L|}}\ ] (\frac{H_m^{(l),j-1}}{C_{\beta_l}^{(l)} h^{|\beta_l|}}). \tag{4.31}$$

Theorem (4.6)[12]. Let u^j be the resulting sequence of j-steps bi-frame shrinkage Eq.(4.29) with $u_k^0 = f(hk), k \in \mathbb{Z}^2$ and $q^{(l)}$ (resp. $\tilde{q}^{(l)}$) have vanishing moments of orders β_l (resp α_l). Then u^j is a discrete approximation of $\{u(hk, j\tau), k \in \mathbb{Z}^2, k = 1, 2, ...\}$ with $u(x,t)$ the solution of Eq.(4.30) provided that the shrinkage function s_l of Eq.(4.29) satisfy:

$$s_l(\xi_1, ..., \xi_L) = \xi_l - \frac{\tau \xi_l}{\tilde{C}_{\alpha_l}^{(l)} C_{\beta_l}^{(l)} h^{|\alpha_l|+|\beta_l|}} g_l [\frac{\xi_1}{C_{\beta_1}^{(1)} h^{|\beta_1|}}, ..., \frac{\xi_L}{C_{\beta_L}^{(L)} h^{|\beta_L|}}], \tag{4.32}$$

where $\xi_1, ..., \xi_L \in \mathbb{R},\ 1 \leq l \leq L$.

4.5 Equivalence between B-spline tight wavelet frame systems and high-order diffusion equation

Suppose that the B-spline tight frame filter bank defines as follow [12]:

$$\begin{aligned}\hat{a}(w) &= \tfrac{1}{4}e^{iw}(1+e^{-iw})^2\\ \hat{b}^{(1)}(w) &= \tfrac{\sqrt{2}}{4}(e^{iw}-e^{-iw})\\ \hat{b}^{(2)}(w) &= \tfrac{1}{4}e^{iw}(1-e^{-iw})^2,\end{aligned} \tag{4.33}$$

then the 2D linear B-spline tight frame filter bank with $w=(w_1,w_2)$ are difened as:

$$\begin{aligned}\hat{p}(w) &= \hat{a}(w_1)\hat{a}(w_2), & \hat{q}^{(1)}(w) &= \hat{b}^{(1)}(w_1)\hat{a}(w_2)\\ \hat{q}^{(2)}(w) &= \hat{a}(w_1)\hat{b}^{(1)}(w_2), & \hat{q}^{(3)}(w) &= \hat{b}^{(2)}(w_1)\hat{a}(w_2)\\ \hat{q}^{(4)}(w) &= \hat{b}^{(1)}(w_1)\hat{b}^{(1)}(w_2), & \hat{q}^{(5)}(w) &= \hat{a}(w_1)\hat{b}^{(2)}(w_2)\\ \hat{q}^{(6)}(w) &= \hat{b}^{(2)}(w_1)\hat{b}^{(1)}(w_2), & \hat{q}^{(7)}(w) &= \hat{b}^{(1)}(w_1)\hat{b}^{(2)}(w_2)\\ \hat{q}^{(8)}(w) &= \hat{b}^{(2)}(w_1)\hat{b}^{(2)}(w_2). & & \end{aligned} \tag{4.34}$$

Orders of vanishing moments of highpass filters $\{q^{(1)},...,q^{(8)}\}$ are given by:

$$\begin{aligned}&\beta_1=(1,0),\beta_2=(0,1),\beta_3=(2,0),\beta_4=(1,1),\beta_5=(0,2),\\ &\qquad\qquad\qquad\qquad\beta_6=(2,1),\beta_7=(1,2),\beta_8=(2,2)\end{aligned} \tag{4.35}$$

The corresponding nonlinear diffusion equation to the separable B-spline tight frame filter bank $\{q^{(1)},...,q^{(8)}\}$ is defined as:

$$\begin{aligned}u_t =&\frac{\partial}{\partial x_1}[g_1((\frac{\partial u}{\partial x_1})^2)\frac{\partial u}{\partial x_1}] + \frac{\partial}{\partial x_2}[g_2((\frac{\partial u}{\partial x_2})^2)\frac{\partial u}{\partial x_2}] - \frac{\partial^2}{\partial {x_1}^2}[g_3((\frac{\partial^2 u}{\partial {x_1}^2})^2)\\ &-\frac{\partial^2 u}{\partial {x_1}^2}]\frac{\partial^2}{\partial x_1 x_2}[g_4((\frac{\partial^2 u}{\partial x_1 x_2})^2)\frac{\partial^2 u}{\partial x_1 x_2}] - \frac{\partial^2}{\partial x_2^2}[g_5((\frac{\partial^2 u}{\partial x_2^2})^2)\frac{\partial^2 u}{\partial x_2^2}]\\ &+\frac{\partial^3}{\partial x_1^2 \partial x_2}[g_6((\frac{\partial^3 u}{\partial x_1^2 \partial x_2})^2)\frac{\partial^3 u}{\partial x_1^2 \partial x_2}] + \frac{\partial^3}{\partial x_1 x_2^2}[g_7((\frac{\partial^3 u}{\partial x_1 x_2^2})^2)\frac{\partial^3 u}{\partial x_1 x_2^2}]\\ &-\frac{\partial^4}{\partial x_1^2 x_2^2}[g_8((\frac{\partial^4 u}{\partial x_1^2 x_2^2})^2)\frac{\partial^4 u}{\partial x_1^2 x_2^2}]\end{aligned} \tag{4.36}$$

let f be the initial condition, $u(x,0)=f(x)$

$$\begin{aligned}C^{(1)}_{\beta_1} = C^{(2)}_{\beta_2} = -\tfrac{\sqrt{2}}{2}, C^{(3)}_{\beta_3} = -\tfrac{1}{4}, C^{(4)}_{\beta_4} = \tfrac{1}{2},\\ C^{(5)}_{\beta_5} = -\tfrac{1}{4}, C^{(6)}_{\beta_6} = C^{(7)}_{\beta_7} = \tfrac{\sqrt{2}}{8}, C^{(8)}_{\beta_8} = \tfrac{1}{16}\end{aligned}$$

From Theorem (4.4), the resulting signals of J-steps wavelet frame shrinkage Eq.(4.19) with filters(4.34), estimates the solution of Eq.(4.36) in discrete setting.Thus, the shrinking functions s_l and the diffusivity g_l have the relationship:

$$\begin{aligned}s_l(\xi) &= \xi\{1-\tfrac{2\tau}{h^2}g_l(\tfrac{2\xi^2}{h^2})\}, && for\ l=1,2,\\ s_l(\xi) &= \xi\{1-\tfrac{32\tau}{h^4}g_l(\tfrac{32\xi^2}{h^4})\}, && for\ l=6,7\\ s_4(\xi) &= \xi\{1-\tfrac{4\tau}{h^4}g_4(\tfrac{4\xi^2}{h^4})\}, && s_8(\xi) = \xi\{1-\tfrac{256\tau}{h^8}g_8(\tfrac{256\xi^2}{h^8})\}\end{aligned}$$

4.6 Equivalence between iterative soft-thresholding algorithms and diffusion equation

Finding the correspondence diffusion equation

$$u_t = \sum_{l=1}^{L} (-1)^{1+|\beta_l|} \frac{\partial^{\beta_l}}{\partial x^{\beta_l}} [g_l(\frac{\partial^{\beta_l} u}{\partial x^{\beta_l}})^2 \frac{\partial^{\beta_l} u}{\partial x^{\beta_l}}] \tag{4.37}$$

to the iterative soft-thresholding algorithm:

$$u^j = W^T T^{\varsigma}_{\alpha^{j-1}}(W u^{j-1}) \tag{4.38}$$

with choices of thresholds is the main goal in this Section. Let consider the soft thresholding:

$$T^1_\alpha(d) = \{T^1_{\alpha_{l,n}(d)}(d_{l,n}) = \frac{d_{l,n}}{|d_{l,n}|} max\{|d_{l,n}| - \alpha_{l,n}(d), 0\} : n \in \mathbb{Z}^2, \xi \in \mathbb{R}, 1 \le l \le L\}$$

The relation between the diffusivity function g_l and the shrinkage s_l is defined by:

$$s_l(\xi) = \xi\{1 - \frac{\tau}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}} g_l(\frac{\xi^2}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}})\},\ \xi \in \mathbb{R},\ 1 \le l \le L \tag{4.39}$$

Then for solving g_l from Eq.(4.39), let $s_\alpha = T^1_\alpha$, and $T^1_{\theta_l}(\xi) = \frac{\xi}{|\xi|} max\{|\xi| - \theta_l, 0\}$, then we have:

$$\frac{\tau}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}} g_l(\frac{\xi^2}{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}}) = 1 - \frac{max\{|\xi| - \theta_l, 0\}}{|\xi|} = min\{1, \frac{\theta_l}{|\xi|}\}.$$

Therefore, the diffusivity function g_l in the nonlinear diffusion equation (4.37) can be written as:

$$g_l(\xi^2) = min\{\frac{(C^{(l)}_{\beta_l})^2 h^{2|\beta_l|}}{\tau}, \frac{C^{(l)}_{\beta_l} h^{|\beta_l|} \theta_l}{\tau\,|\xi|}\}$$

If the threshold θ_l is chosen by

$$\theta_l = \frac{\tau\,|\xi|}{C^{(l)}_{\beta_l} h^{|\beta_l|}} \tilde{g}_l(\xi^2),\ with\ \tilde{g}_l : \mathbb{R} \to \mathbb{R}^+.$$

Where for some constant $C > 0$, $\frac{h^{2m}}{\tau} = C$ with h, τ small enough. Then the diffusivity function g_l is defined as:

$$g_l(\xi^2) = min\{\frac{(C^{(l)}_{\beta_l})^2 C}{h^{2(m-|\beta_l|)}}, \tilde{g}_l(\xi^2)\} = \begin{cases} min\{(C^{(l)}_{\beta_l})^2,\ \tilde{g}_l(\xi^2)\} & for\ \ |\beta_l| = m \\ \tilde{g}_l(\xi^2) & for\ 1 \le |\beta_l| < m \end{cases} \tag{4.40}$$

where m is the largest number among $|\beta_1|, ..., |\beta_L|$ whenever h is small enough.

Theorem (4.7)[12]. Suppose that the tight frame filter bank is $\{p, q^{(1)}, ..., q^{(L)}\}$, and $q^{(l)}$ have vanishing moment β_l with $C^{(l)}_{\beta_l}$ and $m = max\{|\beta_l| : 1 \leq l \leq L\}$. Let the threshold $\alpha(d)$ of frame coefficients d is defined as:

$$\alpha(d) = \{\alpha_{l,n}(d_{l,n}) = \frac{\tau\,|d_{l,n}|}{C^{(l)}_{\beta_l} h^{|\beta_l|}} \tilde{g}_l((d_{l,n})^2) : 1 \leq l \leq L, n \in \mathbb{Z}^2\} \tag{4.41}$$

with $\tilde{g}_l : \mathbb{R} \to \mathbb{R}^+$ being some smooth function, and let $\frac{h^{2m}}{\tau} = C$ for some constant $C > 0$ with h, τ sufficiently small. Suppose that u^j is generated from the J-steps soft-thresholding algorithm Eq.(4.38) with $\varsigma = 1$. Then, u^j is a discrete approximation of $\{u(hk, j\tau), k \in \mathbb{Z}^2, j = 1, 2, ...\}$ with $u(x,t)$ being the solution of the diffusion equation (4.37) with the diffusivity functions given by (4.40).

Chapter 5

Equivalence between multiwavelet shrinkage and nonlinear diffusion

Presenting diffusion in term of shrinkage functions with advance achievement is one among many advantages of the equivalent between multiwavelet shrinkage and diffusion filtering. The Equivalence between both approach helps to design multiwavelet-inspired diffusivity functions that mixes benefits from both approaches.

In this chapter, we present the corresponding between Haar, CL(2), DGHM multiwavelet shrinkage and second-order nonlinear diffusion equation in Sections 5.1, 5.2, 5.3 respectively.

In Section 5.4, we show that multiwavelet shrinkage in general case is associated with high-order nonlinear diffusion equation. The experimental results of one-dimensional multiwavelet shrinkage have been discussed in Section 5.5.

5.1 Equivalence between Haar multiwavelet shrinkage and second-order nonlinear diffusion equation

5.1.1 Haar multiwavelet shrinkage

If $\{P, Q\}$ are FIR multiwavelet filter bank satisfying:

$$P(w)^* P(w) + Q(w)^* Q(w) = I_2$$

and $\underline{c}_k^0 = \binom{c_{2k}}{c_{2k+1}}$ is a signal, then undecimated multiwavelet transform based denoising consist of the undecimated discrete multiwavelet transform algorithm:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \sum_{k\in\mathbb{Z}} P_k \begin{pmatrix} c_{2(k+n)} \\ c_{2(k+n)+1} \end{pmatrix}$$
$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \sum_{k\in\mathbb{Z}} Q_k \begin{pmatrix} c_{2(k+n)} \\ c_{2(k+n)+1} \end{pmatrix}$$

and denoising algorithm:

$$\underline{u}_k = \begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \sum_{n\in\mathbb{Z}} P_n^T \begin{pmatrix} L_{1,k-n} \\ L_{2,k-n} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11}S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21}S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12}S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22}S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \tag{5.1}$$

By using Haar multiscaling coefficients [47]:

$$P_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{pmatrix} \quad , \quad P_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

and Haar multiwavelet coefficients:

$$Q_0 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

Then the undecimated discrete multiwavelet transform algorithm is defined by:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} c_{2n} + c_{2n+1} \\ c_{2n+2} + c_{2n+3} \end{pmatrix}$$

$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} c_{2n} - c_{2n+1} \\ c_{2n+2} - c_{2n+3} \end{pmatrix},$$

and the denoising algorithm is given by:

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} c_{2k} + c_{2k+1} \\ c_{2k} + c_{2k+1} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} c_{2k} + c_{2k+1} \\ c_{2k} + c_{2k+1} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} S_{\theta_{11}}[\frac{1}{2}(c_{2k} - c_{2k+1})] \\ -S_{\theta_{21}}[\frac{1}{2}(c_{2k} - c_{2k+1})] \end{pmatrix}$$
$$+ \frac{1}{2}\begin{pmatrix} S_{\theta_{21}}[\frac{1}{2}(c_{2k} - c_{2k+1})] \\ -S_{\theta_{22}}[\frac{1}{2}(c_{2k} - c_{2k+1})] \end{pmatrix} \tag{5.2}$$

If the shrinkage functions are suitable, then $\underline{u}_k$ is the denoised signal of one-step multiwavelet denoising operation of the initial signal $\underline{c}_k^0$ in addition to noise. If there is no shrinking apply then:

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} c_{2k} + c_{2k+1} \\ c_{2k} + c_{2k+1} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} c_{2k} - c_{2k+1} \\ -(c_{2k} - c_{2k+1}) \end{pmatrix} + \frac{1}{4}\begin{pmatrix} c_{2k} - c_{2k+1} \\ -(c_{2k} - c_{2k+1}) \end{pmatrix}$$

i.e

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} \tag{5.3}$$

5.1.2 Haar multiwavelet shrinkage for nonlinear diffusion

The second-order nonlinear diffusion equation for 1-D signal f with a noise, where

$$u(x,0) = f(x) \ \textit{acts as inital condition}$$

is given by:

$$u_t = \frac{\partial}{\partial x}\{g(u_x^2)u_x\} \tag{5.4}$$

The approximation of $\frac{\partial}{\partial x}u(x,t)$ at $\begin{pmatrix}(2kh, j\tau)\\([2k+1]h, j\tau)\end{pmatrix}$ by using highpass filters (Haar multiwavelet coefficients) is given by:

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix}u(2kh, j\tau)\\u([2k+1]h, j\tau)\end{pmatrix} &\approx \frac{-2}{h}\sum Q_n\begin{pmatrix}u(2(k+n)h, j\tau)\\u([2(k+n)+1]h, j\tau)\end{pmatrix}\\
&\approx \frac{-2}{h}\sum Q_n\begin{pmatrix}u^j_{2(k+n)}\\u^j_{2(k+n)+1}\end{pmatrix}\\
&\approx \frac{-1}{h}\begin{pmatrix}1 & -1\\0 & 0\end{pmatrix}\begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix} + \frac{-1}{h}\begin{pmatrix}0 & 0\\1 & -1\end{pmatrix}\begin{pmatrix}u^j_{2k+2}\\u^j_{2k+3}\end{pmatrix}\\
&\approx \frac{-1}{h}\begin{pmatrix}u^j_{2k} - u^j_{2k+1}\\u^j_{2k+2} - u^j_{2k+3}\end{pmatrix} \qquad (5.5)
\end{aligned}$$

and the approximating partial derivatives of $\frac{\partial}{\partial x}G(x,t)$ where $G(x,t) := g[(\frac{\partial u}{\partial x})^2]\frac{\partial u}{\partial x}$ at $\begin{pmatrix}(2kh, j\tau)\\([2k+1]h, j\tau)\end{pmatrix}$ is defined as:

$$\frac{\partial}{\partial x}\begin{pmatrix}G(2kh, j\tau)\\G([2k+1]h, j\tau)\end{pmatrix} \approx \frac{2}{h}\sum Q^T_m\begin{pmatrix}G(2(k-m)h, j\tau)\\G([2(k-m)+1]h, j\tau)\end{pmatrix} \qquad (5.6)$$

Then the second-order nonlinear diffusion equation (5.4) can be discretized as:

$$\begin{pmatrix}u^{j+1}_{2k}\\u^{j+1}_{2k+1}\end{pmatrix} = \begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix} - \frac{2\tau}{h^2}\sum_m Q^T_m\begin{pmatrix}g[\frac{(u^j_{2(k-m)}-u^j_{2(k-m)+1})^2}{h^2}](u^j_{2(k-m)} - u^j_{2(k-m)+1})\\g[\frac{(u^j_{2(k-m)+2}-u^j_{2(k-m)+3})^2}{h^2}](u^j_{2(k-m)+2} - u^j_{2(k-m)+3})\end{pmatrix}$$

By using Haar multiwavelet coefficients, we get:

$$\begin{aligned}
\begin{pmatrix}u^{j+1}_{2k}\\u^{j+1}_{2k+1}\end{pmatrix} = \begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix} &- \frac{\tau}{h^2}\begin{pmatrix}1 & 0\\-1 & 0\end{pmatrix}\begin{pmatrix}g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1})\\g[\frac{(u^j_{2k+2}-u^j_{2k+3})^2}{h^2}](u^j_{2k+2} - u^j_{2k+3})\end{pmatrix}\\
&- \frac{\tau}{h^2}\begin{pmatrix}0 & 1\\0 & -1\end{pmatrix}\begin{pmatrix}g[\frac{(u^j_{2k-2}-u^j_{2k-1})^2}{h^2}](u^j_{2k-2)} - u^j_{2k-1})\\g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1})\end{pmatrix}
\end{aligned}$$

Then

$$\begin{aligned}
\begin{pmatrix}u^{j+1}_{2k}\\u^{j+1}_{2k+1}\end{pmatrix} = \begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix} &- \frac{\tau}{h^2}\begin{pmatrix}g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1})\\-g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1})\end{pmatrix}\\
&- \frac{\tau}{h^2}\begin{pmatrix}g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1}))\\-g[\frac{(u^j_{2k}-u^j_{2k+1})^2}{h^2}](u^j_{2k} - u^j_{2k+1})\end{pmatrix} \qquad (5.7)
\end{aligned}$$

When $j=0$, the signal after 1-step diffusing with $\binom{u^0_{2k}}{u^0_{2k+1}} = \binom{c_{2k}}{c_{2k+1}}$ can be written as:

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} + \begin{pmatrix} \frac{-\tau}{h^2} g[\frac{(c_{2k}-c_{2k+1})^2}{h^2}](c_{2k} - c_{2k+1}) \\ \frac{\tau}{h^2} g[\frac{(c_{2k}-c_{2k+1})^2}{h^2}](c_{2k} - c_{2k+1}) \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{-\tau}{h^2} g[\frac{(c_{2k}-c_{2k+1})^2}{h^2}](c_{2k} - c_{2k+1}) \\ \frac{\tau}{h^2} g[\frac{(c_{2k}-c_{2k+1})^2}{h^2}](c_{2k} - c_{2k+1}) \end{pmatrix} \tag{5.8}$$

Write $\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix}$ as:

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} c_{2k} + c_{2k+1} \\ c_{2k} + c_{2k+1} \end{pmatrix} + \frac{1}{4}\begin{pmatrix} c_{2k} - c_{2k+1} \\ -(c_{2k} - c_{2k+1}) \end{pmatrix} + \frac{1}{4}\begin{pmatrix} c_{2k} - c_{2k+1} \\ -(c_{2k} - c_{2k+1}) \end{pmatrix}$$

Then the second-order nonlinear diffusion equation after one-step diffusion Eq(5.8) is defined as:

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} c_{2k} + c_{2k+1} \\ c_{2k} + c_{2k+1} \end{pmatrix} + \begin{pmatrix} \{\frac{1}{4} - \frac{\tau}{h^2} g[(\frac{c_{2k}-c_{2k+1}}{h})^2]\}(c_{2k} - c_{2k+1}) \\ \{\frac{\tau}{h^2} g[(\frac{c_{2k}-c_{2k+1}}{h})^2] - \frac{1}{4}\}(c_{2k} - c_{2k+1}) \end{pmatrix}$$
$$+ \begin{pmatrix} \{\frac{1}{4} - \frac{\tau}{h^2} g[(\frac{c_{2k}-c_{2k+1}}{h})^2]\}(c_{2k} - c_{2k+1}) \\ \{\frac{\tau}{h^2} g[(\frac{c_{2k}-c_{2k+1}}{h})^2] - \frac{1}{4}\}(c_{2k} - c_{2k+1}) \end{pmatrix} \tag{5.9}$$

Theorem (5.1): Let $\binom{u_{1,k}}{u_{2,k}}$ in Eq(5.2) be the resulting signal after one-step Haar multiwavelet shrinking with $\underline{c}^0_k = \binom{c_{2k}}{c_{2k+1}} = f(kh), k \in \mathbb{Z}$ as the inital input and $\binom{u^1_{2k}}{u^1_{2k+1}}$ in Eq.(5.9) be the signal of one-step diffusing with initial input $\underline{u}^0_k = \binom{u^0_{2k}}{u^0_{2k+1}} = f(kh)$ If

$$S_{\theta_{11}}(x) = S_{\theta_{21}}(x) = x(1 - \frac{4\tau}{h^2} g\{\frac{4x^2}{h^2}\})$$
$$S_{\theta_{12}}(x) = S_{\theta_{22}}(x) = x(1 - \frac{4\tau}{h^2} g\{\frac{4x^2}{h^2}\}) \tag{5.10}$$

Then

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix}$$

Corollary (5.1): With diffusivity function $g(x)$ and shrinkage functions satisfying Eq.(5.10), iterated multiwavelet shrinking with Haar multifilter banks and nonlinear diffusing with Eq.(5.4) result in the same signal.

5.2 Equivalence between Chui-Lian CL(2) multiwavelet shrinkage and second-order nonlinear diffusion equation

5.2.1 Chui-Lian CL(2) multiwavelet shrinkage

For multiwavelet transform, $\hat{\Phi}(0)$ is a normalized right 1-eigenvector of $P(0)$, and $\hat{\Phi}(0)$ needs not have to be $(1, . . . , 1)^T$. Therefore, one method to deal with this problem is using another pair of multifilter bank $\{P, Q\}$ constructed from $\{H, G\}$. Thus, the scaling functions and multiwavelet with $\hat{\Phi}(0) = (1, . . . , 1)^T$ are generated by using new multifilter banks $\{P, Q\}$.
Let

$$H_{-1} = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4} \end{pmatrix}, \quad H_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad H_1 = \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{7}}{4} & -\frac{\sqrt{7}}{4} \end{pmatrix}$$

$$G_{-1} = \frac{1}{2}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}, \quad G_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \frac{\sqrt{7}}{2} \end{pmatrix}, \quad G_1 = \frac{1}{2}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

be the Chui-Lian CL(2) multifilter bank, and $U = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is an orthogonal matrix then

$$P_{-1} = UH_{-1}U^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 1+\frac{\sqrt{7}}{2} \\ 0 & 1-\frac{\sqrt{7}}{2} \end{pmatrix}, P_0 = UH_0U^{-1} = \frac{1}{4}\begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$P_1 = \frac{1}{4}\begin{pmatrix} 1-\frac{\sqrt{7}}{2} & 0 \\ 1+\frac{\sqrt{7}}{2} & 0 \end{pmatrix}, \; Q_{-1} = \frac{1}{4}\begin{pmatrix} 0 & -\frac{3}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$$

$$Q_0 = \frac{1}{4}\begin{pmatrix} 1+\frac{\sqrt{7}}{2} & 1-\frac{\sqrt{7}}{2} \\ 1-\frac{\sqrt{7}}{2} & 1+\frac{\sqrt{7}}{2} \end{pmatrix}, \; Q_1 = \frac{1}{4}\begin{pmatrix} -\frac{1}{2} & 0 \\ -\frac{3}{2} & 0 \end{pmatrix}$$

are orthogonal and generates the scaling function and multiwavelet with
$\hat{\Phi}(0) = (1 \quad 1)^T$.
Suppose a signal is given by $\underline{c}_k^0 = \binom{c_{2k}}{c_{2k+1}}$. Thus, the shift-invariant multiwavelet decomposition algorithms with Chui-Lian CL(2) multiwavelet coefficients is given by:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & 2+\sqrt{7} \\ 0 & 2-\sqrt{7} \end{pmatrix}\begin{pmatrix} c_{2n-2} \\ c_{2n-1} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} c_{2n} \\ c_{2n+1} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 2-\sqrt{7} & 0 \\ 2+\sqrt{7} & 0 \end{pmatrix}\begin{pmatrix} c_{2n+2} \\ c_{2n+3} \end{pmatrix}$$
$$= \frac{1}{8}\begin{pmatrix} (2+\sqrt{7})c_{2n-1} + 3c_{2n} + c_{2n+1} + (2-\sqrt{7})c_{2n+2} \\ (2-\sqrt{7})c_{2n-1} + c_{2n} + 3c_{2n+1} + (2+\sqrt{7})c_{2n+2} \end{pmatrix}$$

$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 0 & -3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c_{2n-2} \\ c_{2n-1} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 2+\sqrt{7} & 2-\sqrt{7} \\ 2-\sqrt{7} & 2+\sqrt{7} \end{pmatrix}\begin{pmatrix} c_{2n} \\ c_{2n+1} \end{pmatrix} + \frac{1}{8}\begin{pmatrix} -1 & 0 \\ -3 & 0 \end{pmatrix}\begin{pmatrix} c_{2n+2} \\ c_{2n+3} \end{pmatrix}$$
$$= \frac{1}{8}\begin{pmatrix} -3c_{2n-1} + (2+\sqrt{7})c_{2n} + (2-\sqrt{7})c_{2n+1} - c_{2n+2} \\ -c_{2n-1} + (2-\sqrt{7})c_{2n} + (2+\sqrt{7})c_{2n+1} - 3c_{2n+2} \end{pmatrix}$$

and the shrunk data is defined as:

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 3L_{1,k} + L_{2,k} + (2-\sqrt{7})L_{1,k-1} + (2+\sqrt{7})L_{2,k-1} \\ (2+\sqrt{7})L_{1,k+1} + (2-\sqrt{7})L_{2,k+1} + L_{1,k} + 3L_{2,k} \end{pmatrix}$$
$$+ \frac{1}{8}\begin{pmatrix} (2+\sqrt{7})S_{\theta_{11}}H_{1,k} + (2-\sqrt{7})S_{\theta_{21}}H_{2,k} - S_{\theta_{11}}H_{1,k-1} - 3S_{\theta_{21}}H_{2,k-1} \\ -3S_{\theta_{12}}H_{1,k+1} - S_{\theta_{22}}H_{2,k+1} + (2-\sqrt{7})S_{\theta_{12}}H_{1,k} + (2+\sqrt{7})S_{\theta_{22}}H_{2,k} \end{pmatrix}$$

$$\begin{aligned}
\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} &= \frac{1}{64}\begin{pmatrix} 3[(2+\sqrt{7})c_{2k-1} + 3c_{2k} + c_{2k+1} + (2-\sqrt{7})c_{2k+2}] \\ (2+\sqrt{7})[(2+\sqrt{7})c_{2k+1} + 3c_{2k+2} + c_{2k+3} + (2-\sqrt{7})c_{2k+4}] \end{pmatrix} \\
&+ \frac{1}{64}\begin{pmatrix} [(2-\sqrt{7})c_{2k-1} + c_{2k} + 3c_{2k+1} + (2+\sqrt{7})c_{2k+2}] \\ (2-\sqrt{7})[(2-\sqrt{7})c_{2k+1} + c_{2k+2} + 3c_{2k+3} + (2+\sqrt{7})c_{2k+4}] \end{pmatrix} \\
&+ \frac{1}{64}\begin{pmatrix} (2-\sqrt{7})[(2+\sqrt{7})c_{2k-3} + 3c_{2k-2} + c_{2k-1} + (2-\sqrt{7})c_{2k}] \\ [(2+\sqrt{7})c_{2k-1} + 3c_{2k} + c_{2k+1} + (2-\sqrt{7})c_{2k+2}] \end{pmatrix} \\
&+ \frac{1}{64}\begin{pmatrix} (2+\sqrt{7})[(2-\sqrt{7})c_{2k-3} + c_{2k-2} + 3c_{2k-1} + (2+\sqrt{7})c_{2k}] \\ 3[(2-\sqrt{7})c_{2k-1} + c_{2k} + 3c_{2k+1} + (2+\sqrt{7})c_{2k+2}] \end{pmatrix} \\
&+ \frac{1}{8}\begin{pmatrix} (2+\sqrt{7})S_{\theta_{11}}[\frac{1}{8}\{-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}\}] \\ (2-\sqrt{7})S_{\theta_{12}}[\frac{1}{8}\{-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}\} \end{pmatrix} \\
&+ \frac{1}{8}\begin{pmatrix} (2-\sqrt{7})S_{\theta_{21}}[\frac{1}{8}\{-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}\}] \\ (2+\sqrt{7})S_{\theta_{22}}[\frac{1}{8}\{-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}\}] \end{pmatrix} \\
&+ \frac{1}{8}\begin{pmatrix} -S_{\theta_{11}}[\frac{1}{8}\{-3c_{2k-3} + (2+\sqrt{7})c_{2k-2} + (2-\sqrt{7})c_{2k-1} - c_{2k}\}] \\ -3S_{\theta_{12}}\quad[\frac{1}{8}\{-3c_{2k+1} + (2+\sqrt{7})c_{2k+2} + (2-\sqrt{7})c_{2k+3} - c_{2k+4}\}] \end{pmatrix} \\
&+ \frac{1}{8}\begin{pmatrix} -3S_{\theta_{21}}[\frac{1}{8}\{-c_{2k-3} + (2-\sqrt{7})c_{2k-2} + (2+\sqrt{7})c_{2k-1} - 3c_{2k}\}] \\ -S_{\theta_{22}}\quad[\frac{1}{8}\{-c_{2k+1} + (2-\sqrt{7})c_{2k+2} + (2+\sqrt{7})c_{2k+3} - 3c_{2k+4}\}] \end{pmatrix}
\end{aligned} \tag{5.11}$$

If there is no shrinkage apply, then

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{64} \begin{pmatrix} (10+\sqrt{7})c_{2k-1} + (8-2\sqrt{7})c_{2k-2} - 6c_{2k-3} \\ +32c_{2k} + 6c_{2k+1} + (8-2\sqrt{7})c_{2k+2} \\ \\ (8-2\sqrt{7})c_{2k-1} + 6c_{2k} + 32c_{2k+1} \\ +(16+4\sqrt{7})c_{2k+2} + (8-2\sqrt{7})c_{2k+3} - 6c_{2k+4} \end{pmatrix}$$

$$+ \frac{1}{64} \begin{pmatrix} (2+\sqrt{7})[-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}] \\ (2-\sqrt{7})[-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2} \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} (2-\sqrt{7})[-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}] \\ (2+\sqrt{7})[-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}] \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} -[-3c_{2k-3} + (2+\sqrt{7})c_{2k-2} + (2-\sqrt{7})c_{2k-1} - c_{2k}] \\ -3[-3c_{2k+1} + (2+\sqrt{7})c_{2k+2} + (2-\sqrt{7})c_{2k+3} - c_{2k+4}] \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} 3[-c_{2k-3} + (2-\sqrt{7})c_{2k-2} + (2+\sqrt{7})c_{2k-1} - 3c_{2k}] \\ -[-c_{2k+1} + (2-\sqrt{7})c_{2k+2} + (2+\sqrt{7})c_{2k+3} - 3c_{2k+4}] \end{pmatrix}$$

i.e

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix}$$

5.2.2 Nonlinear diffusion in terms of multiwavelet shrinkage

The second-order nonlinear diffusion formula of $u(x,t)$ with noise is given by:

$$u_t = \frac{\partial}{\partial x}\{g(u_x^2)u_x\}, \tag{5.12}$$

with original condition $u(x,0) = f(x)$. The approximation of $\frac{\partial}{\partial x}u(x,t)$ at $(2kh, j\tau)$ is:

$$u_x \approx \frac{-3u^j_{2k+1} + (2+\sqrt{7})u^j_{2k+2} + (2-\sqrt{7})u^j_{2k+3} - u^j_{2k+4}}{(3-\sqrt{7})h}$$

and the approximation of $\frac{\partial}{\partial x}u(x,t)$ at $([2k+1]h, j\tau)$ is:

$$u_x \approx -\left(\frac{-u^j_{2k+1} + (2-\sqrt{7})u^j_{2k+2} + (2+\sqrt{7})u^j_{2k+3} - 3u^j_{2k+4}}{(3-\sqrt{7})h}\right)$$

Then the value of $\frac{\partial}{\partial x}u(x,t)$ at $\begin{pmatrix}(2kh, j\tau)\\([2k+1]h, j\tau)\end{pmatrix}$ by using highpass filters can be written as:

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix}u(2kh, j\tau)\\u([2k+1]h, j\tau)\end{pmatrix} &\approx \begin{pmatrix}\frac{8}{(3-\sqrt{7})h} & 0\\ 0 & -\frac{8}{(3-\sqrt{7})h}\end{pmatrix}\sum Q_n\begin{pmatrix}u(2(k+n)h, j\tau)\\u([2(k+n)+1]h, j\tau)\end{pmatrix}\\
&= \begin{pmatrix}\frac{8}{(3-\sqrt{7})h} & 0\\ 0 & -\frac{8}{(3-\sqrt{7})h}\end{pmatrix}\sum Q_n\begin{pmatrix}u^j_{2(k+n)}\\u^j_{2(k+n)+1}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{(3-\sqrt{7})h} & 0\\ 0 & -\frac{1}{(3-\sqrt{7})h}\end{pmatrix}[\begin{pmatrix}0 & -3\\0 & -1\end{pmatrix}\begin{pmatrix}u^j_{2k}\\u_{2k+1}\end{pmatrix}\\
&+\begin{pmatrix}2+\sqrt{7} & 2-\sqrt{7}\\2-\sqrt{7} & 2+\sqrt{7}\end{pmatrix}\begin{pmatrix}u^j_{2k+2}\\u^j_{2k+3}\end{pmatrix}+\begin{pmatrix}-1 & 0\\-3 & 0\end{pmatrix}\begin{pmatrix}u^j_{2k+4}\\u^j_{2k+5}\end{pmatrix}]\\
&= \begin{pmatrix}\frac{-3u^j_{2k+1}+(2+\sqrt{7})u^j_{2k+2}+(2-\sqrt{7})u^j_{2k+3}-u^j_{2k+4}}{(3-\sqrt{7})h}\\ -(\frac{-u^j_{2k+1}+(2-\sqrt{7})u^j_{2k+2}+(2+\sqrt{7})u^j_{2k+3}-3u^j_{2k+4}}{(3-\sqrt{7})h})\end{pmatrix}
\end{aligned}\tag{5.13}$$

and the approximating partial derivatives of $G(x,t) = g((u_x)^2)u_x$ at $\begin{pmatrix}(2kh, j\tau)\\([2k+1]h, j\tau)\end{pmatrix}$ is defined as:

$$\frac{\partial}{\partial x}\begin{pmatrix}G(2kh, j\tau)\\G([2k+1]h, j\tau)\end{pmatrix} \approx \begin{pmatrix}\frac{-8}{(3-\sqrt{7})h} & 0\\ 0 & \frac{-8}{(-3+\sqrt{7})h}\end{pmatrix}\sum Q^T_m\begin{pmatrix}G(2(k-m)h, j\tau)\\G([2(k-m)+1]h, j\tau)\end{pmatrix}$$

Then the second-order nonlinear diffusion equation can be discretized as:

$$\begin{aligned}
\begin{pmatrix}u^{j+1}_{2k}\\u^{j+1}_{2k+1}\end{pmatrix} &= \begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix}+\begin{pmatrix}\frac{-8\tau}{(3-\sqrt{7})h} & 0\\ 0 & \frac{-8\tau}{(-3+\sqrt{7})h}\end{pmatrix}\\
&\sum Q^T_m\begin{pmatrix}\frac{g}{(3-\sqrt{7})h}[\frac{(-3u^j_{2(k-m)+1}+(2+\sqrt{7})u^j_{2(k-m)+2}+(2-\sqrt{7})u^j_{2(k-m)+3}-u^j_{2(k-m)+4})^2}{(3-\sqrt{7})^2h^2}]\\ \times(-3u^j_{2(k-m)+1}+(2+\sqrt{7})u^j_{2(k-m)+2}+(2-\sqrt{7})u^j_{2(k-m)+3}-u^j_{2(k-m)+4})\\ \\ \frac{g}{(-3+\sqrt{7})h}[\frac{(-u^j_{2(k-m)+1}+(2-\sqrt{7})u^j_{2(k-m)+2}+(2+\sqrt{7})u^j_{2(k-m)+3}-3u^j_{2(k-m)+4})^2}{(-3+\sqrt{7})^2h^2}]\\ \times(-u^j_{2(k-m)+1}+(2-\sqrt{7})u^j_{2(k-m)+2}+(2+\sqrt{7})u^j_{2(k-m)+3}-3u^j_{2(k-m)+4})\end{pmatrix}
\end{aligned}$$

By applying Chui-Lian CL(2) coefficients :

$$
\begin{aligned}
\begin{pmatrix} u_{2k}^{j+1} \\ u_{2k+1}^{j+1} \end{pmatrix} &= \begin{pmatrix} u_{2k}^{j} \\ u_{2k+1}^{j} \end{pmatrix} + \begin{pmatrix} \frac{-\tau}{(3-\sqrt{7})h} & 0 \\ 0 & \frac{-\tau}{(-3+\sqrt{7})h} \end{pmatrix} \Bigg\{ \begin{pmatrix} 0 & 0 \\ -3 & -1 \end{pmatrix} \\
&\times \begin{pmatrix} \frac{g}{(3-\sqrt{7})h}\left[\frac{(-3u_{2k+1}^j+(2+\sqrt{7})u_{2k+2}^j+(2-\sqrt{7})u_{2k+3}^j-u_{2k+4}^j)^2}{(3-\sqrt{7})^2h^2}\right] \\ \times(-3u_{2k+1}^j+(2+\sqrt{7})u_{2k+2}^j+(2-\sqrt{7})u_{2k+3}^j-u_{2k+4}^j) \\ \\ \frac{g}{(-3+\sqrt{7})h}\left[\frac{(-u_{2k+1}^j+(2-\sqrt{7})u_{2k+2}^j+(2+\sqrt{7})u_{2k+3}^j-3u_{2k+4}^j)^2}{(-3+\sqrt{7})^2h^2}\right] \\ \times(-u_{2k+1}^j+(2-\sqrt{7})u_{2k+2}^j+(2+\sqrt{7})u_{2k+3}^j-3u_{2k+4}^j) \end{pmatrix} \\
&+ \begin{pmatrix} 2+\sqrt{7} & 2-\sqrt{7} \\ 2-\sqrt{7} & 2+\sqrt{7} \end{pmatrix} \begin{pmatrix} \frac{g}{(3-\sqrt{7})h}\left[\frac{(-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j)^2}{(3-\sqrt{7})^2h^2}\right] \\ \times(-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j) \\ \\ \frac{g}{(-3+\sqrt{7})h}\left[\frac{(-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j)^2}{(-3+\sqrt{7})^2h^2}\right] \\ \times(-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j) \end{pmatrix} \\
&+ \begin{pmatrix} -1 & -3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{g}{(3-\sqrt{7})h}\left[\frac{(-3u_{2k-3}^j+(2+\sqrt{7})u_{2k-2}^j+(2-\sqrt{7})u_{2k-1}^j-u_{2k}^j)^2}{(3-\sqrt{7})^2h^2}\right] \\ \times(-3u_{2k-3}^j+(2+\sqrt{7})u_{2k-2}^j+(2-\sqrt{7})u_{2k-1}^j-u_{2k}^j) \\ \\ \frac{g}{(-3+\sqrt{7})h}\left[\frac{(-u_{2k-3}^j+(2-\sqrt{7})u_{2k-2}^j+(2+\sqrt{7})u_{2k-1}^j-3u_{2k}^j)^2}{(-3+\sqrt{7})^2h^2}\right] \\ \times(-u_{2k-3}^j+(2-\sqrt{7})u_{2k-2}^j+(2+\sqrt{7})u_{2k-1}^j-3u_{2k}^j) \end{pmatrix} \Bigg\}
\end{aligned}
$$

Then the second-order nonlinear diffusion can be written as:

$$\begin{pmatrix} u_{2k}^{j+1} \\ u_{2k+1}^{j+1} \end{pmatrix} = \begin{pmatrix} u_{2k}^{j} \\ u_{2k+1}^{j} \end{pmatrix} + \begin{pmatrix} \frac{-\tau}{(3-\sqrt{7})h} & 0 \\ 0 & \frac{-\tau}{(-3+\sqrt{7})h} \end{pmatrix}$$

$$\times \left\{ \begin{pmatrix} \frac{(2+\sqrt{7})g}{(3-\sqrt{7})h}[\frac{(-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j)^2}{(3-\sqrt{7})^2h^2}] \\ (-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j) \\ \\ \frac{(2-\sqrt{7})g}{(3-\sqrt{7})h}[\frac{(-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j)^2}{(3-\sqrt{7})^2h^2}] \\ (-3u_{2k-1}^j+(2+\sqrt{7})u_{2k}^j+(2-\sqrt{7})u_{2k+1}^j-u_{2k+2}^j) \end{pmatrix} \right.$$

$$+ \begin{pmatrix} \frac{(2-\sqrt{7})g}{(-3+\sqrt{7})h}[\frac{(-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j)^2}{(-3+\sqrt{7})^2h^2}] \\ (-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j) \\ \\ \frac{(2+\sqrt{7})g}{(-3+\sqrt{7})h}[\frac{(-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j)^2}{(-3+\sqrt{7})^2h^2}] \\ (-u_{2k-1}^j+(2-\sqrt{7})u_{2k}^j+(2+\sqrt{7})u_{2k+1}^j-3u_{2k+2}^j) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-g}{(3-\sqrt{7})h}[\frac{(-3u_{2k-3}^j+(2+\sqrt{7})u_{2k-2}^j+(2-\sqrt{7})u_{2k-1}^j-u_{2k}^j)^2}{(3-\sqrt{7})^2h^2}] \\ (-3u_{2k-3}^j+(2+\sqrt{7})u_{2k-2}^j+(2-\sqrt{7})u_{2k-1}^j-u_{2k}^j) \\ \\ \frac{-3g}{(3-\sqrt{7})h}[\frac{(-3u_{2k+1}^j+(2+\sqrt{7})u_{2k+2}^j+(2-\sqrt{7})u_{2k+3}^j-u_{2k+4}^j)^2}{(3-\sqrt{7})^2h^2}] \\ (-3u_{2k+1}^j+(2+\sqrt{7})u_{2k+2}^j+(2-\sqrt{7})u_{2k+3}^j-u_{2k+4}^j) \end{pmatrix}$$

$$\left. + \begin{pmatrix} \frac{-3g}{(-3+\sqrt{7})h}[\frac{(-u_{2k-3}^j+(2-\sqrt{7})u_{2k-2}^j+(2+\sqrt{7})u_{2k-1}^j-3u_{2k}^j)^2}{(-3+\sqrt{7})^2h^2}] \\ (-u_{2k-3}^j+(2-\sqrt{7})u_{2k-2}^j+(2+\sqrt{7})u_{2k-1}^j-3u_{2k}^j) \\ \\ \frac{-g}{(-3+\sqrt{7})h}[\frac{(-u_{2k+1}^j+(2-\sqrt{7})u_{2k+2}^j+(2+\sqrt{7})u_{2k+3}^j-3u_{2k+4}^j)^2}{(-3+\sqrt{7})^2h^2}] \\ (-u_{2k+1}^j+(2-\sqrt{7})u_{2k+2}^j+(2+\sqrt{7})u_{2k+3}^j-3u_{2k+4}^j) \end{pmatrix} \right\}.$$

When $j = 0$ with $\binom{u_{2k}^0}{u_{2k+1}^0} = \binom{c_{2k}}{c_{2k+1}}$, we have

$$\begin{pmatrix} u_{2k}^1 \\ u_{2k+1}^1 \end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} + \begin{pmatrix} \frac{-(2+\sqrt{7})\tau g}{(3-\sqrt{7})^2h^2}\left[\frac{(-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2})^2}{(3-\sqrt{7})^2h^2}\right] \\ (-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2}) \\ \\ \frac{-(2-\sqrt{7})\tau g}{(-3+\sqrt{7})(3-\sqrt{7})h^2}\left[\frac{(-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2})^2}{(3-\sqrt{7})^2h^2}\right] \\ (-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2}) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-(2-\sqrt{7})\tau g}{(3-\sqrt{7})(-3+\sqrt{7})h^2}\left[\frac{(-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2})^2}{(-3+\sqrt{7})^2h^2}\right] \\ (-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2}) \\ \\ \frac{-(2+\sqrt{7})\tau g}{(-3+\sqrt{7})^2h^2}\left[\frac{(-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2})^2}{(-3+\sqrt{7})^2h^2}\right] \\ (-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2}) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{\tau g}{(3-\sqrt{7})^2h^2}\left[\frac{(-3c_{2k-3}+(2+\sqrt{7})c_{2k-2}+(2-\sqrt{7})c_{2k-1}-c_{2k})^2}{(3-\sqrt{7})^2h^2}\right] \\ (-3c_{2k-3}+(2+\sqrt{7})c_{2k-2}+(2-\sqrt{7})c_{2k-1}-c_{2k}) \\ \\ \frac{3\tau g}{(-3+\sqrt{7})(3-\sqrt{7})h^2}\left[\frac{(-3c_{2k+1}+(2+\sqrt{7})c_{2k+2}+(2-\sqrt{7})c_{2k+3}-c_{2k+4})^2}{(3-\sqrt{7})^2h^2}\right] \\ (-3c_{2k+1}+(2+\sqrt{7})c_{2k+2}+(2-\sqrt{7})c_{2k+3}-c_{2k+4}) \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{3\tau g}{(3-\sqrt{7})(-3+\sqrt{7})h^2}\left[\frac{(-c_{2k-3}+(2-\sqrt{7})c_{2k-2}+(2+\sqrt{7})c_{2k-1}-3c_{2k})^2}{(-3+\sqrt{7})^2h^2}\right] \\ (-c_{2k-3}+(2-\sqrt{7})c_{2k-2}+(2+\sqrt{7})c_{2k-1}-3c_{2k}) \\ \\ \frac{\tau g}{(-3+\sqrt{7})^2h^2}\left[\frac{(-c_{2k+1}+(2-\sqrt{7})c_{2k+2}+(2+\sqrt{7})c_{2k+3}-3c_{2k+4})^2}{(-3+\sqrt{7})^2h^2}\right] \\ (-c_{2k+1}+(2-\sqrt{7})c_{2k+2}+(2+\sqrt{7})c_{2k+3}-3c_{2k+4}) \end{pmatrix}.$$

write $\binom{c_{2k}}{c_{2k+1}}$ as:

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = \frac{1}{64} \begin{pmatrix} (10+\sqrt{7})c_{2k-1} + (8-2\sqrt{7})c_{2k-2} - 6c_{2k-3} \\ +32c_{2k} + 6c_{2k+1} + (8-2\sqrt{7})c_{2k+2} \\ \\ \\ (8-2\sqrt{7})c_{2k-1} + 6c_{2k} + 32c_{2k+1} \\ +(16+4\sqrt{7})c_{2k+2} + (8-2\sqrt{7})c_{2k+3} - 6c_{2k+4} \end{pmatrix}$$

$$+ \frac{1}{64} \begin{pmatrix} (2+\sqrt{7})[-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}] \\ (2-\sqrt{7})[-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2} \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} (2-\sqrt{7})[-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}] \\ (2+\sqrt{7})[-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}] \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} -[-3c_{2k-3} + (2+\sqrt{7})c_{2k-2} + (2-\sqrt{7})c_{2k-1} - c_{2k}] \\ -3[-3c_{2k+1} + (2+\sqrt{7})c_{2k+2} + (2-\sqrt{7})c_{2k+3} - c_{2k+4}] \end{pmatrix}$$
$$+ \frac{1}{64} \begin{pmatrix} -3[-c_{2k-3} + (2-\sqrt{7})c_{2k-2} + (2+\sqrt{7})c_{2k-1} - 3c_{2k}] \\ -[-c_{2k+1} + (2-\sqrt{7})c_{2k+2} + (2+\sqrt{7})c_{2k+3} - 3c_{2k+4}] \end{pmatrix}$$

Then the second-order nonlinear diffusion equation after one-step diffusion can be written as:

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix} = \frac{1}{64}\begin{pmatrix} (10+\sqrt{7})c_{2k-1} + (8-2\sqrt{7})c_{2k-2} - 6c_{2k-3} \\ +32c_{2k} + 6c_{2k+1} + (8-2\sqrt{7})c_{2k+2} \\ \\ (8-2\sqrt{7})c_{2k-1} + 6c_{2k} + 32c_{2k+1} \\ +(16+4\sqrt{7})c_{2k+2} + (8-2\sqrt{7})c_{2k+3} - 6c_{2k+4} \end{pmatrix}$$

$$+\begin{pmatrix} \{\frac{(2+\sqrt{7})}{64} - \frac{(2+\sqrt{7})\tau g}{(3-\sqrt{7})^2h^2}[\frac{(-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2})^2}{(3-\sqrt{7})^2h^2}]\} \\ (-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}) \\ \\ \{\frac{(2-\sqrt{7})}{64} - \frac{(2-\sqrt{7})\tau g}{(-3+\sqrt{7})(3-\sqrt{7})h^2}[\frac{(-3c_{2k-1}+(2+\sqrt{7})c_{2k}+(2-\sqrt{7})c_{2k+1}-c_{2k+2})^2}{(3-\sqrt{7})^2h^2}]\} \\ (-3c_{2k-1} + (2+\sqrt{7})c_{2k} + (2-\sqrt{7})c_{2k+1} - c_{2k+2}) \end{pmatrix}$$

$$+\begin{pmatrix} \{\frac{(2-\sqrt{7})}{64} - \frac{(2-\sqrt{7})\tau g}{(3-\sqrt{7})(-3+\sqrt{7})h^2}[\frac{(-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2})^2}{(-3+\sqrt{7})^2h^2}]\} \\ (-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}) \\ \\ \{\frac{(2+\sqrt{7})}{64} - \frac{(2+\sqrt{7})\tau g}{(-3+\sqrt{7})^2h^2}[\frac{(-c_{2k-1}+(2-\sqrt{7})c_{2k}+(2+\sqrt{7})c_{2k+1}-3c_{2k+2})^2}{(-3+\sqrt{7})^2h^2}]\} \\ (-c_{2k-1} + (2-\sqrt{7})c_{2k} + (2+\sqrt{7})c_{2k+1} - 3c_{2k+2}) \end{pmatrix}$$

$$+\begin{pmatrix} \{\frac{-1}{64} - \frac{-\tau g}{(3-\sqrt{7})^2h^2}[\frac{(-3c_{2k-3}+(2+\sqrt{7})c_{2k-2}+(2-\sqrt{7})c_{2k-1}-c_{2k})^2}{(3-\sqrt{7})^2h^2}]\} \\ (-3c_{2k-3} + (2+\sqrt{7})c_{2k-2} + (2-\sqrt{7})c_{2k-1} - c_{2k}) \\ \\ \{\frac{-3}{64} - \frac{-3\tau g}{(-3+\sqrt{7})(3-\sqrt{7})h^2}[\frac{(-3c_{2k+1}+(2+\sqrt{7})c_{2k+2}+(2-\sqrt{7})c_{2k+3}-c_{2k+4})^2}{(3-\sqrt{7})^2h^2}]\} \\ (-3c_{2k+1} + (2+\sqrt{7})c_{2k+2} + (2-\sqrt{7})c_{2k+3} - c_{2k+4}) \end{pmatrix}$$

$$+\begin{pmatrix} \{\frac{-3}{64} - \frac{-3\tau g}{(3-\sqrt{7})(-3+\sqrt{7})h^2}[\frac{(-c_{2k-3}+(2-\sqrt{7})c_{2k-2}+(2+\sqrt{7})c_{2k-1}-3c_{2k})^2}{(-3+\sqrt{7})^2h^2}]\} \\ (-c_{2k-3} + (2-\sqrt{7})c_{2k-2} + (2+\sqrt{7})c_{2k-1} - 3c_{2k}) \\ \\ \{\frac{-1}{64} - \frac{-\tau g}{(-3+\sqrt{7})^2h^2}[\frac{(-c_{2k+1}+(2-\sqrt{7})c_{2k+2}+(2+\sqrt{7})c_{2k+3}-3c_{2k+4})^2}{(-3+\sqrt{7})^2h^2}]\} \\ (-c_{2k+1} + (2-\sqrt{7})c_{2k+2} + (2+\sqrt{7})c_{2k+3} - 3c_{2k+4}) \end{pmatrix} \quad (5.14)$$

Theorem (5.2): Let $\binom{u_{1,k}}{u_{2,k}}$ in Eq(5.11) be resulting signal after 1-step Chui-Lian CL(2) multiwavelet shrinking with $\underline{c}^0_k = \binom{c_{2k}}{c_{2k+1}} = f(kh), k \in \mathbb{Z}$ as the inital data, and

$\binom{u^1_{2k}}{u^1_{2k+1}}$ in Eq(5.14) be the signal after 1-step diffusing with original input $\underline{u}^0_k = \binom{u^0_{2k}}{u^0_{2k+1}} = f(kh), k \in \mathbb{Z}$. Thus,

$$u_{1,k} = u^1_{2k} \quad if \quad \begin{cases} S_{\theta_{11}}(x) = x[1 - \frac{64\tau}{(3-\sqrt{7})^2 h^2} g(\frac{64x^2}{(3-\sqrt{7})^2h^2})] \\ S_{\theta_{21}}(x) = x[1 + \frac{64\tau}{(3-\sqrt{7})^2 h^2} g(\frac{64x^2}{(3-\sqrt{7})^2h^2})] \end{cases},$$

and

$$u_{2,k} = u^1_{2k+1} \quad if \quad \begin{cases} S_{\theta_{12}}(x) = x[1 + \frac{64\tau}{(3-\sqrt{7})^2 h^2} g(\frac{64x^2}{(3-\sqrt{7})^2h^2})] \\ S_{\theta_{22}}(x) = [1 - \frac{64\tau}{(3-\sqrt{7})^2 h^2} g(\frac{64x^2}{(3-\sqrt{7})^2h^2})] \end{cases}.$$

Suppose h=1, and Perona-Malik diffusivity defines as[46]:

$$g(x^2) = \frac{c}{1 + (\frac{x}{\lambda})^2}$$

where c is a constant, then the corresponding shrinkage function are

$$\begin{cases} S_{\theta_{11}}(x) = x[1 - \frac{64\tau}{(3-\sqrt{7})^2} \frac{c}{[1+[8x/(3-\sqrt{7})\theta_{11}]^2]}] \\ S_{\theta_{21}}(x) = x[1 + \frac{64\tau}{(3-\sqrt{7})^2} \frac{c}{[1+[8x/(3-\sqrt{7})\theta_{21}]^2]}] \end{cases},$$

$$\begin{cases} S_{\theta_{12}}(x) = x[1 + \frac{64\tau}{(3-\sqrt{7})^2} \frac{c}{[1+[8x/(3-\sqrt{7})\theta_{12}]^2]}] \\ S_{\theta_{22}}(x) = x[1 - \frac{64\tau}{(3-\sqrt{7})^2} \frac{c}{[1+[8x/(3-\sqrt{7})\theta_{22}]^2]}] \end{cases}.$$

If the Weickert diffusivity g is defined as [71] :

$$g(x^2) = \begin{cases} 1 & if \quad x = 0 \\ 1 - exp(-3.31488\lambda^8/x^8) & if \quad x \neq 0, \end{cases}$$

then the corresponding shrinkage function are defined as:

$$S_{\theta_{11}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 - \frac{64\tau}{(3-\sqrt{7})^2}[1 - exp(-3.31488\ \theta_{11}^8/\ (\frac{8}{(3-\sqrt{7})}x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

$$S_{\theta_{21}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 + \frac{64\tau}{(3-\sqrt{7})^2}[1 - exp(-3.31488\ \theta_{21}^8/\ (\frac{8}{(3-\sqrt{7})}x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

and

$$S_{\theta_{12}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 + \frac{64\tau}{(3-\sqrt{7})^2}[1 - exp(-3.31488\ \theta_{12}^8/(\frac{8}{(3-\sqrt{7})}x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

$$S_{\theta_{22}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 - \frac{64\tau}{(3-\sqrt{7})^2}[1 - exp(-3.31488\ \theta_{22}^8/\ (\frac{8}{(3-\sqrt{7})}x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

5.3 Equivalence between Donovan-Geronimo-Hardin-Massopust(DGHM) multiwavelet shrinkage and second-order nonlinear diffusion equation

5.3.1 Donovan-Geronimo-Hardin-Massopust(DGHM) multiwavelet shrinkage

Let

$$H_0 = \frac{1}{40}\begin{pmatrix} 12 & 16\sqrt{2} \\ -\sqrt{2} & -6 \end{pmatrix}, \quad H_1 = \frac{1}{40}\begin{pmatrix} 12 & 0 \\ 9\sqrt{2} & 20 \end{pmatrix}, \quad H_2 = \frac{1}{40}\begin{pmatrix} 0 & 0 \\ 9\sqrt{2} & -6 \end{pmatrix},$$
$$H_3 = \frac{1}{40}\begin{pmatrix} 0 & 0 \\ -\sqrt{2} & 0 \end{pmatrix}, \quad G_0 = \frac{1}{40}\begin{pmatrix} -\sqrt{2} & -6 \\ 2 & 6\sqrt{2} \end{pmatrix}, \quad G_1 = \frac{1}{40}\begin{pmatrix} 9\sqrt{2} & -20 \\ -18 & 0 \end{pmatrix}$$
$$G_2 = \frac{1}{40}\begin{pmatrix} 9\sqrt{2} & -6 \\ 18 & -6\sqrt{2} \end{pmatrix}, \quad G_3 = \frac{1}{40}\begin{pmatrix} -\sqrt{2} & 0 \\ -2 & 0 \end{pmatrix}$$

be DGHM-multifilter bank and $U = \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ -1+\sqrt{2} & 1+\sqrt{2} \end{pmatrix}$,then

$$P_0 = UH_0U^{-1} = \frac{1}{240}\begin{pmatrix} 18+21\sqrt{2} & 78+51\sqrt{2} \\ 78-51\sqrt{2} & 18-21\sqrt{2} \end{pmatrix}, P_1 = UH_1U^{-1} = \frac{1}{240}\begin{pmatrix} 96-25\sqrt{2} & 28-27\sqrt{2} \\ 28+27\sqrt{2} & 96+25\sqrt{2} \end{pmatrix}$$
$$P_2 = \frac{1}{240}\begin{pmatrix} -18+3\sqrt{2} & 42-27\sqrt{2} \\ 42+27\sqrt{2} & -18-3\sqrt{2} \end{pmatrix}, \quad P_3 = \frac{1}{240}\begin{pmatrix} \sqrt{2} & -4+3\sqrt{2} \\ -4-3\sqrt{2} & -\sqrt{2} \end{pmatrix}$$
$$Q_0 = \frac{1}{240}\begin{pmatrix} -24+15\sqrt{2} & -24-15\sqrt{2} \\ 24-15\sqrt{2} & 24+15\sqrt{2} \end{pmatrix}, Q_1 = \frac{1}{240}\begin{pmatrix} 74+27\sqrt{2} & -6-67\sqrt{2} \\ 6-67\sqrt{2} & -74+27\sqrt{2} \end{pmatrix}$$
$$Q_2 = \frac{1}{240}\begin{pmatrix} 48+9\sqrt{2} & -72+39\sqrt{2} \\ 72+39\sqrt{2} & -48+9\sqrt{2} \end{pmatrix}, \quad Q_3 = \frac{1}{240}\begin{pmatrix} -2-3\sqrt{2} & 6-5\sqrt{2} \\ -6-5\sqrt{2} & 2-3\sqrt{2} \end{pmatrix}$$

is orthogonal and generates

$$(P_0+P_1+P_2+P_3)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{240}\begin{pmatrix} 96 & 144 \\ 144 & 96 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$(Q_0+Q_1+Q_2+Q_3)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The undecimated DGHM multiwavelet decomposition algorithm can be defined as:

$$\begin{aligned}\begin{pmatrix}L_{1,n}\\L_{2,n}\end{pmatrix} &= \frac{1}{240}\begin{pmatrix}18+21\sqrt{2} & 78+51\sqrt{2}\\78-51\sqrt{2} & 18-21\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n}\\c_{2n+1}\end{pmatrix}+\frac{1}{240}\begin{pmatrix}96-25\sqrt{2} & 28-27\sqrt{2}\\28+27\sqrt{2} & 96+25\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+2}\\c_{2n+3}\end{pmatrix}\\&+\frac{1}{240}\begin{pmatrix}-18+3\sqrt{2} & 42-27\sqrt{2}\\42+27\sqrt{2} & -18-3\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+4}\\c_{2n+5}\end{pmatrix}+\frac{1}{240}\begin{pmatrix}\sqrt{2} & -4+3\sqrt{2}\\-4-3\sqrt{2} & -\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+6}\\c_{2n+7}\end{pmatrix}\\\begin{pmatrix}H_{1,n}\\H_{2,n}\end{pmatrix} &= \frac{1}{240}\begin{pmatrix}-24+15\sqrt{2} & -24-15\sqrt{2}\\24-15\sqrt{2} & 24+15\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n}\\c_{2n+1}\end{pmatrix}+\frac{1}{240}\begin{pmatrix}74+27\sqrt{2} & -6-67\sqrt{2}\\6-67\sqrt{2} & -74+27\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+2}\\c_{2n+3}\end{pmatrix}\\&+\frac{1}{240}\begin{pmatrix}48+9\sqrt{2} & -72+39\sqrt{2}\\72+39\sqrt{2} & -48+9\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+4}\\c_{2n+5}\end{pmatrix}+\frac{1}{240}\begin{pmatrix}-2-3\sqrt{2} & 6-5\sqrt{2}\\-6-5\sqrt{2} & 2-3\sqrt{2}\end{pmatrix}\begin{pmatrix}c_{2n+6}\\c_{2n+7}\end{pmatrix}\end{aligned}$$

then

$$\begin{pmatrix}L_{1,n}\\L_{2,n}\end{pmatrix}=\frac{1}{240}\begin{pmatrix}(18+21\sqrt{2})c_{2n}+(78+51\sqrt{2})c_{2n+1}+(96-25\sqrt{2})c_{2n+2}+(28-27\sqrt{2})c_{2n+3}\\+(-18+3\sqrt{2})c_{2n+4}+(42-27\sqrt{2})c_{2n+5}+\sqrt{2}c_{2n+6}+(-4+3\sqrt{2})c_{2n+7}\\ \\(78-51\sqrt{2})c_{2n}+(18-21\sqrt{2})c_{2n+1}(28+27\sqrt{2})c_{2n+2}+(96+25\sqrt{2})c_{2n+3}\\+(42+27\sqrt{2})c_{2n+4}+(-18-3\sqrt{2})c_{2n+5}+(-4-3\sqrt{2})c_{2n+6}-\sqrt{2}c_{2n+7}\end{pmatrix}$$

$$\begin{pmatrix}H_{1,n}\\H_{2,n}\end{pmatrix}=\frac{1}{240}\begin{pmatrix}(-24+15\sqrt{2})c_{2n}+(-24-15\sqrt{2})c_{2n+1}+(74+27\sqrt{2})c_{2n+2}+(-6-67\sqrt{2})c_{2n+3}\\+(48+9\sqrt{2})c_{2n+4}+(-72+39\sqrt{2})c_{2n+5}+(-2-3\sqrt{2})c_{2n+6}+(6-5\sqrt{2})c_{2n+7}\\ \\(24-15\sqrt{2})c_{2n}+(24+15\sqrt{2})c_{2n+1}+(6-67\sqrt{2})c_{2n+2}+(-74+27\sqrt{2})c_{2n+3}\\+(72+39\sqrt{2})c_{2n+4}+(-48+9\sqrt{2})c_{2n+5}+(-6-5\sqrt{2})c_{2n+6}+(2-3\sqrt{2})c_{2n+7}\end{pmatrix} \tag{5.15}$$

and the denoising algorithm is given by: -

$$\begin{aligned}\begin{pmatrix}u_{1,k}\\u_{2,k}\end{pmatrix}&=\frac{1}{240}\begin{pmatrix}(18+21\sqrt{2})L_{1,k}+(78-51\sqrt{2})L_{2,k}+(96-25\sqrt{2})L_{1,k-1}+(28+27\sqrt{2})L_{2,k-1}\\+(-18+3\sqrt{2})L_{1,k-2}+(42+27\sqrt{2})L_{2,k-2}+\sqrt{2}L_{1,k-3}+(-4-3\sqrt{2})L_{2,k-3}\\ \\(78+51\sqrt{2})L_{1,k}+(18-21\sqrt{2})L_{2,k}+(28-27\sqrt{2})L_{1,k-1}+(96+25\sqrt{2})L_{2,k-1}\\+(42-27\sqrt{2})L_{1,k-2}+(-18-3\sqrt{2})L_{2,k-2}+(-4+3\sqrt{2})L_{1,k-3}-\sqrt{2}L_{2,k-3}\end{pmatrix}\\&+\frac{1}{240}\begin{pmatrix}(-24+15\sqrt{2})S_{\theta_{11}}H_{1,k}+(24-15\sqrt{2})S_{\theta_{21}}H_{2,k}+(74+27\sqrt{2})S_{\theta_{11}}H_{1,k-1}\\+(6-67\sqrt{2})S_{\theta_{21}}H_{2,k-1}+(48+9\sqrt{2})S_{\theta_{11}}H_{1,k-2}+(72+39\sqrt{2})S_{\theta_{21}}H_{2,k-2}\\+(-2-3\sqrt{2})S_{\theta_{11}}H_{1,k-3}+(-6-5\sqrt{2})S_{\theta_{21}}H_{2,k-3}\\ \\(-24-15\sqrt{2})S_{\theta_{12}}H_{1,k}+(24+15\sqrt{2})S_{\theta_{22}}H_{2,k}+(-6-67\sqrt{2})S_{\theta_{12}}H_{1,k-1}\\+(-74+27\sqrt{2})S_{\theta_{22}}H_{2,k-1}+(-72+39\sqrt{2})S_{\theta_{12}}H_{1,k-2}+(-48+9\sqrt{2})S_{\theta_{22}}H_{2,k-2}\\+(6-5\sqrt{2})S_{\theta_{12}}H_{1,k-3}+(2-3\sqrt{2})S_{\theta_{22}}H_{2,k-3}\end{pmatrix}\end{aligned}$$

With $L_{1,k}, L_{2,k}$ and $H_{1,k}, H_{2,k}$ given in Eq(5.15), the synthesis step can be defined as:

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{(240)^2} \begin{pmatrix} (18+21\sqrt{2})[(18+21\sqrt{2})c_{2k} + (78+51\sqrt{2})c_{2k+1} + (96-25\sqrt{2})c_{2k+2} \\ +(28-27\sqrt{2})c_{2k+3} + (-18+3\sqrt{2})c_{2k+4} + (42-27\sqrt{2})c_{2k+5} \\ +\sqrt{2}c_{2k+6} + (-4+3\sqrt{2})c_{2k+7}] \\ \\ (78+51\sqrt{2})[(18+21\sqrt{2})c_{2k} + (78+51\sqrt{2})c_{2k+1} + (96-25\sqrt{2})c_{2k+2} \\ +(28-27\sqrt{2})c_{2k+3} + (-18+3\sqrt{2})c_{2k+4} + (42-27\sqrt{2})c_{2k+5} + \sqrt{2}c_{2k+6} \\ +(-4+3\sqrt{2})c_{2k+7}] \end{pmatrix}$$

$$+ \frac{1}{(240)^2} \begin{pmatrix} (78-51\sqrt{2})[(78-51\sqrt{2})c_{2k} + (18-21\sqrt{2})c_{2k+1} + (28+27\sqrt{2})c_{2k+2} + \\ (96+25\sqrt{2})c_{2k+3} + (42+27\sqrt{2})c_{2k+4} + (-18-3\sqrt{2})c_{2k+5} \\ +(-4-3\sqrt{2})c_{2k+6} - \sqrt{2}c_{2k+7}] \\ \\ (18-21\sqrt{2})[(78-51\sqrt{2})c_{2k} + (18-21\sqrt{2})c_{2k+1} + (28+27\sqrt{2})c_{2k+2} \\ +(96+25\sqrt{2})c_{2k+3} + (42+27\sqrt{2})c_{2k+4} + (-18-3\sqrt{2})c_{2k+5} \\ +(-4-3\sqrt{2})c_{2k+6} - \sqrt{2}c_{2k+7}] \end{pmatrix}$$

$$+ \frac{1}{(240)^2} \begin{pmatrix} (96-25\sqrt{2})[(18+21\sqrt{2})c_{2k-2} + (78+51\sqrt{2})c_{2k-1} + (96-25\sqrt{2})c_{2k} + \\ (28-27\sqrt{2})c_{2k+1} + (-18+3\sqrt{2})c_{2k+2} + (42-27\sqrt{2})c_{2k+3} + \sqrt{2}c_{2k+4} \\ +(-4+3\sqrt{2})c_{2k+5}] \\ \\ (28-27\sqrt{2})[(18+21\sqrt{2})c_{2k-2} + (78+51\sqrt{2})c_{2k-1} + (96-25\sqrt{2})c_{2k} \\ +(28-27\sqrt{2})c_{2k+1} + (-18+3\sqrt{2})c_{2k+2} + (42-27\sqrt{2})c_{2k+3} + \sqrt{2}c_{2k+4} \\ +(-4+3\sqrt{2})c_{2k+5}] \end{pmatrix}$$

$$+ \frac{1}{(240)^2} \begin{pmatrix} (28+27\sqrt{2})[(78-51\sqrt{2})c_{2k-2} + (18-21\sqrt{2})c_{2k-1} + (28+27\sqrt{2})c_{2k} \\ +(96+25\sqrt{2})c_{2k+1} + (42+27\sqrt{2})c_{2k+2} + (-18-3\sqrt{2})c_{2k+3} \\ +(-4-3\sqrt{2})c_{2k+4} - \sqrt{2}c_{2k+5}] \\ \\ (96+25\sqrt{2})[(78-51\sqrt{2})c_{2k-2} + (18-21\sqrt{2})c_{2k-1} + (28+27\sqrt{2})c_{2k} \\ +(96+25\sqrt{2})c_{2k+1} + (42+27\sqrt{2})c_{2k+2} + (-18-3\sqrt{2})c_{2k+3} \\ +(-4-3\sqrt{2})c_{2k+4} - \sqrt{2}c_{2k+5}] \end{pmatrix}$$

$$+\frac{1}{(240)^2}\begin{pmatrix}(-18+3\sqrt{2})[(18+21\sqrt{2})c_{2k-4}+(78+51\sqrt{2})c_{2k-3}+(96-25\sqrt{2})c_{2k-2}\\ +(28-27\sqrt{2})c_{2k-1}+(-18+3\sqrt{2})c_{2k}+(42-27\sqrt{2})c_{2k+1}+\sqrt{2}c_{2k+2}\\ +(-4+3\sqrt{2})c_{2k+3}]\\ \\ (42-27\sqrt{2})[(18+21\sqrt{2})c_{2k-4}+(78+51\sqrt{2})c_{2k-3}+(96-25\sqrt{2})c_{2k-2}\\ +(28-27\sqrt{2})c_{2k-1}+(-18+3\sqrt{2})c_{2k}+(42-27\sqrt{2})c_{2k+1}+\sqrt{2}c_{2k+2}\\ +(-4+3\sqrt{2})c_{2k+3}]\end{pmatrix}$$

$$+\frac{1}{(240)^2}\begin{pmatrix}(42+27\sqrt{2})[(78-51\sqrt{2})c_{2k-4}+(18-21\sqrt{2})c_{2k-3}+(28+27\sqrt{2})c_{2k-2}\\ +(96+25\sqrt{2})c_{2k-1}+(42+27\sqrt{2})c_{2k}+(-18-3\sqrt{2})c_{2k+1}\\ +(-4-3\sqrt{2})c_{2k+2}-\sqrt{2}c_{2k+3}]\\ \\ (-18-3\sqrt{2})[(78-51\sqrt{2})c_{2k-4}+(18-21\sqrt{2})c_{2k-3}+(28+27\sqrt{2})c_{2k-2}\\ +(96+25\sqrt{2})c_{2k-1}+(42+27\sqrt{2})c_{2k}+(-18-3\sqrt{2})c_{2k+1}\\ +(-4-3\sqrt{2})c_{2k+2}-\sqrt{2}c_{2k+3}]\end{pmatrix}$$

$$+\frac{1}{(240)^2}\begin{pmatrix}\sqrt{2}[(18+21\sqrt{2})c_{2k-6}+(78+51\sqrt{2})c_{2k-5}(96-25\sqrt{2})c_{2k-4}\\ +(28-27\sqrt{2})c_{2k-3}+(-18+3\sqrt{2})c_{2k-2}+(42-27\sqrt{2})c_{2k-1}+\sqrt{2}c_{2k}\\ +(-4+3\sqrt{2})c_{2k+1}]\\ \\ (-4+3\sqrt{2})[(18+21\sqrt{2})c_{2k-6}+(78+51\sqrt{2})c_{2k-5}+(96-25\sqrt{2})c_{2k-4}\\ +(28-27\sqrt{2})c_{2k-3}+(-18+3\sqrt{2})c_{2k-2}+(42-27\sqrt{2})c_{2k-1}\\ +\sqrt{2}c_{2k}+(-4+3\sqrt{2})c_{2k+1}]\end{pmatrix}$$

$$+\frac{1}{(240)^2}\begin{pmatrix}(-4-3\sqrt{2})[(78-51\sqrt{2})c_{2k-6}+(18-21\sqrt{2})c_{2k-5}+(28+27\sqrt{2})c_{2k-4}\\ +(96+25\sqrt{2})c_{2k-3}+(42+27\sqrt{2})c_{2k-2}+(-18-3\sqrt{2})c_{2k-1}\\ +(-4-3\sqrt{2})c_{2k}-\sqrt{2}c_{2k+1}]\\ \\ -\sqrt{2}[(78-51\sqrt{2})c_{2k-6}+(18-21\sqrt{2})c_{2k-5}+(28+27\sqrt{2})c_{2k-4}\\ +(96+25\sqrt{2})c_{2k-3}+(42+27\sqrt{2})c_{2k-2}+(-18-3\sqrt{2})c_{2k-1}\\ +(-4-3\sqrt{2})c_{2k}-\sqrt{2}c_{2k+1}]\end{pmatrix}$$

$$
+\frac{1}{240}\begin{pmatrix}
(-24+15\sqrt{2})S_{\theta_{11}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k}+(-24-15\sqrt{2})c_{2k+1} \\
+(74+27\sqrt{2})c_{2k+2}+(-6-67\sqrt{2})c_{2k+3}+(48+9\sqrt{2})c_{2k+4}+ \\
(-72+39\sqrt{2})c_{2n+5}+(-2-3\sqrt{2})c_{2n+6}+(6-5\sqrt{2})c_{2n+7}]\} \\
\\
(-24-15\sqrt{2})S_{\theta_{12}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k}+(-24-15\sqrt{2})c_{2k+1} \\
+(74+27\sqrt{2})c_{2k+2}+(-6-67\sqrt{2})c_{2k+3}+(48+9\sqrt{2})c_{2k+4}+ \\
(-72+39\sqrt{2})c_{2k+5}+(-2-3\sqrt{2})c_{2k+6}+(6-5\sqrt{2})c_{2k+7}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(24-15\sqrt{2})S_{\theta_{21}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k}+(24+15\sqrt{2})c_{2k+1} \\
(6-67\sqrt{2})c_{2k+2}+(-74+27\sqrt{2})c_{2k+3}+(72+39\sqrt{2})c_{2k+4}+ \\
(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6}+(2-3\sqrt{2})c_{2k+7}]\} \\
\\
(24+15\sqrt{2})S_{\theta_{22}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k}+(24+15\sqrt{2})c_{2k+1} \\
(6-67\sqrt{2})c_{2k+2}+(-74+27\sqrt{2})c_{2k+3}+(72+39\sqrt{2})c_{2k+4}+ \\
(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6}+(2-3\sqrt{2})c_{2k+7}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(74+27\sqrt{2})S_{\theta_{11}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\
(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2} \\
+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}]\} \\
\\
(-6-67\sqrt{2})S_{\theta_{12}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\
(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2} \\
+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(6-67\sqrt{2})S_{\theta_{21}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}]\} \\
\\
(-74+27\sqrt{2})S_{\theta_{22}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(48+9\sqrt{2})S_{\theta_{11}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\
(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k} \\
+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}]\} \\
\\
(-72+39\sqrt{2})S_{\theta_{12}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\
(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k} \\
+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(72+39\sqrt{2})S_{\theta_{21}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\
(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\
+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}]\} \\
\\
(-48+9\sqrt{2})S_{\theta_{22}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\
(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\
+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(-2-3\sqrt{2})S_{\theta_{11}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2} \\
+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}]\} \\
\\
(6-5\sqrt{2})S_{\theta_{12}}\{\frac{1}{240}[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2} \\
+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}]\}
\end{pmatrix}
$$

$$
+\frac{1}{240}\begin{pmatrix}
(-6-5\sqrt{2})S_{\theta_{21}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}]\} \\
\\
(2-3\sqrt{2})S_{\theta_{22}}\{\frac{1}{240}[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}]\}
\end{pmatrix}. \qquad (5.16)
$$

If there is no shrinking apply then:

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \frac{1}{57600} \begin{pmatrix} (36-12\sqrt{2})c_{2k-6} + (156+108\sqrt{2})c_{2k-5} - 456\sqrt{2}c_{2k-4} \\ +(-2064-1440\sqrt{2})c_{2k-3} + (540+4620\sqrt{2})c_{2k-2} \\ +(9060+7164\sqrt{2})c_{2k-1} + (28800-8304\sqrt{2})c_{2k} + 13344\ c_{2k+1} \\ +(540+4620\sqrt{2})c_{2k+2} + (9060-7164)c_{2k+3} - 456\sqrt{2}c_{2k+4} \\ +(-2064+1440\sqrt{2})c_{2k+5} + (36-12\sqrt{2})c_{2k+6} + (156-108\sqrt{2})c_{2k+7} \\ \\ (156-108\sqrt{2})c_{2k-6} + (36+12\sqrt{2})c_{2k-5} + (-2064+1440\sqrt{2})c_{2k-4} \\ +456\sqrt{2}\ c_{2k-3} + (9060-7164\sqrt{2})c_{2k-2} + (540-4620\sqrt{2})c_{2k-1} \\ +13344\ c_{2k} + (28800+8304\sqrt{2})c_{2k+1} + (9060+7164\sqrt{2})c_{2k+2} \\ +(540-4620\sqrt{2})c_{2k+3} + (-2064-1440\sqrt{2})c_{2k+4} + 456\ c_{2k+5} \\ +(156+108\sqrt{2})c_{2k+6} + (36+12\sqrt{2})c_{2k+7} \end{pmatrix}$$

$$+\frac{1}{57600} \begin{pmatrix} (-36+12\sqrt{2})c_{2k-6} + (-156-108\sqrt{2})c_{2k-5} + 456\sqrt{2}c_{2k-4} \\ +(2064+1440\sqrt{2})c_{2k-3} + (-540-4620\sqrt{2})c_{2k-2} \\ +(-9060-7164\sqrt{2})c_{2k-1} + (28800+8304\sqrt{2})c_{2k} - 13344\ c_{2k+1} \\ +(-540-4620\sqrt{2})c_{2k+2} + (-9060+7164)c_{2k+3} + 456\sqrt{2}c_{2k+4} \\ +(2064-1440\sqrt{2})c_{2k+5} + (-36+12\sqrt{2})c_{2k+6} + (-156+108\sqrt{2})c_{2k+7} \\ \\ (-156+108\sqrt{2})c_{2k-6} + (-36-12\sqrt{2})c_{2k-5} + (2064-1440\sqrt{2})c_{2k-4} \\ -456\sqrt{2}\ c_{2k-3} + (-9060+7164\sqrt{2})c_{2k-2} + (-540+4620\sqrt{2})c_{2k-1} \\ -13344\ c_{2k} + (28800-8304\sqrt{2})c_{2k+1} + (-9060-7164\sqrt{2})c_{2k+2} \\ +(-540+4620\sqrt{2})c_{2k+3} + (2064+1440\sqrt{2})c_{2k+4} - 456\ c_{2k+5} \\ +(-156-108\sqrt{2})c_{2k+6} + (-36-12\sqrt{2})c_{2k+7} \end{pmatrix}$$

i.e

$$\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix}$$

5.3.2 DGHM multiwavelet shrinkage for nonlinear diffusion

Let

$$u_t = \frac{\partial}{\partial x}\{g(u_x^2)u_x\} \tag{5.17}$$

be the second-order nonlinear diffusion formula for $u(x,t)$ with $u(x,0) = f(x)$ as an initial condition.

Approximation of $\frac{\partial}{\partial x}u(x,t)$ at $(2kh, j\tau)$ is given by:

$$\begin{aligned} u_x \approx \frac{1}{(-32+16\sqrt{2})h}\{&(-24+15\sqrt{2})u^j_{2k} + (-24-15\sqrt{2})u^j_{2k+1} + (74+27\sqrt{2})u^j_{2k+2} \\ &+ (-6-67\sqrt{2})u^j_{2k+3} + (48+9\sqrt{2})u^j_{2k+4} + (-72+39\sqrt{2})u^j_{2k+5} \\ &+ (-2-3\sqrt{2})u^j_{2k+6} + (6-5\sqrt{2})u^j_{2k+7}\} \end{aligned}$$

Also, $\frac{\partial}{\partial x}u(x,t)$ at $([2k+1]h, j\tau)$ can be written as:

$$u_x \approx \frac{1}{(-160+112\sqrt{2})h}\{(24-15\sqrt{2})u^j_{2k} + (24+15\sqrt{2})u^j_{2k+1}(6-67\sqrt{2})u^j_{2k+2} + (-74+27\sqrt{2})u^j_{2k+3} + (72+39\sqrt{2})u^j_{2k+4} + (-48+9\sqrt{2})u^j_{2k+5} + (-6-5\sqrt{2})u^j_{2k+6} + (2-3\sqrt{2})u^j_{2k+7}\}$$

Therefore, the value of $\frac{\partial}{\partial x}u(x,t)$ at $\begin{pmatrix}(2kh, j\tau)\\([2k+1]h, j\tau)\end{pmatrix}$ by using highpass filters can be defined as:

$$\begin{aligned}
\frac{\partial}{\partial x}\begin{pmatrix}u(2kh,j\tau)\\u([2k+1]h,j\tau)\end{pmatrix} &\approx \begin{pmatrix}\frac{240}{(-32+16\sqrt{2})h} & 0\\ 0 & \frac{240}{(-160+112\sqrt{2})h}\end{pmatrix}\sum Q_n\begin{pmatrix}u(2(k+n)h,j\tau)\\u([2(k+n)+1]h,j\tau)\end{pmatrix}\\
&= \begin{pmatrix}\frac{240}{(-32+16\sqrt{2})h} & 0\\ 0 & \frac{240}{(-160+112\sqrt{2})h}\end{pmatrix}\sum Q_n\begin{pmatrix}u^j_{2(k+n)}\\u^j_{2(k+n)+1}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{(-32+16\sqrt{2})h} & 0\\ 0 & \frac{1}{(-160+112\sqrt{2})h}\end{pmatrix}\{\begin{pmatrix}-24+15\sqrt{2} & -24-15\sqrt{2}\\ 24-15\sqrt{2} & 24+15\sqrt{2}\end{pmatrix}\begin{pmatrix}u^j_{2k}\\u^j_{2k+1}\end{pmatrix}\\
&+\begin{pmatrix}74+27\sqrt{2} & -6-67\sqrt{2}\\ 6-67\sqrt{2} & -74+27\sqrt{2}\end{pmatrix}\begin{pmatrix}u^j_{2k+2}\\u^j_{2k+3}\end{pmatrix}+\begin{pmatrix}48+9\sqrt{2} & -72+39\sqrt{2}\\ 72+39\sqrt{2} & -48+9\sqrt{2}\end{pmatrix}\\
&\times\begin{pmatrix}u^j_{2k+4}\\u^j_{2k+5}\end{pmatrix}+\begin{pmatrix}-2-3\sqrt{2} & 6-5\sqrt{2}\\ -6-5\sqrt{2} & 2-3\sqrt{2}\end{pmatrix}\begin{pmatrix}u^j_{2k+6}\\u^j_{2k+7}\end{pmatrix}\}\\
&= \begin{pmatrix}\frac{1}{(-32+16\sqrt{2})h} & 0\\ 0 & \frac{1}{(-160+112\sqrt{2})h}\end{pmatrix}\times\\
&\begin{pmatrix}(-24+15\sqrt{2})u^j_{2k}+(-24-15\sqrt{2})u^j_{2k+1}\\ +(74+27\sqrt{2})u^j_{2k+2}+(-6-67\sqrt{2})u^j_{2n+3}+(48+9\sqrt{2})u^j_{2k+4}\\ +(-72+39\sqrt{2})u^j_{2k+5}+(-2-3\sqrt{2})u^j_{2k+6}+(6-5\sqrt{2})u^j_{2k+7}\\ \\ (24-15\sqrt{2})u^j_{2k}+(24+15\sqrt{2})u^j_{2k+1}\\ +(6-67\sqrt{2})u^j_{2k+2}+(-74+27\sqrt{2})u^j_{2k+3}+(72+39\sqrt{2})u^j_{2k+4}\\ +(-48+9\sqrt{2})u^j_{2k+5}+(-6-5\sqrt{2})u^j_{2k+6}+(2-3\sqrt{2})u^j_{2k+7}\end{pmatrix}\\
&= \begin{pmatrix}\frac{\mathbf{1}}{\mathbf{(-32+16\sqrt{2})h}}\{(-24+15\sqrt{2})u^j_{2k}+(-24-15\sqrt{2})u^j_{2k+1}\\ +(74+27\sqrt{2})u^j_{2k+2}+(-6-67\sqrt{2})u^j_{2k+3}+(48+9\sqrt{2})u^j_{2k+4}\\ +(-72+39\sqrt{2})u^j_{2k+5}+(-2-3\sqrt{2})u^j_{2k+6}+(6-5\sqrt{2})u^j_{2k+7}\}\\ \\ \frac{\mathbf{1}}{\mathbf{(-160+112\sqrt{2})h}}\{(24-15\sqrt{2})u^j_{2k}+(24+15\sqrt{2})u^j_{2k+1}\\ +(6-67\sqrt{2})u^j_{2k+2}+(-74+27\sqrt{2})u^j_{2k+3}+(72+39\sqrt{2})u^j_{2k+4}\\ +(-48+9\sqrt{2})u^j_{2k+5}+(-6-5\sqrt{2})u^j_{2k+6}+(2-3\sqrt{2})u^j_{2k+7}\}\end{pmatrix},
\end{aligned}$$

and the approximating partial derivatives of $G(x,t) = g((u_x)^2)u_x$ at $\begin{pmatrix} (2kh, j\tau) \\ ([2k+1]h, j\tau) \end{pmatrix}$ is defined as:

$$\frac{\partial}{\partial x}\begin{pmatrix} G(2kh, j\tau) \\ G([2k+1]h, j\tau) \end{pmatrix} = \begin{pmatrix} \frac{-240}{(-32+16\sqrt{2})h} & 0 \\ 0 & \frac{-240}{(-160+112\sqrt{2})h} \end{pmatrix} \sum Q_m^T \begin{pmatrix} G(2(k-m)h, j\tau) \\ G([2(k-m)+1]h, j\tau) \end{pmatrix}$$

Then the second-order nonlinear diffusion equation can be discretized as:

$$\begin{pmatrix} u_{2k}^{j+1} \\ u_{2k+1}^{j+1} \end{pmatrix} = \begin{pmatrix} u_{2k}^{j} \\ u_{2k+1}^{j} \end{pmatrix} + \begin{pmatrix} \frac{-240\tau}{(-32+16\sqrt{2})h} & 0 \\ 0 & \frac{-240\tau}{(-160+112\sqrt{2})h} \end{pmatrix} \sum Q_m^T$$

$$\times \begin{pmatrix} \frac{g}{(-32+16\sqrt{2})h}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u_{2(k-m)}^j + (-24-15\sqrt{2})u_{2(k-m)+1}^j \\ +(74+27\sqrt{2})u_{2(k-m)+2}^j + (-6-67\sqrt{2})u_{2(k-m)+3}^j + (48+9\sqrt{2})u_{2(k-m)+4}^j \\ +(-72+39\sqrt{2})u_{2(k-m)+5}^j + (-2-3\sqrt{2})u_{2(k-m)+6}^j + (6-5\sqrt{2})u_{2(k-m)+7}^j]^2\} \\ \times[(-24+15\sqrt{2})u_{2(k-m)}^j + (-24-15\sqrt{2})u_{2(k-m)+1}^j \\ +(74+27\sqrt{2})u_{2(k-m)+2}^j + (-6-67\sqrt{2})u_{2(k-m)+3}^j + (48+9\sqrt{2})u_{2(k-m)+4}^j \\ +(-72+39\sqrt{2})u_{2(k-m)+5}^j + (-2-3\sqrt{2})u_{2(k-m)+6}^j + (6-5\sqrt{2})u_{2(k-m)+7}^j] \\ \\ \frac{g}{(-160+112\sqrt{2})h}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u_{2(k-m)}^j + (24+15\sqrt{2})u_{2(k-m)+1}^j \\ +(6-67\sqrt{2})u_{2(k-m)+2}^j + (-74+27\sqrt{2})u_{2(k-m)+3}^j + (72+39\sqrt{2})u_{2(k-m)+4}^j \\ +(-48+9\sqrt{2})u_{2(k-m)+5}^j + (-6-5\sqrt{2})u_{2(k-m)+6}^j + (2-3\sqrt{2})u_{2(k-m)+7}^j]^2\} \\ \times(24-15\sqrt{2})u_{2(k-m)}^j + (24+15\sqrt{2})u_{2(k-m)+1}^j \\ +(6-67\sqrt{2})u_{2(k-m)+2}^j + (-74+27\sqrt{2})u_{2(k-m)+3}^j + (72+39\sqrt{2})u_{2(k-m)+4}^j \\ +(-48+9\sqrt{2})u_{2(k-m)+5}^j + (-6-5\sqrt{2})u_{2(k-m)+6}^j + (2-3\sqrt{2})u_{2(k-m)+7}^j] \end{pmatrix}$$

Then by applying(DGHM) multiwavelet coefficients, we have

$$
\begin{pmatrix} u_{2k}^{j+1} \\ u_{2k+1}^{j+1} \end{pmatrix} = \begin{pmatrix} u_{2k}^{j} \\ u_{2k+1}^{j} \end{pmatrix} + \begin{pmatrix} \frac{-\tau}{(-32+16\sqrt{2})h} & 0 \\ 0 & \frac{-\tau}{(-160+112\sqrt{2})h} \end{pmatrix} \ \{ \begin{pmatrix} -24+15\sqrt{2} & 24-15\sqrt{2} \\ -24-15\sqrt{2} & 24+15\sqrt{2} \end{pmatrix}
$$

$$
\times \begin{pmatrix}
\frac{g}{(-32+16\sqrt{2})h}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u_{2k}^j + (-24-15\sqrt{2})u_{2k+1}^j \\
+(74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j + (48+9\sqrt{2})u_{2k+4}^j \\
+(-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j + (6-5\sqrt{2})u_{2k+7}^j]^2\} \\
\times[(-24+15\sqrt{2})u_{2k}^j + (-24-15\sqrt{2})u_{2k+1}^j \\
+(74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j + (48+9\sqrt{2})u_{2k+4}^j \\
+(-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j + (6-5\sqrt{2})u_{2k+7}^j] \\
\\
\\
\frac{g}{(-160+112\sqrt{2})h}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u_{2k}^j + (24+15\sqrt{2})u_{2k+1}^j \\
+(6-67\sqrt{2})u_{2k+2}^j + (-74+27\sqrt{2})u_{2k+3}^j + (72+39\sqrt{2})u_{2k+4}^j \\
+(-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j + (2-3\sqrt{2})u_{2k+7}^j]^2\} \\
\times(24-15\sqrt{2})u_{2k}^j + (24+15\sqrt{2})u_{2k+1}^j \\
+(6-67\sqrt{2})u_{2k+2}^j + (-74+27\sqrt{2})u_{2k+3}^j + (72+39\sqrt{2})u_{2k+4}^j \\
+(-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j + (2-3\sqrt{2})u_{2k+7}^j]
\end{pmatrix}
$$

$$
+ \begin{pmatrix} 74+27\sqrt{2} & 6-67\sqrt{2} \\ -6-67\sqrt{2} & -74+27\sqrt{2} \end{pmatrix}
$$

$$
\times \begin{pmatrix}
\frac{g}{(-32+16\sqrt{2})h}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u_{2k-2}^j + (-24-15\sqrt{2})u_{2k-1}^j \\
+(74+27\sqrt{2})u_{2k}^j + (-6-67\sqrt{2})u_{2k+1}^j + (48+9\sqrt{2})u_{2k+2}^j \\
+(-72+39\sqrt{2})u_{2k+3}^j + (-2-3\sqrt{2})u_{2k+4}^j + (6-5\sqrt{2})u_{2k+5}^j]^2\} \\
\times[(-24+15\sqrt{2})u_{2k-2}^j + (-24-15\sqrt{2})u_{2k-1}^j \\
+(74+27\sqrt{2})u_{2k}^j + (-6-67\sqrt{2})u_{2k+1}^j + (48+9\sqrt{2})u_{2k+2}^j \\
+(-72+39\sqrt{2})u_{2k+3}^j + (-2-3\sqrt{2})u_{2k+4}^j + (6-5\sqrt{2})u_{2k+5}^j] \\
\\
\\
\frac{g}{(-160+112\sqrt{2})h}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u_{2k-2}^j + (24+15\sqrt{2})u_{2k-1}^j \\
+(6-67\sqrt{2})u_{2k}^j + (-74+27\sqrt{2})u_{2k+1}^j + (72+39\sqrt{2})u_{2k+2}^j \\
+(-48+9\sqrt{2})u_{2k+3}^j + (-6-5\sqrt{2})u_{2k+4}^j + (2-3\sqrt{2})u_{2k+5}^j]^2\} \\
\times(24-15\sqrt{2})u_{2k-2}^j + (24+15\sqrt{2})u_{2k-1}^j \\
+(6-67\sqrt{2})u_{2k}^j + (-74+27\sqrt{2})u_{2k+1}^j + (72+39\sqrt{2})u_{2k+2}^j \\
+(-48+9\sqrt{2})u_{2k+3}^j + (-6-5\sqrt{2})u_{2k+4}^j + (2-3\sqrt{2})u_{2k+5}^j]
\end{pmatrix}
$$

$$
+\begin{pmatrix} 48+9\sqrt{2} & 72+39\sqrt{2} \\ -72+39\sqrt{2} & -48+9\sqrt{2} \end{pmatrix}
$$

$$
\times\begin{pmatrix}
\frac{g}{(-32+16\sqrt{2})h}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-4}+(-24-15\sqrt{2})u^j_{2k-3} \\
+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1}+(48+9\sqrt{2})u^j_{2k} \\
+(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2}+(6-5\sqrt{2})u^j_{2k+3}]^2\} \\
\times[(-24+15\sqrt{2})u^j_{2k-4}+(-24-15\sqrt{2})u^j_{2k-3} \\
+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1}+(48+9\sqrt{2})u^j_{2k} \\
+(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2}+(6-5\sqrt{2})u^j_{2k+3}] \\
\\
\frac{g}{(-160+112\sqrt{2})h}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-4}+(24+15\sqrt{2})u^j_{2k-3} \\
+(6-67\sqrt{2})u^j_{2k-2}+(-74+27\sqrt{2})u^j_{2k-1}+(72+39\sqrt{2})u^j_{2k} \\
+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2}+(2-3\sqrt{2})u^j_{2k+3}]^2\} \\
\times(24-15\sqrt{2})u^j_{2k-4}+(24+15\sqrt{2})u^j_{2k-3} \\
+(6-67\sqrt{2})u^j_{2k-2}+(-74+27\sqrt{2})u^j_{2k-1}+(72+39\sqrt{2})u^j_{2k} \\
+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2}+(2-3\sqrt{2})u^j_{2k+3}]
\end{pmatrix}
$$

$$
+\begin{pmatrix} -2-3\sqrt{2} & -6-5\sqrt{2} \\ 6-5\sqrt{2} & 2-3\sqrt{2} \end{pmatrix}
$$

$$
\times\begin{pmatrix}
\frac{g}{(-32+16\sqrt{2})h}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-6}+(-24-15\sqrt{2})u^j_{2k-5} \\
+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3}+(48+9\sqrt{2})u^j_{2k-2} \\
+(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k}+(6-5\sqrt{2})u^j_{2k+1}]^2\} \\
\times[(-24+15\sqrt{2})u^j_{2k-6}+(-24-15\sqrt{2})u^j_{2k-5} \\
+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3}+(48+9\sqrt{2})u^j_{2k-2} \\
+(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k}+(6-5\sqrt{2})u^j_{2k+1}] \\
\\
\frac{g}{(-160+112\sqrt{2})h}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-6}+(24+15\sqrt{2})u^j_{2k-5} \\
+(6-67\sqrt{2})u^j_{2k-4}+(-74+27\sqrt{2})u^j_{2k-3}+(72+39\sqrt{2})u^j_{2k-2} \\
+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k}+(2-3\sqrt{2})u^j_{2k+1}]^2\} \\
\times(24-15\sqrt{2})u^j_{2k-6}+(24+15\sqrt{2})u^j_{2k-5} \\
+(6-67\sqrt{2})u^j_{2k-4}+(-74+27\sqrt{2})u^j_{2k-3}+(72+39\sqrt{2})u^j_{2k-2} \\
+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k}+(2-3\sqrt{2})u^j_{2k+1}]
\end{pmatrix}
$$

Then

$$\begin{pmatrix} u_{2k}^{j+1} \\ u_{2k+1}^{j+1} \end{pmatrix} = \begin{pmatrix} u_{2k}^{j} \\ u_{2k+1}^{j} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-(-24+15\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u_{2k}^j \\ +(-24-15\sqrt{2})u_{2k+1}^j + (74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j \\ +(48+9\sqrt{2})u_{2k+4}^j + (-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j \\ +(6-5\sqrt{2})u_{2k+7}^j]^2\}[(-24+15\sqrt{2})u_{2k}^j + (-24-15\sqrt{2})u_{2k+1}^j \\ +(74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j + (48+9\sqrt{2})u_{2k+4}^j + \\ (-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j + (6-5\sqrt{2})u_{2k+7}^j] \\ \\ \frac{-(-24-15\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u_{2k}^j \\ +(-24-15\sqrt{2})u_{2k+1}^j + (74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j \\ +(48+9\sqrt{2})u_{2k+4}^j + (-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j \\ +(6-5\sqrt{2})u_{2k+7}^j]^2\}[(-24+15\sqrt{2})u_{2k}^j + (-24-15\sqrt{2})u_{2k+1}^j \\ +(74+27\sqrt{2})u_{2k+2}^j + (-6-67\sqrt{2})u_{2k+3}^j + (48+9\sqrt{2})u_{2k+4}^j + \\ (-72+39\sqrt{2})u_{2k+5}^j + (-2-3\sqrt{2})u_{2k+6}^j + (6-5\sqrt{2})u_{2k+7}^j] \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-(24-15\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u_{2k}^j \\ +(24+15\sqrt{2})u_{2k+1}^j + (6-67\sqrt{2})u_{2n+2}^j + (-74+27\sqrt{2})u_{2k+3}^j \\ +(72+39\sqrt{2})u_{2k+4}^j + (-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j \\ +(2-3\sqrt{2})u_{2k+7}^j]^2\}[(24-15\sqrt{2})u_{2k}^j + (24+15\sqrt{2})u_{2k+1}^j \\ +(6-67\sqrt{2})u_{2n+2}^j + (-74+27\sqrt{2})u_{2k+3}^j + (72+39\sqrt{2})u_{2k+4}^j \\ +(-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j + (2-3\sqrt{2})u_{2k+7}^j] \\ \\ \frac{-(24+15\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u_{2k}^j \\ +(24+15\sqrt{2})u_{2k+1}^j + (6-67\sqrt{2})u_{2k+2}^j + (-74+27\sqrt{2})u_{2k+3}^j \\ +(72+39\sqrt{2})u_{2k+4}^j + (-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j \\ +(2-3\sqrt{2})u_{2k+7}^j]^2\}[(24-15\sqrt{2})u_{2k}^j + (24+15\sqrt{2})u_{2k+1}^j \\ +(6-67\sqrt{2})u_{2n+2}^j + (-74+27\sqrt{2})u_{2k+3}^j + (72+39\sqrt{2})u_{2k+4}^j \\ +(-48+9\sqrt{2})u_{2k+5}^j + (-6-5\sqrt{2})u_{2k+6}^j + (2-3\sqrt{2})u_{2k+7}^j] \end{pmatrix}$$

$$
+\begin{pmatrix}
\frac{-(74+27\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-2} \\
+(-24-15\sqrt{2})u^j_{2k-1}+(74+27\sqrt{2})u^j_{2k}+(-6-67\sqrt{2})u^j_{2k+1} \\
+(48+9\sqrt{2})u^j_{2k+2}+(-72+39\sqrt{2})u^j_{2k+3}+(-2-3\sqrt{2})u^j_{2k+4} \\
+(6-5\sqrt{2})u^j_{2k+5}]^2\}[(-24+15\sqrt{2})u^j_{2k-2}+(-24-15\sqrt{2})u^j_{2k-1} \\
+(74+27\sqrt{2})u^j_{2k}+(-6-67\sqrt{2})u^j_{2k+1}+(48+9\sqrt{2})u^j_{2k+2}+ \\
(-72+39\sqrt{2})u^j_{2k+3}+(-2-3\sqrt{2})u^j_{2k+4}+(6-5\sqrt{2})u^j_{2k+5}] \\
\\
\frac{-(-6-67\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-2} \\
+(-24-15\sqrt{2})u^j_{2k-1}+(74+27\sqrt{2})u^j_{2k}+(-6-67\sqrt{2})u^j_{2k+1} \\
+(48+9\sqrt{2})u^j_{2k+2}+(-72+39\sqrt{2})u^j_{2k+3}+(-2-3\sqrt{2})u^j_{2k+4} \\
+(6-5\sqrt{2})u^j_{2k+5}]^2\}[(-24+15\sqrt{2})u^j_{2k-2}+(-24-15\sqrt{2})u^j_{2k-1} \\
+(74+27\sqrt{2})u^j_{2k}+(-6-67\sqrt{2})u^j_{2k+1}+(48+9\sqrt{2})u^j_{2k+2}+ \\
(-72+39\sqrt{2})u^j_{2k+3}+(-2-3\sqrt{2})u^j_{2k+4}+(6-5\sqrt{2})u^j_{2k+5}]
\end{pmatrix}
$$

$$
+\begin{pmatrix}
\frac{-(6-67\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-2} \\
+(24+15\sqrt{2})u^j_{2k-1}+(6-67\sqrt{2})u^j_{2k}+(-74+27\sqrt{2})u^j_{2k+1} \\
+(72+39\sqrt{2})u^j_{2k+2}+(-48+9\sqrt{2})u^j_{2k+3}+(-6-5\sqrt{2})u^j_{2k+4} \\
+(2-3\sqrt{2})u^j_{2k+5}]^2\}[(24-15\sqrt{2})u^j_{2k-2}+(24+15\sqrt{2})u^j_{2k-1} \\
+(6-67\sqrt{2})u^j_{2k}+(-74+27\sqrt{2})u^j_{2k+1}+(72+39\sqrt{2})u^j_{2k+2} \\
+(-48+9\sqrt{2})u^j_{2k+3}+(-6-5\sqrt{2})u^j_{2k+4}+(2-3\sqrt{2})u^j_{2k+5}] \\
\\
\frac{-(-74+27\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-2} \\
+(24+15\sqrt{2})u^j_{2k-1}+(6-67\sqrt{2})u^j_{2k}+(-74+27\sqrt{2})u^j_{2k+1} \\
+(72+39\sqrt{2})u^j_{2k+2}+(-48+9\sqrt{2})u^j_{2k+3}+(-6-5\sqrt{2})u^j_{2k+4} \\
+(2-3\sqrt{2})u^j_{2k+5}]^2\}[(24-15\sqrt{2})u^j_{2k-2}+(24+15\sqrt{2})u^j_{2k-1} \\
+(6-67\sqrt{2})u^j_{2k}+(-74+27\sqrt{2})u^j_{2k+1}+(72+39\sqrt{2})u^j_{2k+2} \\
+(-48+9\sqrt{2})u^j_{2k+3}+(-6-5\sqrt{2})u^j_{2k+4}+(2-3\sqrt{2})u^k_{2k+5}]
\end{pmatrix}
$$

$$
+\left(\begin{array}{c}
\frac{-(48+9\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-4} \\
+(-24-15\sqrt{2})u^j_{2k-3}+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1} \\
+(48+9\sqrt{2})u^j_{2k}+(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2} \\
+(6-5\sqrt{2})u^j_{2k+3}]^2\}[(-24+15\sqrt{2})u^j_{2k-4}+(-24-15\sqrt{2})u^j_{2k-3} \\
+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1}+(48+9\sqrt{2})u^j_{2k}+ \\
(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2}+(6-5\sqrt{2})u^j_{2k+3}] \\
\\
\frac{-(-72+39\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-4} \\
+(-24-15\sqrt{2})u^j_{2k-3}+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1} \\
+(48+9\sqrt{2})u^j_{2k}+(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2} \\
+(6-5\sqrt{2})u^j_{2k+3}]^2\}[(-24+15\sqrt{2})u^j_{2k-4}+(-24-15\sqrt{2})u^j_{2k-3} \\
+(74+27\sqrt{2})u^j_{2k-2}+(-6-67\sqrt{2})u^j_{2k-1}+(48+9\sqrt{2})u^j_{2k}+ \\
(-72+39\sqrt{2})u^j_{2k+1}+(-2-3\sqrt{2})u^j_{2k+2}+(6-5\sqrt{2})u^j_{2k+3}]
\end{array}\right)
$$

$$
+\left(\begin{array}{c}
\frac{-(72+39\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-4} \\
+(24+15\sqrt{2})u^j_{2k-3}+(6-67\sqrt{2})u^j_{2n-2}+(-74+27\sqrt{2})u^j_{2k-1} \\
+(72+39\sqrt{2})u^j_{2k}+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2} \\
+(2-3\sqrt{2})u^j_{2k+3}]^2\}[(24-15\sqrt{2})u^j_{2k-4}+(24+15\sqrt{2})u^j_{2k-3} \\
+(6-67\sqrt{2})u^j_{2k-2}+(-74+27\sqrt{2})u^j_{2k-1}+(72+39\sqrt{2})u^j_{2k} \\
+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2}+(2-3\sqrt{2})u^j_{2k+3}] \\
\\
\frac{-(-48+9\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-4} \\
+(24+15\sqrt{2})u^j_{2k-3}+(6-67\sqrt{2})u^j_{2n-2}+(-74+27\sqrt{2})u^j_{2k-1} \\
+(72+39\sqrt{2})u^j_{2k}+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2} \\
+(2-3\sqrt{2})u^j_{2k+3}]^2\}[(24-15\sqrt{2})u^j_{2k-4}+(24+15\sqrt{2})u^j_{2k-3} \\
+(6-67\sqrt{2})u^j_{2k-2}+(-74+27\sqrt{2})u^j_{2k-1}+(72+39\sqrt{2})u^j_{2k} \\
+(-48+9\sqrt{2})u^j_{2k+1}+(-6-5\sqrt{2})u^j_{2k+2}+(2-3\sqrt{2})u^j_{2k+3}]
\end{array}\right)
$$

$$
+\begin{pmatrix}
\begin{array}{c}
\frac{-(-2-3\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-6} \\
+(-24-15\sqrt{2})u^j_{2k-5}+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3} \\
+(48+9\sqrt{2})u^j_{2k-2}+(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k} \\
+(6-5\sqrt{2})u^j_{2k+1}]^2\}[(-24+15\sqrt{2})u^j_{2k-6}+(-24-15\sqrt{2})u^j_{2k-5} \\
+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3}+(48+9\sqrt{2})u^j_{2k-2}+ \\
(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k}+(6-5\sqrt{2})u^j_{2k+1}]
\end{array} \\
\\
\begin{array}{c}
\frac{-(6-5\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})u^j_{2k-6} \\
+(-24-15\sqrt{2})u^j_{2k-5}+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3} \\
+(48+9\sqrt{2})u^j_{2k-2}+(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k} \\
+(6-5\sqrt{2})u^j_{2k+1}]^2\}[(-24+15\sqrt{2})u^j_{2k-6}+(-24-15\sqrt{2})u^j_{2k-5} \\
+(74+27\sqrt{2})u^j_{2k-4}+(-6-67\sqrt{2})u^j_{2k-3}+(48+9\sqrt{2})u^j_{2k-2}+ \\
(-72+39\sqrt{2})u^j_{2k-1}+(-2-3\sqrt{2})u^j_{2k}+(6-5\sqrt{2})u^j_{2k+1}]
\end{array}
\end{pmatrix}
$$

$$
+\begin{pmatrix}
\begin{array}{c}
\frac{-(-6-5\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-6} \\
+(24+15\sqrt{2})u^j_{2k-5}+(6-67\sqrt{2})u^j_{2k-4}+(-74+27\sqrt{2})u^j_{2k-3} \\
+(72+39\sqrt{2})u^j_{2k-2}+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k} \\
+(2-3\sqrt{2})u^j_{2k+1}]^2\}[(24-15\sqrt{2})u^j_{2k-6}+(24+15\sqrt{2})u^j_{2k-5} \\
+(6-67\sqrt{2})u^j_{2n-4}+(-74+27\sqrt{2})u^j_{2k-3}+(72+39\sqrt{2})u^j_{2k-2} \\
+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k}+(2-3\sqrt{2})u^j_{2k+1}]
\end{array} \\
\\
\begin{array}{c}
\frac{-(2-3\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})u^j_{2k-6} \\
+(24+15\sqrt{2})u^j_{2k-5}+(6-67\sqrt{2})u^j_{2k-4}+(-74+27\sqrt{2})u^j_{2k-3} \\
+(72+39\sqrt{2})u^j_{2k-2}+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k} \\
+(2-3\sqrt{2})u^k_{2k+1}]^2\}[(24-15\sqrt{2})u^j_{2k-6}+(24+15\sqrt{2})u^j_{2k-5} \\
+(6-67\sqrt{2})u^j_{2k-4}+(-74+27\sqrt{2})u^j_{2k-3}+(72+39\sqrt{2})u^j_{2k-2} \\
+(-48+9\sqrt{2})u^j_{2k-1}+(-6-5\sqrt{2})u^j_{2k}+(2-3\sqrt{2})u^j_{2k+1}]
\end{array}
\end{pmatrix}
$$

When $j = 0$ with $\binom{u^0_{2k}}{u^0_{2k+1}} = \binom{c_{2k}}{c_{2k+1}}$, we get:

$$
\begin{aligned}
\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix} &= \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} \\
&+ \begin{pmatrix}
\begin{matrix}
\frac{-(-24+15\sqrt{2})\tau g}{(-32+16\sqrt{2})^2 h^2}\{\frac{1}{(-32+16\sqrt{2})^2 h^2}[(-24+15\sqrt{2})c_{2k} \\
+(-24-15\sqrt{2})c_{2k+1} + (74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} \\
+(48+9\sqrt{2})c_{2k+4} + (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} \\
+(6-5\sqrt{2})c_{2k+7}]^2\}[(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\
+(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4}+ \\
(-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} + (6-5\sqrt{2})c_{2k+7}]
\end{matrix} \\
\\
\begin{matrix}
\frac{-(-24-15\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2 h^2}[(-24+15\sqrt{2})c_{2k} \\
+(-24-15\sqrt{2})c_{2k+1} + (74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} \\
+(48+9\sqrt{2})c_{2k+4} + (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} \\
+(6-5\sqrt{2})c_{2k+7}]^2\}[(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\
+(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4}+ \\
(-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} + (6-5\sqrt{2})c_{2k+7}]
\end{matrix}
\end{pmatrix} \\
&+ \begin{pmatrix}
\begin{matrix}
\frac{-(24-15\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2 h^2}[(24-15\sqrt{2})c_{2k} \\
+(24+15\sqrt{2})c_{2k+1} + (6-67\sqrt{2})c_{2n+2} + (-74+27\sqrt{2})c_{2k+3} \\
+(72+39\sqrt{2})c_{2k+4} + (-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} \\
+(2-3\sqrt{2})c_{2k+7}]^2\}[(24-15\sqrt{2})c_{2k} + (24+15\sqrt{2})c_{2k+1} \\
+(6-67\sqrt{2})c_{2n+2} + (-74+27\sqrt{2})c_{2k+3} + (72+39\sqrt{2})c_{2k+4} \\
+(-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} + (2-3\sqrt{2})c_{2k+7}]
\end{matrix} \\
\\
\begin{matrix}
\frac{-(24+15\sqrt{2})\tau g}{(-160+112\sqrt{2})^2 h^2}\{\frac{1}{(-160+112\sqrt{2})^2 h^2}[(24-15\sqrt{2})c_{2k} \\
+(24+15\sqrt{2})c_{2k+1} + (6-67\sqrt{2})c_{2k+2} + (-74+27\sqrt{2})c_{2k+3} \\
+(72+39\sqrt{2})c_{2k+4} + (-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} \\
+(2-3\sqrt{2})c_{2k+7}]^2\}[(24-15\sqrt{2})c_{2k} + (24+15\sqrt{2})c_{2k+1} \\
+(6-67\sqrt{2})c_{2n+2} + (-74+27\sqrt{2})c_{2k+3} + (72+39\sqrt{2})c_{2k+4} \\
+(-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} + (2-3\sqrt{2})c_{2k+7}]
\end{matrix}
\end{pmatrix}
\end{aligned}
$$

$$
+\begin{pmatrix}
\begin{array}{c}
\frac{-(74+27\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-2} \\
+(-24-15\sqrt{2})c_{2k-1}+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1} \\
+(48+9\sqrt{2})c_{2k+2}+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4} \\
+(6-5\sqrt{2})c_{2k+5}]^2\}[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\
+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2}+ \\
(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}]
\end{array} \\
\\
\begin{array}{c}
\frac{-(-6-67\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-2} \\
+(-24-15\sqrt{2})c_{2k-1}+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1} \\
+(48+9\sqrt{2})c_{2k+2}+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4} \\
+(6-5\sqrt{2})c_{2k+5}]^2\}[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\
+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2}+ \\
(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}]
\end{array}
\end{pmatrix}
$$

$$
+\begin{pmatrix}
\begin{array}{c}
\frac{-(6-67\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-2} \\
+(24+15\sqrt{2})c_{2k-1}+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1} \\
+(72+39\sqrt{2})c_{2k+2}+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4} \\
+(2-3\sqrt{2})c_{2k+5}]^2\}[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}]
\end{array} \\
\\
\begin{array}{c}
\frac{-(-74+27\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-2} \\
+(24+15\sqrt{2})c_{2k-1}+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1} \\
+(72+39\sqrt{2})c_{2k+2}+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4} \\
+(2-3\sqrt{2})c_{2k+5}]^2\}[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})u^k_{2k+5}]
\end{array}
\end{pmatrix}
$$

$$+\left(\begin{array}{c}\frac{-(48+9\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-4}\\ +(-24-15\sqrt{2})c_{2k-3}+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}\\ +(48+9\sqrt{2})c_{2k}+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}\\ +(6-5\sqrt{2})c_{2k+3}]^2\}[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3}\\ +(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k}+\\ (-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}]\\ \\ \frac{-(-72+39\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-4}\\ +(-24-15\sqrt{2})c_{2k-3}+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}\\ +(48+9\sqrt{2})c_{2k}+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}\\ +(6-5\sqrt{2})c_{2k+3}]^2\}[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3}\\ +(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k}+\\ (-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}]\end{array}\right)$$

$$+\left(\begin{array}{c}\frac{-(72+39\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-4}\\ +(24+15\sqrt{2})c_{2k-3}+(6-67\sqrt{2})c_{2n-2}+(-74+27\sqrt{2})c_{2k-1}\\ +(72+39\sqrt{2})c_{2k}+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}\\ +(2-3\sqrt{2})c_{2k+3}]^2\}[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3}\\ +(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k}\\ +(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}]\\ \\ \frac{-(-48+9\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-4}\\ +(24+15\sqrt{2})c_{2k-3}+(6-67\sqrt{2})c_{2n-2}+(-74+27\sqrt{2})c_{2k-1}\\ +(72+39\sqrt{2})c_{2k}+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}\\ +(2-3\sqrt{2})c_{2k+3}]^2\}[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3}\\ +(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k}\\ +(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}]\end{array}\right)$$

$$
+\left(\begin{array}{c}
\frac{-(-2-3\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-6} \\
+(-24-15\sqrt{2})c_{2k-5}+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3} \\
+(48+9\sqrt{2})c_{2k-2}+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k} \\
+(6-5\sqrt{2})c_{2k+1}]^2\}[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2}+ \\
(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}] \\
\\
\frac{-(6-5\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-6} \\
+(-24-15\sqrt{2})c_{2k-5}+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3} \\
+(48+9\sqrt{2})c_{2k-2}+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k} \\
+(6-5\sqrt{2})c_{2k+1}]^2\}[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2}+ \\
(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}]
\end{array}\right)
$$

$$
+\left(\begin{array}{c}
\frac{-(-6-5\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-6} \\
+(24+15\sqrt{2})c_{2k-5}+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3} \\
+(72+39\sqrt{2})c_{2k-2}+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k} \\
+(2-3\sqrt{2})c_{2k+1}]^2\}[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2n-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}] \\
\\
\frac{-(2-3\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-6} \\
+(24+15\sqrt{2})c_{2k-5}+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3} \\
+(72+39\sqrt{2})c_{2k-2}+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k} \\
+(2-3\sqrt{2})c_{2k+1}]^2\}[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}]
\end{array}\right)
$$

write:

$$\begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} = \frac{1}{57600} \begin{pmatrix} (36-12\sqrt{2})c_{2k-6} + (156+108\sqrt{2})c_{2k-5} - 456\sqrt{2}c_{2k-4} \\ +(-2064-1440\sqrt{2})c_{2k-3} + (540+4620\sqrt{2})c_{2k-2} \\ +(9060+7164\sqrt{2})c_{2k-1} + (28800-8304\sqrt{2})c_{2k} + 13344\ c_{2k+1} \\ +(540+4620\sqrt{2})c_{2k+2} + (9060-7164)c_{2k+3} - 456\sqrt{2}c_{2k+4} \\ +(-2064+1440\sqrt{2})c_{2k+5} + (36-12\sqrt{2})c_{2k+6} + (156-108\sqrt{2})c_{2k+7} \\ \\ \\ (156-108\sqrt{2})c_{2k-6} + (36+12\sqrt{2})c_{2k-5} + (-2064+1440\sqrt{2})c_{2k-4} \\ +456\sqrt{2}\ c_{2k-3} + (9060-7164\sqrt{2})c_{2k-2} + (540-4620\sqrt{2})c_{2k-1} \\ +13344\ c_{2k} + (28800+8304\sqrt{2})c_{2k+1} + (9060+7164\sqrt{2})c_{2k+2} \\ +(540-4620\sqrt{2})c_{2k+3} + (-2064-1440\sqrt{2})c_{2k+4} + 456\ c_{2k+5} \\ +(156+108\sqrt{2})c_{2k+6} + (36+12\sqrt{2})c_{2k+7} \end{pmatrix}$$

$$+ \frac{1}{57600} \begin{pmatrix} (-24+15\sqrt{2})[(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\ +(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4}+ \\ (-72+39\sqrt{2})c_{2n+5} + (-2-3\sqrt{2})c_{2n+6} + (6-5\sqrt{2})c_{2n+7}] \\ \\ (-24-15\sqrt{2})[(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\ +(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4}+ \\ (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} + (6-5\sqrt{2})c_{2k+7}] \end{pmatrix}$$

$$+ \frac{1}{57600} \begin{pmatrix} (24-15\sqrt{2})[(24-15\sqrt{2})c_{2k} + (24+15\sqrt{2})c_{2k+1} \\ (6-67\sqrt{2})c_{2k+2} + (-74+27\sqrt{2})c_{2k+3} + (72+39\sqrt{2})c_{2k+4}+ \\ (-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} + (2-3\sqrt{2})c_{2k+7}] \\ \\ (24+15\sqrt{2})[(24-15\sqrt{2})c_{2k} + (24+15\sqrt{2})c_{2k+1} \\ (6-67\sqrt{2})c_{2k+2} + (-74+27\sqrt{2})c_{2k+3} + (72+39\sqrt{2})c_{2k+4}+ \\ (-48+9\sqrt{2})c_{2k+5} + (-6-5\sqrt{2})c_{2k+6} + (2-3\sqrt{2})c_{2k+7}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (74+27\sqrt{2})[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\ (74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2} \\ +(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}] \\ \\ (-6-67\sqrt{2})[(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\ (74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2} \\ +(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (6-67\sqrt{2})[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\ (6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\ +(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}] \\ \\ (-74+27\sqrt{2})[(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\ (6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\ +(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (48+9\sqrt{2})[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\ (74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k} \\ +(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}] \\ \\ (-72+39\sqrt{2})[(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\ (74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k} \\ +(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (72+39\sqrt{2})[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\ (6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\ +(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}] \\ \\ (-48+9\sqrt{2})[(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\ (6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\ +(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (-2-3\sqrt{2})[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\ +(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2} \\ +(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}] \\ \\ (6-5\sqrt{2})[(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\ +(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2} \\ +(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}] \end{pmatrix}$$

$$+\frac{1}{57600}\begin{pmatrix} (-6-5\sqrt{2})[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\ (6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\ +(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}] \\ \\ (2-3\sqrt{2})[(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\ (6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\ +(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}] \end{pmatrix}.$$

Then the second-order nonlinear diffusion equation can be written as:

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1} \end{pmatrix} = \frac{1}{57600} \begin{pmatrix} (36-12\sqrt{2})c_{2k-6} + (156+108\sqrt{2})c_{2k-5} - 456\sqrt{2}c_{2k-4} \\ +(-2064-1440\sqrt{2})c_{2k-3} + (540+4620\sqrt{2})c_{2k-2} \\ +(9060+7164\sqrt{2})c_{2k-1} + (28800-8304\sqrt{2})c_{2k} + 13344\ c_{2k+1} \\ +(540+4620\sqrt{2})c_{2k+2} + (9060-7164)c_{2k+3} - 456\sqrt{2}c_{2k+4} \\ +(-2064+1440\sqrt{2})c_{2k+5} + (36-12\sqrt{2})c_{2k+6} + (156-108\sqrt{2})c_{2k+7} \\ \\ (156-108\sqrt{2})c_{2k-6} + (36+12\sqrt{2})c_{2k-5} + (-2064+1440\sqrt{2})c_{2k-4} \\ +456\sqrt{2}\ c_{2k-3} + (9060-7164\sqrt{2})c_{2k-2} + (540-4620\sqrt{2})c_{2k-1} \\ +13344\ c_{2k} + (28800+8304\sqrt{2})c_{2k+1} + (9060+7164\sqrt{2})c_{2k+2} \\ +(540-4620\sqrt{2})c_{2k+3} + (-2064-1440\sqrt{2})c_{2k+4} + 456\ c_{2k+5} \\ +(156+108\sqrt{2})c_{2k+6} + (36+12\sqrt{2})c_{2k+7} \end{pmatrix}$$

$$+ \begin{pmatrix} [\frac{(-24+15\sqrt{2})}{57600} - \frac{(-24+15\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k} \\ +(-24-15\sqrt{2})c_{2k+1} + (74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} \\ +(48+9\sqrt{2})c_{2k+4} + (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} \\ +(6-5\sqrt{2})c_{2k+7}]^2\}]\ \ [(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\ +(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4} + \\ (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} + (6-5\sqrt{2})c_{2k+7}] \\ \\ [\frac{(-24-15\sqrt{2})}{57600} - \frac{(-24-15\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k} \\ +(-24-15\sqrt{2})c_{2k+1} + (74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} \\ +(48+9\sqrt{2})c_{2k+4} + (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} \\ +(6-5\sqrt{2})c_{2k+7}]^2\}]\ \ [(-24+15\sqrt{2})c_{2k} + (-24-15\sqrt{2})c_{2k+1} \\ +(74+27\sqrt{2})c_{2k+2} + (-6-67\sqrt{2})c_{2k+3} + (48+9\sqrt{2})c_{2k+4} + \\ (-72+39\sqrt{2})c_{2k+5} + (-2-3\sqrt{2})c_{2k+6} + (6-5\sqrt{2})c_{2k+7}] \end{pmatrix}$$

$$+\begin{pmatrix} [\frac{(24-15\sqrt{2})}{57600}-\frac{(24-15\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k} \\ +(24+15\sqrt{2})c_{2k+1}+(6-67\sqrt{2})c_{2n+2}+(-74+27\sqrt{2})c_{2k+3} \\ +(72+39\sqrt{2})c_{2k+4}+(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6} \\ +(2-3\sqrt{2})c_{2k+7}]^2\}] \quad [(24-15\sqrt{2})c_{2k}+(24+15\sqrt{2})c_{2k+1} \\ +(6-67\sqrt{2})c_{2n+2}+(-74+27\sqrt{2})c_{2k+3}+(72+39\sqrt{2})c_{2k+4} \\ +(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6}+(2-3\sqrt{2})c_{2k+7}] \\ \\ [\frac{(24+15\sqrt{2})}{57600}-\frac{(24+15\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k} \\ +(24+15\sqrt{2})c_{2k+1}+(6-67\sqrt{2})c_{2k+2}+(-74+27\sqrt{2})c_{2k+3} \\ +(72+39\sqrt{2})c_{2k+4}+(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6} \\ +(2-3\sqrt{2})c_{2k+7}]^2\}] \quad [(24-15\sqrt{2})c_{2k}+(24+15\sqrt{2})c_{2k+1} \\ +(6-67\sqrt{2})c_{2n+2}+(-74+27\sqrt{2})c_{2k+3}+(72+39\sqrt{2})c_{2k+4} \\ +(-48+9\sqrt{2})c_{2k+5}+(-6-5\sqrt{2})c_{2k+6}+(2-3\sqrt{2})c_{2k+7}] \end{pmatrix}$$

$$+\begin{pmatrix} [\frac{(74+27\sqrt{2})}{57600}-\frac{(74+27\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-2} \\ +(-24-15\sqrt{2})c_{2k-1}+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1} \\ +(48+9\sqrt{2})c_{2k+2}+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4} \\ +(6-5\sqrt{2})c_{2k+5}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\ +(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2}+ \\ (-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}] \\ \\ [\frac{(-6-67\sqrt{2})}{57600}-\frac{(-6-67\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-2} \\ +(-24-15\sqrt{2})c_{2k-1}+(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1} \\ +(48+9\sqrt{2})c_{2k+2}+(-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4} \\ +(6-5\sqrt{2})c_{2k+5}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-2}+(-24-15\sqrt{2})c_{2k-1} \\ +(74+27\sqrt{2})c_{2k}+(-6-67\sqrt{2})c_{2k+1}+(48+9\sqrt{2})c_{2k+2}+ \\ (-72+39\sqrt{2})c_{2k+3}+(-2-3\sqrt{2})c_{2k+4}+(6-5\sqrt{2})c_{2k+5}] \end{pmatrix}$$

$$
+\begin{pmatrix}
[\frac{(6-67\sqrt{2})}{57600}-\frac{(6-67\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-2} \\
+(24+15\sqrt{2})c_{2k-1}+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1} \\
+(72+39\sqrt{2})c_{2k+2}+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4} \\
+(2-3\sqrt{2})c_{2k+5}]^2\}] \quad [(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})c_{2k+5}] \\
\\
[\frac{(-74+27\sqrt{2})}{57600}-\frac{(-74+27\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-2} \\
+(24+15\sqrt{2})c_{2k-1}+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1} \\
+(72+39\sqrt{2})c_{2k+2}+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4} \\
+(2-3\sqrt{2})c_{2k+5}]^2\}] \quad [(24-15\sqrt{2})c_{2k-2}+(24+15\sqrt{2})c_{2k-1} \\
+(6-67\sqrt{2})c_{2k}+(-74+27\sqrt{2})c_{2k+1}+(72+39\sqrt{2})c_{2k+2} \\
+(-48+9\sqrt{2})c_{2k+3}+(-6-5\sqrt{2})c_{2k+4}+(2-3\sqrt{2})u^k_{2k+5}]
\end{pmatrix}
$$

$$
+\begin{pmatrix}
[\frac{(48+9\sqrt{2})}{57600}-\frac{(48+9\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-4} \\
+(-24-15\sqrt{2})c_{2k-3}+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1} \\
+(48+9\sqrt{2})c_{2k}+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2} \\
+(6-5\sqrt{2})c_{2k+3}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\
+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k}+ \\
(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}] \\
\\
[\frac{(-72+39\sqrt{2})}{57600}-\frac{(-72+39\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-4} \\
+(-24-15\sqrt{2})c_{2k-3}+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1} \\
+(48+9\sqrt{2})c_{2k}+(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2} \\
+(6-5\sqrt{2})c_{2k+3}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-4}+(-24-15\sqrt{2})c_{2k-3} \\
+(74+27\sqrt{2})c_{2k-2}+(-6-67\sqrt{2})c_{2k-1}+(48+9\sqrt{2})c_{2k}+ \\
(-72+39\sqrt{2})c_{2k+1}+(-2-3\sqrt{2})c_{2k+2}+(6-5\sqrt{2})c_{2k+3}]
\end{pmatrix}
$$

$$
+\left(\begin{array}{c}
[\frac{(72+39\sqrt{2})}{57600}-\frac{(72+39\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-4} \\
+(24+15\sqrt{2})c_{2k-3}+(6-67\sqrt{2})c_{2n-2}+(-74+27\sqrt{2})c_{2k-1} \\
+(72+39\sqrt{2})c_{2k}+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2} \\
+(2-3\sqrt{2})c_{2k+3}]^2\}] \quad [(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\
+(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\
+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}] \\
\\
[\frac{(-48+9\sqrt{2})}{57600}-\frac{(-48+9\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-4} \\
+(24+15\sqrt{2})c_{2k-3}+(6-67\sqrt{2})c_{2n-2}+(-74+27\sqrt{2})c_{2k-1} \\
+(72+39\sqrt{2})c_{2k}+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2} \\
+(2-3\sqrt{2})c_{2k+3}]^2\}] \quad [(24-15\sqrt{2})c_{2k-4}+(24+15\sqrt{2})c_{2k-3} \\
+(6-67\sqrt{2})c_{2k-2}+(-74+27\sqrt{2})c_{2k-1}+(72+39\sqrt{2})c_{2k} \\
+(-48+9\sqrt{2})c_{2k+1}+(-6-5\sqrt{2})c_{2k+2}+(2-3\sqrt{2})c_{2k+3}]
\end{array}\right)
$$

$$
+\left(\begin{array}{c}
[\frac{(-2-3\sqrt{2})}{57600}-\frac{(-2-3\sqrt{2})\tau g}{(-32+16\sqrt{2})^2h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-6} \\
+(-24-15\sqrt{2})c_{2k-5}+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3} \\
+(48+9\sqrt{2})c_{2k-2}+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k} \\
+(6-5\sqrt{2})c_{2k+1}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2}+ \\
(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}] \\
\\
[\frac{(6-5\sqrt{2})}{57600}-\frac{(6-5\sqrt{2})\tau g}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}\{\frac{1}{(-32+16\sqrt{2})^2h^2}[(-24+15\sqrt{2})c_{2k-6} \\
+(-24-15\sqrt{2})c_{2k-5}+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3} \\
+(48+9\sqrt{2})c_{2k-2}+(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k} \\
+(6-5\sqrt{2})c_{2k+1}]^2\}] \quad [(-24+15\sqrt{2})c_{2k-6}+(-24-15\sqrt{2})c_{2k-5} \\
+(74+27\sqrt{2})c_{2k-4}+(-6-67\sqrt{2})c_{2k-3}+(48+9\sqrt{2})c_{2k-2}+ \\
(-72+39\sqrt{2})c_{2k-1}+(-2-3\sqrt{2})c_{2k}+(6-5\sqrt{2})c_{2k+1}]
\end{array}\right)
$$

$$
+\begin{pmatrix}
[\frac{(-6-5\sqrt{2})}{57600}-\frac{(-6-5\sqrt{2})\tau g}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-6} \\
+(24+15\sqrt{2})c_{2k-5}+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3} \\
+(72+39\sqrt{2})c_{2k-2}+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k} \\
+(2-3\sqrt{2})c_{2k+1}]^2\}] \;\; [(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2n-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}] \\
\\
[\frac{(2-3\sqrt{2})}{57600}-\frac{(2-3\sqrt{2})\tau g}{(-160+112\sqrt{2})^2h^2}\{\frac{1}{(-160+112\sqrt{2})^2h^2}[(24-15\sqrt{2})c_{2k-6} \\
+(24+15\sqrt{2})c_{2k-5}+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3} \\
+(72+39\sqrt{2})c_{2k-2}+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k} \\
+(2-3\sqrt{2})c_{2k+1}]^2\}] \; [(24-15\sqrt{2})c_{2k-6}+(24+15\sqrt{2})c_{2k-5} \\
+(6-67\sqrt{2})c_{2k-4}+(-74+27\sqrt{2})c_{2k-3}+(72+39\sqrt{2})c_{2k-2} \\
+(-48+9\sqrt{2})c_{2k-1}+(-6-5\sqrt{2})c_{2k}+(2-3\sqrt{2})c_{2k+1}]
\end{pmatrix} \tag{5.18}
$$

Theorem (5.3): If $\binom{u_{1,k}}{u_{2,k}}$ in Eq.(5.16) is the denoised signal after one-step (DGHM) multiwavelet shrinking with $\underline{c}_k^0 = \binom{c_{2k}}{c_{2k+1}} = f(kh), k \in \mathbb{Z}$ as the original input and $\binom{u_{2k}^1}{u_{2k+1}^1}$ in Eq.(5.18) is the signal after 1-step diffusing with original data $\underline{u}_k^0 = \binom{u_{2k}^0}{u_{2k+1}^0} = f(kh)$, then

$$
u_{1,k} = u_{2k}^1 \quad if \quad \begin{cases} S_{\theta_{11}}(x) = x[1-\frac{57600\tau}{(-32+16\sqrt{2})^2h^2}g(\frac{57600\ x^2}{(-32+16\sqrt{2})^2h^2})] \\ S_{\theta_{21}}(x) = x[1-\frac{57600\tau}{(-32+16\sqrt{2})(-160+112\sqrt{2})h^2}g(\frac{57600\ x^2}{(-160+112\sqrt{2})^2h^2})] \end{cases},
$$

and

$$
u_{2,k} = u_{2k+1}^1 \quad if \quad \begin{cases} S_{\theta_{12}}(x) = x[1-\frac{57600\tau}{(-160+112\sqrt{2})(-32+16\sqrt{2})h^2}g(\frac{57600\ x^2}{(-32+16\sqrt{2})^2h^2})] \\ S_{\theta_{22}}(x) = x[1-\frac{57600\tau}{(-160+112\sqrt{2})^2h^2}g(\frac{57600\ x^2}{(-160+112\sqrt{2})^2h^2})] \end{cases}.
$$

5.4 Equivalence between multiwavelet shrinkage and high-order nonlinear diffusion equation

5.4.1 Multiwavelet shrinkage in general case

Let $P(w)$, $Q(w)$ be orthogonal multiwavelet filter bank satisfying:

$$
P^*(w)\ P(w) + Q^*(w)\ Q(w) = I_2, \tag{5.19}
$$

and suppose that

$$P(0)\binom{1}{1} = \binom{1}{1} \quad and \quad Q(0)\binom{1}{1} = \binom{0}{0}.$$

Let $\underline{c}_k^0 = \binom{c_{2k}}{c_{2k+1}}$ be an initial input, then undecimated discrete multiwavelet transform is defined as:

$$\begin{pmatrix} L_{1,n} \\ L_{2,n} \end{pmatrix} = \sum_k P_k \begin{pmatrix} c_{2(k+n)} \\ c_{2(k+n)+1} \end{pmatrix}$$
$$\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} = \sum_k Q_k \begin{pmatrix} c_{2(k+n)} \\ c_{2(k+n)+1} \end{pmatrix}$$

and the denoising algorithm is:

$$\begin{aligned}
\begin{pmatrix} u_{1,k} \\ u_{2,k} \end{pmatrix} &= \sum_n P_n^T \begin{pmatrix} L_{1,(k-n)} \\ L_{2,(k-n)} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= \sum_n P_n^T \sum_m P_m \begin{pmatrix} c_{2(m+k-n)} \\ c_{2(m+k-n)+1} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= \sum_n \sum_j P_n^T P_{n+j} \begin{pmatrix} c_{2(k+j)} \\ c_{2(k+j)+1} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= [\begin{pmatrix} \delta(j) & 0 \\ 0 & \delta(j) \end{pmatrix} - \sum_n \sum_j Q_n^T Q_{n+j}] \begin{pmatrix} c_{2(k+j)} \\ c_{2(k+j)+1} \end{pmatrix} \\
&+ \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} - \sum_n \sum_j Q_n^T Q_{n+j} \begin{pmatrix} c_{2(k+j)} \\ c_{2(k+j)+1} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} - \sum_n Q_n^T \begin{pmatrix} H_{1,(k-n)} \\ H_{2,(k-n)} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11} S_{\theta_{11}}(H_{1,(k-n)}) + q_{n,21} S_{\theta_{21}}(H_{2,(k-n)}) \\ q_{n,12} S_{\theta_{12}}(H_{1,(k-n)}) + q_{n,22} S_{\theta_{22}}(H_{2,(k-n)}) \end{pmatrix} \\
&= \begin{pmatrix} c_{2k} \\ c_{2k+1} \end{pmatrix} + \sum_n \begin{pmatrix} q_{n,11}\ \{S_{\theta_{11}}(H_{1,(k-n)}) - H_{1,(k-n)}\}\ + q_{n,21}\ \{S_{\theta_{21}}(H_{2,(k-n)}) - H_{2,(k-n)}\} \\ q_{n,12}\ \{S_{\theta_{12}}(H_{1,(k-n)}) - H_{1,(k-n)}\}\ + q_{n,22}\ \{S_{\theta_{22}}(H_{2,(k-n)}) - H_{2,(k-n)}\} \end{pmatrix}
\end{aligned} \tag{5.20}$$

5.4.2 Multiwavelet shrinkage for high order-nonlinear diffusion

Suppose

$$u_t = (-1)^{1+\beta} \frac{\partial^\beta}{\partial x^\beta} \{ g(\frac{\partial^\beta u}{\partial x^\beta})^2 \frac{\partial^\beta u}{\partial x^\beta} \} \tag{5.21}$$

is the high-order nonlinear diffusion equation, and $u(x,0) = f(x)$ is the the initial condition.

For a highpass filter, suppose that

$$Q_k(w) = \begin{pmatrix} a_k(w) & b_k(w) \\ s_k(w) & d_k(w) \end{pmatrix}$$

then the highpass outputs can be written as:

$$\begin{aligned}
\begin{pmatrix} H_{1,n} \\ H_{2,n} \end{pmatrix} &= \sum_{k=0}^{N} \begin{pmatrix} a_k & b_k \\ s_k & d_k \end{pmatrix} \begin{pmatrix} c_{2(k+n)} \\ c_{2(k+n)+1} \end{pmatrix} \\
&= \sum_{k=0}^{N} \begin{pmatrix} a_k c_{2(k+n)} + b_k c_{2(k+n)+1} \\ s_k c_{2(k+n)} + d_k c_{2(k+n)+1} \end{pmatrix} \\
&= \sum_{k=0}^{N} \begin{pmatrix} e_k c_{k+n} \\ f_k c_{k+n} \end{pmatrix}
\end{aligned}$$

with the high-pass multifilter bank Q, the normalized highpass frequency response can be defined as:

$$e(w) = \sum_{k=0}^{N} (a_k e^{-2ikw} + b_k e^{-i(2k+1)w})$$

$$f(w) = \sum_{k=0}^{N} (s_{k+1} e^{-i(2k+1)w} + d_k e^{-2ikw})$$

which equivalent to

$$e(w) = \sum_k e_k e^{-ikw}$$

$$f(w) = \sum_k f_k e^{-ikw}$$

The highpass filter $e(w) = \sum_k e_k e^{-ikw}$ has vanishing moment order J if

$$\sum k^j e_k = 0 \quad with \quad 0 \leq j < J$$

Denote $C_J = \frac{1}{J!} \sum k^J e_k$
Similarly, the highpass filter $f(w) = \sum_k f_k e^{-ikw}$ has vanishing moment order J if

$$\sum k^j f_k = 0 \quad with \quad 0 \leq j < J$$

Denote $D_J = \frac{1}{J!} \sum k^J f_k$
We can use a highpass filters to approximate the derivative of a function, so the

approximation of $\frac{\partial^\beta}{\partial x^\beta}u(x,t)$ at $(2kh, j\tau)$ and $[(2k+1)h, j\tau]$ by using highpass filters are defined respectively by:

$$\begin{aligned}
\frac{\partial^\beta}{\partial x^\beta}(u(2kh, j\tau)) &\approx \frac{1}{C_\beta h^\beta}\sum_n [a_n \ b_n] \begin{pmatrix} u(2(k+n)h, j\tau) \\ u([2(k+n)+1]h, j\tau) \end{pmatrix} \\
&= \frac{1}{C_\beta h^\beta}\sum_n [a_n \ b_n] \begin{pmatrix} u^j_{2(k+n)} \\ u^j_{2(k+n)+1} \end{pmatrix}
\end{aligned}$$

let $j = 0$, with $u^0_{2k} = c_{2k}$ and $u^0_{2k+1} = c_{2k+1}$

$$\begin{aligned}
\frac{\partial^\beta}{\partial x^\beta}(u(2kh)) &\approx \frac{1}{C_\beta h^\beta}\sum_n (a_n c_{2n+2k} + b_n c_{2n+1+2k}) \\
&= \frac{1}{C_\beta h^\beta}\sum_n (e_{2n} c_{2n+2k} + e_{2n+1} c_{2n+1+2k}) \\
&= \frac{1}{C_\beta h^\beta}\sum_m (e_m c_{m+2k}) \\
&= \frac{(e^- * c)_{(2k)}}{C_\beta h^\beta} = \frac{H_{1,k}}{C_\beta h^\beta} \\
\frac{\partial^\beta}{\partial x^\beta}(u([2k+1]h)) &\approx \frac{1}{D_\beta h^\beta}\sum_n [s_n \ d_n] \begin{pmatrix} u^0_{2(k+n)} \\ u^0_{2(k+n)+1} \end{pmatrix} \\
&= \frac{1}{D_\beta h^\beta}\sum_n (s_n c_{2n+2k} + d_n c_{2n+2k+1}) \\
&= \frac{1}{D_\beta h^\beta}\sum_n (s_n c_{2n-1+2k+1} + d_n c_{2n+2k+1}) \\
&= \frac{1}{D_\beta h^\beta}\sum_n (s_{n+1} c_{2n+1+2k+1} + d_n c_{2n+2k+1}) \\
&= \frac{1}{D_\beta h^\beta}\sum_n (f_{2n+1} c_{2n+1+2k+1} + f_{2n} c_{2n+2k+1}) \\
&= \frac{1}{D_\beta h^\beta}\sum_m (f_m c_{m+2k+1}) \\
&= \frac{(f^- * c)_{(2k+1)}}{D_\beta h^\beta} = \frac{H_{2,k}}{D_\beta h^\beta}
\end{aligned}$$

and the approximating partial derivatives of $\frac{\partial^\beta}{\partial x^\beta}G(x,t)$ where $G(x,t) := g[(\frac{\partial^\beta u}{\partial x^\beta})^2]\frac{\partial^\beta u}{\partial x^\beta}$ at $(2kh, j\tau)$,and $([2k+1]h, j\tau)$ are defined respectively as:

$$\frac{\partial^\beta}{\partial x^\beta}\big(G(2kh, j\tau)\big) \approx \frac{(-1)^\beta}{C_\beta h^\beta}\sum_m [a_m \ s_m]\begin{pmatrix} G(2(k-m)h, j\tau) \\ G([2(k-m)+1]h, j\tau)\end{pmatrix}, \quad let\ j=0$$

$$\begin{aligned}
\frac{\partial^\beta}{\partial x^\beta}\big(G(2kh)\big) &= \frac{(-1)^\beta}{C_\beta h^\beta}\sum_m [a_m G_{-2m+2k} + s_m G_{-2m+1+2k}] \\
&= \frac{(-1)^\beta}{C_\beta h^\beta}\sum_m [e_{-2m} G_{-2m+2k} + e_{-2m+1} G_{-2m+1+2k}] \\
&= \frac{(-1)^\beta}{C_\beta h^\beta}\sum_n (e_{-n} G_{-n+2k}) \\
&= \frac{(-1)^\beta}{C_\beta h^\beta}(e * G)_{(2k)}
\end{aligned}$$

$$\frac{\partial^\beta}{\partial x^\beta}\big(G([2k+1]h, j\tau)\big) \approx \frac{(-1)^\beta}{D_\beta h^\beta}\sum [b_m \ d_m]\begin{pmatrix} G(2(k-m)h, j\tau) \\ G([2(k-m)+1]h, j\tau)\end{pmatrix}; \quad j=0$$

$$\begin{aligned}
\frac{\partial^\beta}{\partial x^\beta}\big(G([2k+1]h)\big) &= \frac{(-1)^\beta}{D_\beta h^\beta}\sum_m [b_m G_{-2m-1+2k+1} + d_m G_{-2m+2k+1}] \\
&= \frac{(-1)^\beta}{D_\beta h^\beta}\sum_m [b_{m-1} G_{-2m+1+2k+1} + d_m G_{-2m+2k+1}] \\
&= \frac{(-1)^\beta}{D_\beta h^\beta}\sum_m [f_{-2m+1} G_{-2m+1+2k+1} + f_{-2m} G_{-2m+2k+1}] \\
&= \frac{(-1)^\beta}{D_\beta h^\beta}\sum_n (f_{-n} G_{-n+2k+1}) \\
&= \frac{(-1)^\beta}{D_\beta h^\beta}(f * G)_{(2k+1)}
\end{aligned}$$

With $\begin{pmatrix} u^0_{2k} \\ u^0_{2k+1}\end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1}\end{pmatrix}$, the high-order nonlinear diffusion equation (5.21) can be discretized as:

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1}\end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1}\end{pmatrix} + \begin{pmatrix} \frac{-\tau}{C_\beta h^\beta} & 0 \\ 0 & \frac{-\tau}{D_\beta h^\beta}\end{pmatrix}\sum_m Q^T_m \begin{pmatrix} \frac{g}{C_\beta h^\beta}(\frac{H_{1,(k-m)}}{C_\beta h^\beta})^2 \ H_{1,(k-m)} \\ \frac{g}{D_\beta h^\beta}(\frac{H_{2,(k-m)}}{D_\beta h^\beta})^2 \ H_{2,(k-m)}\end{pmatrix}$$

$$\begin{pmatrix} u^1_{2k} \\ u^1_{2k+1}\end{pmatrix} = \begin{pmatrix} c_{2k} \\ c_{2k+1}\end{pmatrix} - \tau\sum_m \begin{pmatrix} \frac{a_m}{(C_\beta)^2 h^{2\beta}} g(\frac{H_{1,(k-m)}}{C_\beta h^\beta})^2 \ H_{1,(k-m)} + \frac{s_m}{D_\beta C_\beta h^{2\beta}} g(\frac{H_{2,(k-m)}}{D_\beta h^\beta})^2 \ H_{2,(k-m)} \\ \frac{b_m}{D_\beta C_\beta h^{2\beta}} g(\frac{H_{1,(k-m)}}{C_\beta h^\beta})^2 \ H_{1,(k-m)} + \frac{d_m}{(D_\beta)^2 h^{2\beta}} g(\frac{H_{2,(k-m)}}{D_\beta h^\beta})^2 \ H_{2,(k-m)}\end{pmatrix} \tag{5.22}$$

Theorem (5.4): Suppose $\binom{u_{1,k}}{u_{2,k}}$ in Eq.(5.20) is the resulting signal after 1-step multiwavelet shrinking with $\underline{c}^0_k = \binom{c_{2k}}{c_{2k+1}}, k \in \mathbb{Z}$ as the original input, and $\binom{u^1_{2k}}{u^1_{2k+1}}$ in

Eq(5.22) is the signal after 1-step diffusing with original data $\underline{u}_k^0 = \binom{u_{2k}^0}{u_{2k+1}^0}$. If

$$S_{\theta_{11}}(x) = x[1 - \frac{\tau}{(C_\beta)^2 h^{2\beta}} g\ (\frac{x^2}{(C_\beta)^2 h^{2\beta}})]$$
$$S_{\theta_{21}}(x) = x[1 - \frac{\tau}{D_\beta\ C_\beta h^{2\beta}} g(\frac{x^2}{(D_\beta)^2 h^{2\beta}})],$$

then $u_{1,k} = u_{2k}^1$. Also, if

$$S_{\theta_{12}}(x) = x[1 - \frac{\tau}{D_\beta\ C_\beta h^{2\beta}} g(\frac{x^2}{(C_\beta)^2 h^{2\beta}})]$$
$$S_{\theta_{22}}(x) = x[1 - \frac{\tau}{(D_\beta)^2 h^{2\beta}} g(\frac{x^2}{(D_\beta)^2 h^{2\beta}})],$$

then $u_{2,k} = u_{2k+1}^1$.

5.5 Denoising of signals using diffusion-inspired multiwavelet shrinkage method

Denoising is a very significant technique used for signal enhancement applications of the multiwavelet. In multiwavelet, the signal is analyzed in a multi-dimensional way. Then the procedure is to apply wavelet shrinkage to each dimension of the multiwavelet coefficients for signal denoising instead of thresholding single elements. We apply signal denoising based on CL(2) multiwavelet filter banks with different shrinkage functions to different signals.

S_1, (a) in figure. 5.1, is the first signal. We generate five noised signals with signal to noise ratio SNR=6 by adding noise five times to S_1. The signal to noise ratio is given by:

$$SNR = 20(log_{10} |s - \bar{s}|_2 - log_{10} |n|_2)$$

where $\bar{s}$ is the mean of the signal s, and the noise is n. After adding the noise, we apply CL(2) multiwavelet shrinking iteratively 50 times to each noised signal.

The second signal is a chirp signal S_2,(c) in figure(5.1) which is defined as the signal that sweeps linearly from a low to a high frequency.

$$x(t) = cos(f_0 t + \frac{k}{2} t^2)$$

where f_0 is the starting frequency and $k = \frac{f_1 - f_0}{T}$ is the rate of the frequency. Also, we generate five noised signals with signal to noise ratio SNR=16 by adding noise five times to the original signal S_2. Then, we apply CL(2) multiwavelet shrinking iteratively 15 times to each noised signal.

Table 5.1 and Table 5.2 present the average of the signal to noise ratio of the signal denoising results of the five noised signals with different shrinkage functions for

Table 5.1: Signal denoising results using diffusion-inspired multiwavelet shrinkage functions.

Shrinkage Method	$S_{\theta_{11}} = S_{\theta_{21}} = PM$ $S_{\theta_{12}} = S_{\theta_{22}} = PM$	$S_{\theta_{11}} = S_{\theta_{21}} = Weickert$ $S_{\theta_{12}} = S_{\theta_{22}} = Weickert$	$S_{\theta_{11}} = S_{\theta_{21}} = Hard$ $S_{\theta_{12}} = S_{\theta_{22}} = Hard$	$S_{\theta_{11}} = S_{\theta_{21}} = Soft$ $S_{\theta_{12}} = S_{\theta_{22}} = Soft$	$S_{\theta_{11}} = S_{\theta_{21}} = PM$ $S_{\theta_{12}} = S_{\theta_{22}} = Weickert$	$S_{\theta_{11}} = S_{\theta_{21}} = Weickert$ $S_{\theta_{12}} = S_{\theta_{22}} = PM$
SNR(for S_1)	17.2881	17.7518	14.4435	14.4465	17.1656	17.7093
SNR(for S_2)	21.0010	21.1429	20.9975	20.9448	21.0151	20.9295

Table 5.2: Signal denoising results using diffusion-inspired wavelet shrinkage functions.

Shrinkage Method	$S_\theta = PM$	$S_\theta = Weickert$	$S_\theta = Hard$	$S_\theta = Soft$
SNR(for S_1)	15.1173	15.2341	14.9891	15.1766
SNR(for S_2)	19.5700	19.8975	18.6549	18.5475

diffusivity-inspired multiwavelet-shrinkage and diffusivity-inspired wavelet-shrinkage methods, respectively. We set $h = 1, \tau = \frac{(3-\sqrt{7})^2}{64}, \ and \ c = 1$ if we are applying Perona-Malik(PM) diffusivity-based and Weickert diffusivity-based to the highpass output. The parameters $\theta_{11}, \theta_{12}, \theta_{21}, \theta_{22}$ are selected such that the signal to noise ratio are big.

In figure 5.2 and figure 5.3, we compare the results of signals denoising using diffusivity-inspired multiwavelet-shrinkages method with signals denoising using nonlinear diffusion from D_4 wavelet shrinkage approach. We see that the signals denoising results using diffusivity-inspired multiwavelet-shrinkage gives better smoothness and SNRs results than signals denoising using diffusivity-inspired wavelet shrinkage method.

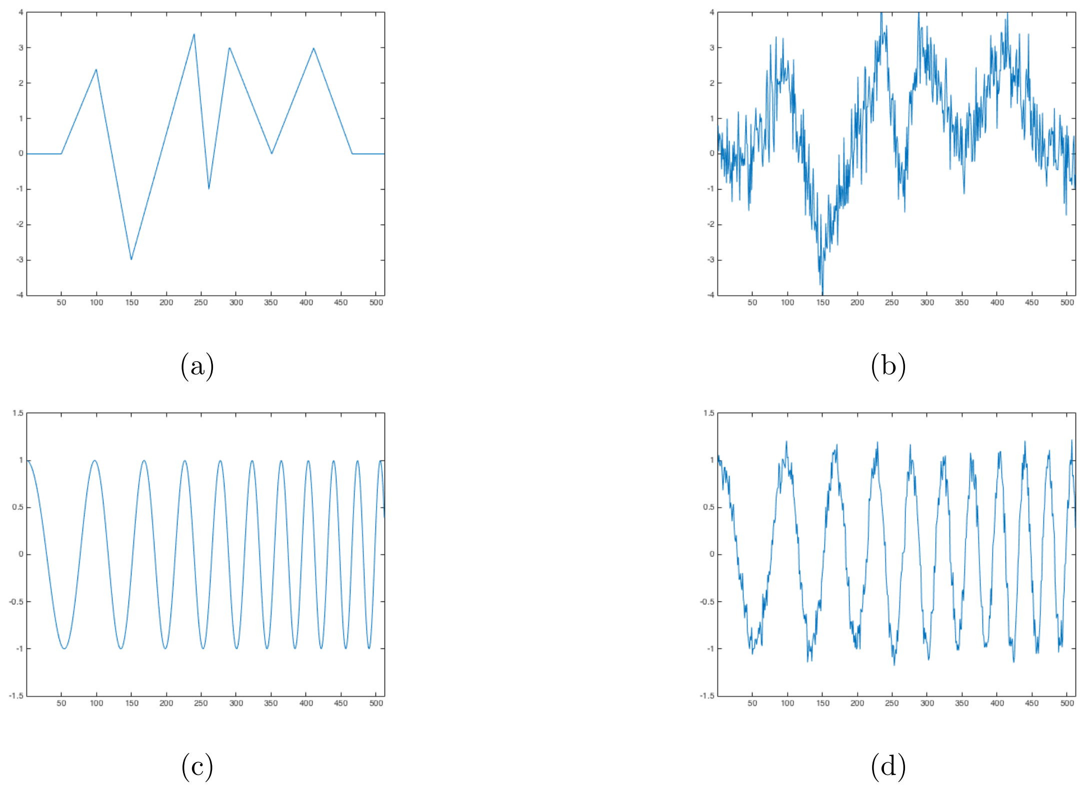
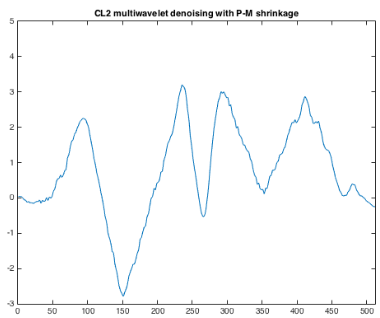
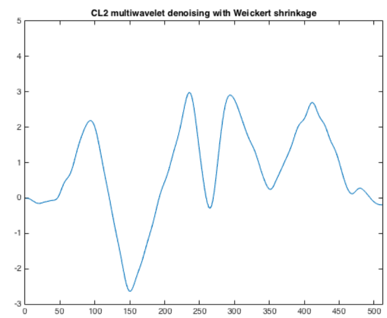


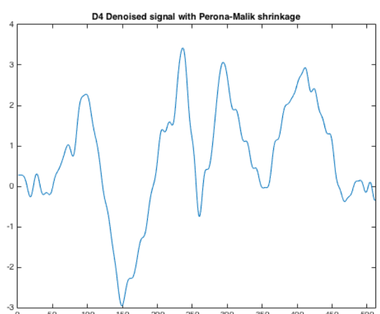
Figure 5.1: (a):Original signal S_1, (b):Noised signal S_1 with SNR=6, (c):Original signal S_2, (d):Noised signal S_2 with SNR=16.



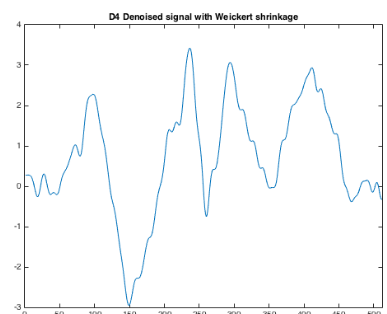
(a)



(b)

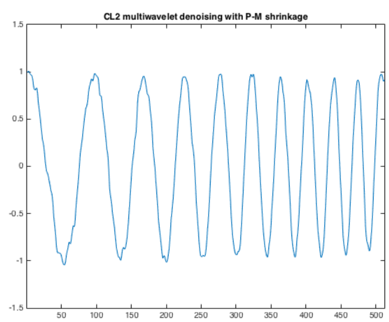


(c)

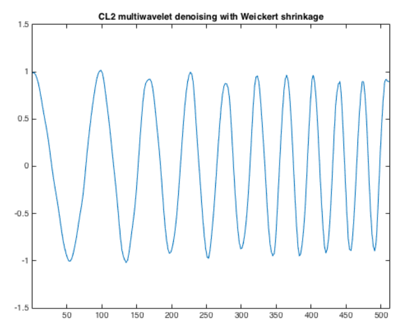


(d)

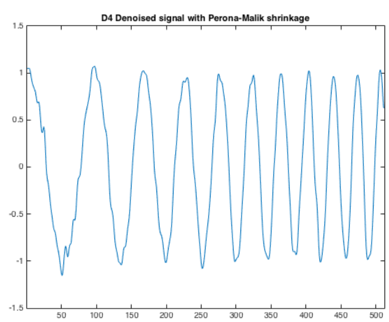
Figure 5.2: (a):Denoised signal S_1 with PM shrinkage using CL(2) multiwavelet filter banks, (b):Denoised signal S_1 with Weickert shrinkage using CL(2) multiwavelet filter banks, (c):Denoised signal S_1 with PM shrinkage using D_4 wavelet filter banks, (d):Denoised signal S_1 with Weickert shrinkage using D_4 wavelet filter banks.



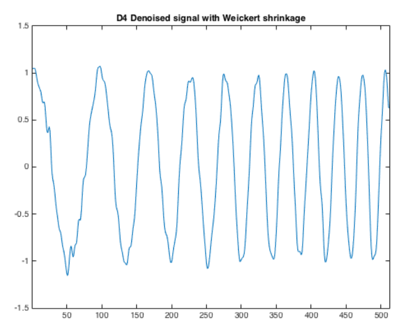
(a)



(b)



(c)



(d)

Figure 5.3: (a):Denoised signal S_2 with PM shrinkage using CL(2) multifilter, (b):Denoised signal S_2 with Weickert shrinkage using CL(2) multifilter, (c):Denoised signal S_2 with PM shrinkage using D_4 filter, (d):Denoised signal S_2 with Weickert shrinkage using D_4 filter.

Chapter 6

Equivalence between multiple frame shrinkage and nonlinear diffusion

In the previous chapter, we have discussed how to choose shrinkage functions so that the multiwavelet shrinkage is a discrete of a given nonlinear diffusion equation.

In this chapter, we present a nonlinear diffusion equation that is derived from multiple frame shrinkage using B-spline tight multiple frame systems as shown in section 6.1. Denoising of signals are presented in section 6.2.

6.1 Diffusion from B-spline multiple frame filter banks

Suppose that $\phi = B_2(. - 2)$ where B_2 is the B-Spline of order 2:

$$B_2(w) := (\frac{1 - e^{-iw}}{iw})^2, \quad w \in \mathbb{R}$$

$\hat{\phi}(2w) = \hat{P}(w)\hat{\phi}(w)$ with $\hat{P}(w) := cos^2(\frac{w}{2})$ and $\phi := [\phi(2.), \phi(2. - 1)]^T$. where:

$$\begin{aligned}
\phi(x) &= \frac{1}{4}\phi(2x+1) + \frac{1}{2}\phi(2x) + \frac{1}{4}\phi(2x-1) \\
\phi(2x) &= \frac{1}{4}\phi(4x+1) + \frac{1}{2}\phi(4x) + \frac{1}{4}\phi(4x-1) \\
\phi(2x-1) &= \frac{1}{4}\phi(4x-1) + \frac{1}{2}\phi(4x-2) + \frac{1}{4}\phi(4x-3)
\end{aligned}$$

then

$$\begin{aligned}
\begin{pmatrix} \phi(2x) \\ \phi(2x-1) \end{pmatrix} &= P_{-1}\begin{pmatrix} \phi(4x+2) \\ \phi(4x+1) \end{pmatrix} + P_0\begin{pmatrix} \phi(4x) \\ \phi(4x-1) \end{pmatrix} + P_1\begin{pmatrix} \phi(4x-2) \\ \phi(4x-3) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \phi(4x+2) \\ \phi(4x+1) \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \phi(4x) \\ \phi(4x-1) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} \phi(4x-2) \\ \phi(4x-3) \end{pmatrix}
\end{aligned}$$

Then the B-Spline tight multiple frame filter $\{P(w), Q^{(1)}(w), Q^{(2)}(w)\}$ are given by:

$$
\begin{aligned}
P_{-1} &= \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}, \; P_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} \end{pmatrix}, \; P_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} \\
Q^{(1)}_{-1} &= \begin{pmatrix} 0 & \frac{\sqrt{2}}{4} \\ 0 & 0 \end{pmatrix}, \; Q^{(1)}_0 = \begin{pmatrix} 0 & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{4} \end{pmatrix}, \; Q^{(1)}_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{\sqrt{2}}{4} \end{pmatrix} \\
Q^{(2)}_{-1} &= \begin{pmatrix} 0 & -\frac{1}{4} \\ 0 & 0 \end{pmatrix}, \; Q^{(2)}_0 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ 0 & -\frac{1}{4} \end{pmatrix}, \; Q^{(2)}_1 = \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & -\frac{1}{4} \end{pmatrix}
\end{aligned} \tag{6.1}
$$

and satisfying the condition:

$$
P(w)^* \; P(w) + \sum_{l=1}^{2} Q^{(l)}(w)^* \; Q^{(l)}(w) = I_2.
$$

The corresponding nonlinear diffusion equation to the B-spline tight multiple frame filter $\{Q^{(1)}, Q^{(2)}\}$ is defined as:

$$
u_t = \frac{\partial}{\partial x}(g_1(u_x^2)u_x) - \frac{\partial^2}{\partial x^2}(g_2(u_{xx}^2)u_{xx}), \tag{6.2}
$$

with original condition $u(x, 0) = f(x)$.

$C^{(l)}_{\beta_l}$ is defined as

$$
C^{(l)}_{\beta_l} = \frac{1}{\beta_l!} \sum k^{\beta_l} q_k
$$

Then

$$
\begin{aligned}
\beta_1 = 1, \;\; C^{(1)}_{\beta_1} &= D^{(1)}_{\beta_1} = -\frac{\sqrt{2}}{2} \\
\beta_2 = 2, \;\; C^{(2)}_{\beta_2} &= D^{(2)}_{\beta_2} = -\frac{1}{4}
\end{aligned}
$$

The approximation solution of Eq.(6.2) in discrete setting provides that $S^{(l)}$ and g_l have the relationship:

$$
\begin{aligned}
u_{1,k} = u^1_{2k} \;\; if \;\; S^{(1)}_{\theta^1_{11}}(x) &= S^{(1)}_{\theta^1_{21}}(x) = x[1 - \frac{2\tau}{h^2} g_1(\frac{2x^2}{h^2})] \\
S^{(2)}_{\theta^2_{11}}(x) &= S^{(2)}_{\theta^2_{21}}(x) = x[1 - \frac{16\tau}{h^4} g_2(\frac{16x^2}{h^4})]
\end{aligned}
$$

and

$$
\begin{aligned}
u_{2,k} = u^1_{2k+1} \;\; if \;\; S^{(1)}_{\theta^1_{12}}(x) &= S^{(1)}_{\theta^1_{22}}(x) = x[1 - \frac{2\tau}{h^2} g_1(\frac{2x^2}{h^2})] \\
S^{(2)}_{\theta^2_{12}}(x) &= S^{(2)}_{\theta^2_{22}}(x) = x[1 - \frac{16\tau}{h^4} g_2(\frac{16x^2}{h^4})].
\end{aligned}
$$

Theorem (6.1): Let $[u_{1,k}\ \ u_{2,k}]^T$ be the denoised signal after one step B-Spline multiple frame shrinking with $\underline{c}_k^0 = [c_{2k}\ \ c_{2k+1}]^T = f(kh),\ \ k \in \mathbb{Z}$ as the original input, and let $[u_{2k}^1\ \ u_{2k+1}^1]^T$ be the signal after 1-step diffusing with original data $\underline{u}_k^0 = [u_{2k}^0\ \ u_{2k+1}^0]^T = f(kh)$. Then the solution of nonlinear diffusion Eq(6.2) provided that the shrinkage functions of the multiple frame denoising algorithm are chosen as:

$$S^{(1)}_{\theta^1_{11}}(x) = S^{(1)}_{\theta^1_{21}}(x) = S^{(1)}_{\theta^1_{12}}(x) = S^{(1)}_{\theta^1_{22}}(x) = x[1 - \frac{2\tau}{h^2} g_1(\frac{2x^2}{h^2})]$$
$$S^{(2)}_{\theta^2_{11}}(x) = S^{(2)}_{\theta^2_{21}}(x) = S^{(2)}_{\theta^2_{12}}(x) = S^{(2)}_{\theta^2_{22}}(x) = x[1 - \frac{16\tau}{h^4} g_2(\frac{16x^2}{h^4})].$$

Let Perona-Malik diffusivity is defined as:

$$g(x^2) = \frac{c}{1 + (\frac{x}{\lambda})^2}$$

where c is a constant. Then the corresponding multiple frame shrinkage functions to Perona-Malik diffusivity are given by:

$$S^{(1)}_{\theta^1_{11}}(x) = S^{(1)}_{\theta^1_{21}}(x) = S^{(1)}_{\theta^1_{12}}(x) = S^{(1)}_{\theta^1_{22}}(x) = x[1 - \frac{2\tau c_1}{[1 + (\sqrt{2}x/\theta^1_{11})^2]}] \tag{6.3}$$
$$S^{(2)}_{\theta^2_{11}}(x) = S^{(2)}_{\theta^2_{21}}(x) = S^{(2)}_{\theta^2_{12}}(x) = S^{(2)}_{\theta^2_{22}}(x) = x[1 - \frac{16\tau c_2}{[1 + (4x/\theta^2_{11})^2]}] \tag{6.4}$$

where the spatial step size h=1. If the Weickert diffusivity g is defined as:

$$g(x^2) = \begin{cases} 1 & if \quad x = 0 \\ 1 - exp(-3.31488\lambda^8/x^8) & if \quad x \neq 0, \end{cases}$$

then the corresponding multiple frame shrinkage functions are defined as:

$$S^{(1)}_{\theta^1_{11}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 2\tau[1 - exp(-3.31488\ (\theta^1_{11})^8/(\sqrt{2}x)^8)\ ]) & if \quad x \neq 0 \end{cases} \tag{6.5}$$

$$S^{(2)}_{\theta^2_{11}} = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 16\tau[1 - exp(-3.31488\ (\theta^2_{11})^8/(4x)^8)\ ]) & if \quad x \neq 0 \end{cases} \tag{6.6}$$

Table 6.1: Signal denoising results using diffusion-inspired multiple frame shrinkage functions.

Shrinkage Method	$S^{(1)}_{\theta^1_{11}} = S^{(1)}_{\theta^1_{21}} = S^{(1)}_{\theta^1_{12}} = S^{(1)}_{\theta^1_{22}} = PM$ $S^{(2)}_{\theta^2_{11}} = S^{(2)}_{\theta^2_{21}} = S^{(2)}_{\theta^2_{12}} = S^{(2)}_{\theta^2_{22}} = PM$	Weickert Weickert	PM Weick-ert	Weickert PM
SNR(for S_1)	18.6934	18.6873	18.7291	18.6619
SNR(for S_3)	26.0158	27.2339	26.5611	26.5274

Table 6.2: Signal denoising results using diffusion-inspired frame shrinkage functions.

Shrinkage Method	$S^{(1)}_{\theta} = PM$ $S^{(2)}_{\sigma} = PM$	$S^{(1)}_{\theta} = Weickert$ $S^{(2)}_{\sigma} = Weickert$	$S^{(1)}_{\theta} = PM$ $S^{(2)}_{\sigma} = Weickert$	$S^{(1)}_{\theta} = Weickert$ $S^{(2)}_{\sigma} = PM$
SNR(for S_1)	18.0560	18.0643	18.0589	18.0539
SNR(for S_3)	26.1805	25.6973	26.3698	25.6202

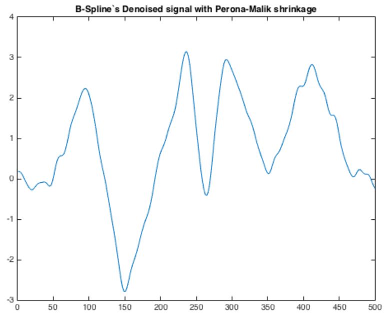
6.2 Denoising of signals using diffusion inspired-multiple frame shrinkage method

We discuss signal denoising based on B-Spline multiple frame filter banks with different signals and shrinkage functions.

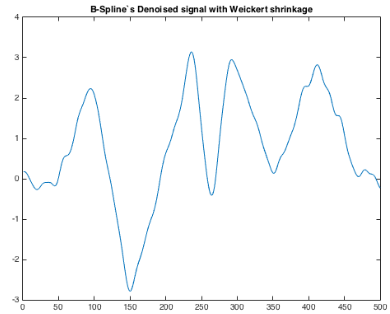
In figure 5.1, (a) is the first original signal S_1, we generate five noised signals with SNR=6 by adding Gaussian noise five time to S_1, (b) in figure 5.1. Then we apply B-Spline multiple frame shrinking iteratively 50 times to each noised signal.
Table 6.1 presents the average of the signal to noise ratio of the signal denoising results of the five noised signals with different shrinkage functions. We set $h = 1, \tau = \frac{1}{4}, c_1 = 1, c_2 = \frac{1}{8}$, if we are applying Perona-Malik(PM) diffusivity functions as shown in Eq.(6.3) and (6.4). However, we set $h = 1, \tau = \frac{1}{4}$ if Weickert diffusivity-based is applied to the first highpass output Eq.(6.5), and $h = 1, \tau = \frac{1}{16}$ if Weickert diffusivity-based is applied to the second highpass output Eq.(6.6).

The second example is the signal S_3, (a) in figure 6.2. Also, we generate five noised signals but with signal to noise ratio SNR=16, (b) in figure 6.2. Then we apply B-Spline multiple frame shrinking iteratively 40 times to each noised signal, and we set the value of h, τ, c_1, c_2 as above. Also, for S_3 we average the five results of SNRs of the denoised signals as shown in Table 6.1.

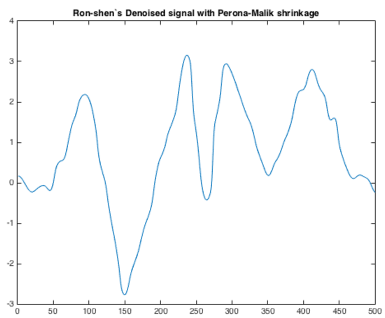
In figures 6.1 and 6.3, we display denoising signal with Perona-Malik(PM) shrinkage and Weickert shrinkage using B-Spline multiple frame filter banks, and we compare the results with denoising signal with Perona-Malik(PM) shrinkage and Weickert shrinkage using Ron-Shen filter banks. The performances of the diffusion inspired multiple frame shrinking with two toy signals S_1 and S_3 are slightly better compared to diffusion inspired frame shrinking.



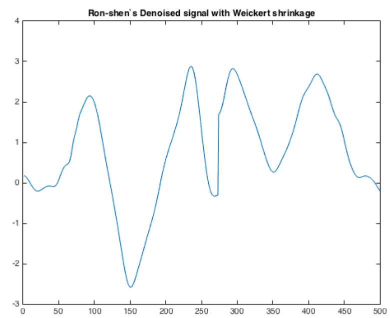
(a)



(b)

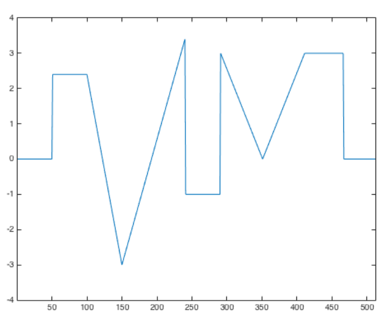


(c)

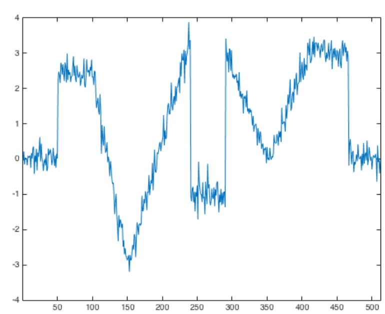


(d)

Figure 6.1: (a),(b): Denoised signal S_1 with Perona-Malik shrinkage and Weickert shrinkage, respectively, using multiple frame filter banks, (c),(d): Denoised signal S_1 with PM shrinkage and Weickert shrinkage, respectively, using frame filter banks.

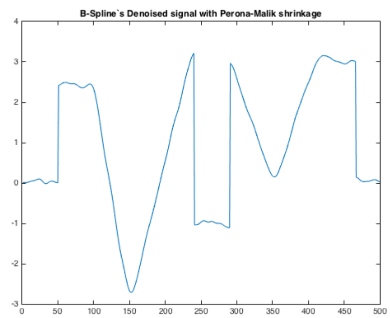


(a)



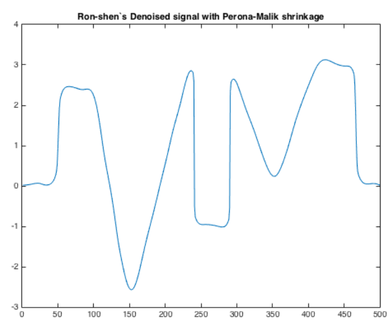
(b)

Figure 6.2: (a):Original signal S_3, (b):Noised signal S_3 with SNR=16.

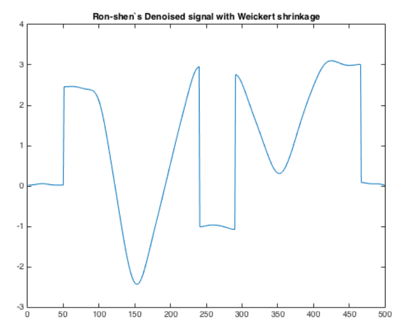


(a)

(b)



(c)



(d)

Figure 6.3: (a),(b): Denoised signal S_3 with PM shrinkage and Weickert shrinkage using B-Spline multiple frame filter banks, (c),(d): Denoised signal S_3 with PM shrinkage and Weickert shrinkage, respectively, using Ron-Shen frame filter banks.

Chapter 7

Correspondence between multiwavelet shrinkage/multiple frame shrinkage and nonlinear diffusion in two dimension

Multiwavelet and multiple frame present higher achievements for image processing in comparison with wavelets and frames in a scalar case.

In this chapter, we illustrate that the two-dimensional algorithm can be decomposed by taking a tensor product of one-dimensional methods as shown in Section 7.1, in other words, by applying the one dimensional algorithm in each dimension separately.

Section 7.2 formulates new algorithms of the correspondence between 2D-multiwavelet shrinkage and nonlinear diffusion. Non-linear diffusion derived from two-dimensional multiple frame shrinkage is presented in Section 7.3. The performance of 2D-multiple frame shrinkage is discussed in Section 7.4.

7.1 Multiwavelet shrinkage in two dimension

In two-dimension the construction of Φ and Ψ can be performed from one-dimensional by using the tensor product method.

Let consider r = 2, $\Phi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$,and $\Psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$, then we have

$$\Phi(x,y) = \Phi(x) \otimes \Phi(y) = \begin{pmatrix} \phi_1(x)\phi_1(y) \\ \phi_1(x)\phi_2(y) \\ \phi_2(x)\phi_1(y) \\ \phi_2(x)\phi_2(y) \end{pmatrix}$$

$$\Psi_1(x,y) = \Phi(x) \otimes \Psi(y) = \begin{pmatrix} \phi_1(x)\psi_1(y) \\ \phi_1(x)\psi_2(y) \\ \phi_2(x)\psi_1(y) \\ \phi_2(x)\psi_2(y) \end{pmatrix}$$

$$\Psi_2(x,y) = \Psi(x) \otimes \Phi(y) = \begin{pmatrix} \psi_1(x)\phi_1(y) \\ \psi_1(x)\phi_2(y) \\ \psi_2(x)\phi_1(y) \\ \psi_2(x)\phi_2(y) \end{pmatrix}$$

$$\Psi_3(x,y) = \Psi(x) \otimes \Psi(y) = \begin{pmatrix} \psi_1(x)\psi_1(y) \\ \psi_1(x)\psi_2(y) \\ \psi_2(x)\psi_1(y) \\ \psi_2(x)\psi_2(y) \end{pmatrix}$$

$$\begin{aligned} \Phi(x,y) = \Phi(x) \otimes \Phi(y) &= \sum_k [P_k \Phi(2x-k)] \otimes \sum_j [P_j \Phi(2y-j)] \\ &= \sum_{k,j} (P_k \otimes P_j)[\Phi(2x-k) \otimes \Phi(2y-j)] \\ &= \sum_{k,j} P_{k,j} \Phi(2(x,y)-(k,j)) \end{aligned}$$

where $P_k \otimes P_j$ denote the kronecker product of P_k and P_j, and define as:

$$\begin{aligned} P_k \otimes P_j &= \begin{pmatrix} p_{k,11}P_j & p_{k,12}P_j \\ p_{k,21}P_j & p_{k,22}P_j \end{pmatrix} \\ &= \begin{pmatrix} p_{k,11}p_{j,11} & p_{k,11}p_{j,12} & p_{k,12}p_{j,11} & p_{k,12}p_{j,12} \\ p_{k,11}p_{j,21} & p_{k,11}p_{j,22} & p_{k,12}p_{j,21} & p_{k,12}p_{j,22} \\ p_{k,21}p_{j,11} & p_{k,21}p_{j,12} & p_{k,22}p_{j,11} & p_{k,22}p_{j,12} \\ p_{k,21}p_{j,21} & p_{k,21}p_{j,22} & p_{k,22}p_{j,21} & p_{k,22}p_{j,22} \end{pmatrix} \end{aligned}$$

then the decomposition algorithm is:

$$\begin{aligned}
\begin{pmatrix} L_{1,(n_1,n_2)} \\ L_{2,(n_1,n_2)} \\ L_{3,(n_1,n_2)} \\ L_{4,(n_1,n_2)} \end{pmatrix} &= \sum_{k_1,k_2} P_{k_1} \otimes P_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix} \\
&= \sum_{k_1} \begin{pmatrix} p_{k_1,11} \sum_{k_2} P_{k_2} & p_{k_1,12} \sum_{k_2} P_{k_2} \\ p_{k_1,21} \sum_{k_2} P_{k_2} & p_{k_1,22} \sum_{k_2} P_{k_2} \end{pmatrix} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix} \\
&= \sum_{k_1} \begin{pmatrix} p_{k_1,11} \sum_{k_2} P_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \end{pmatrix} + p_{k_1,12} \sum_{k_2} P_{k_2} \begin{pmatrix} c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix} \\ p_{k_1,21} \sum_{k_2} P_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \end{pmatrix} + p_{k_1,22} \sum_{k_2} P_{k_2} \begin{pmatrix} c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix} \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} H^1_{1,(n_1,n_2)} \\ H^1_{2,(n_1,n_2)} \\ H^1_{3,(n_1,n_2)} \\ H^1_{4,(n_1,n_2)} \end{pmatrix} = \sum_{k_1,k_2} P_{k_1} \otimes Q_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix}$$

$$\begin{pmatrix} H^2_{1,(n_1,n_2)} \\ H^2_{2,(n_1,n_2)} \\ H^2_{3,(n_1,n_2)} \\ H^2_{4,(n_1,n_2)} \end{pmatrix} = \sum_{k_1,k_2} Q_{k_1} \otimes P_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix}$$

$$\begin{pmatrix} H^3_{1,(n_1,n_2)} \\ H^3_{2,(n_1,n_2)} \\ H^3_{3,(n_1,n_2)} \\ H^3_{4,(n_1,n_2)} \end{pmatrix} = \sum_{k_1,k_2} Q_{k_1} \otimes Q_{k_2} \begin{pmatrix} c_{2(k_1+n_1),2(k_2+n_2)} \\ c_{2(k_1+n_1),2(k_2+n_2)+1} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)} \\ c_{2(k_1+n_1)+1,2(k_2+n_2)+1} \end{pmatrix}$$

and the denoising algorithm is given by:

$$
\begin{pmatrix} u_{1,(k_1,k_2)} \\ u_{2,(k_1,k_2)} \\ u_{3,(k_1,k_2)} \\ u_{4,(k_1,k_2)} \end{pmatrix} = \sum_{n_1,n_2\in\mathbb{Z}} (P_{n_1}\otimes P_{n_2})^T \begin{pmatrix} L_{1,(k_1-n_1,k_2-n_2)} \\ L_{2,(k_1-n_1,k_2-n_2)} \\ L_{3,(k_1-n_1,k_2-n_2)} \\ L_{4,(k_1-n_1,k_2-n_2)} \end{pmatrix}
$$

$$
+ \sum_{n_1,n_2\in\mathbb{Z}} \begin{pmatrix}
p_{n_1,11}q_{n_2,11}S_{\theta_{11}}(H^1_{1,(k_1-n_1,k_2-n_2)}) + p_{n_1,11}q_{n_2,21}S_{\theta_{21}}(H^1_{2,(k_1-n_1,k_2-n_2)}) \\
+p_{n_1,21}q_{n_2,11}S_{\theta_{31}}(H^1_{3,(k_1-n_1,k_2-n_2)}) + p_{n_1,21}p_{n_2,21}S_{\theta_{41}}(H^1_{4,(k_1-n_1,k_2-n_2)}) \\
\\
p_{n_1,11}q_{n_2,12}S_{\theta_{12}}(H^1_{1,(k_1-n_1,k_2-n_2)}) + p_{n_1,11}q_{n_2,22}S_{\theta_{22}}(H^1_{2,(k_1-n_1,k_2-n_2)}) \\
+p_{n_1,21}q_{n_2,12}S_{\theta_{32}}(H^1_{3,(k_1-n_1,k_2-n_2)}) + p_{n_1,21}q_{n_2,22}S_{\theta_{42}}(H^1_{4,(k_1-n_1,k_2-n_2)}) \\
\\
p_{n_1,12}q_{n_2,11}S_{\theta_{13}}(H^1_{1,(k_1-n_1,k_2-n_2)}) + p_{n_1,12}q_{n_2,21}S_{\theta_{23}}(H^1_{2,(k_1-n_1,k_2-n_2)}) \\
+p_{n_1,22}q_{n_2,11}S_{\theta_{33}}(H^1_{3,(k_1-n_1,k_2-n_2)}) + p_{n_1,22}p_{n_2,21}S_{\theta_{43}}(H^1_{4,(k_1-n_1,k_2-n_2)}) \\
\\
p_{n_1,12}q_{n_2,12}S_{\theta_{14}}(H^1_{1,(k_1-n_1,k_2-n_2)}) + p_{n_1,12}q_{n_2,22}S_{\theta_{24}}(H^1_{2,(k_1-n_1,k_2-n_2)}) \\
+p_{n_1,22}q_{n_2,12}S_{\theta_{34}}(H^1_{3,(k_1-n_1,k_2-n_2)}) + p_{n_1,22}p_{n_2,22}S_{\theta_{44}}(H^1_{4,(k_1-n_1,k_2-n_2)})
\end{pmatrix}
$$

$$
+ \sum_{n_1,n_2\in\mathbb{Z}} \begin{pmatrix}
q_{n_1,11}p_{n_2,11}S_{\sigma_{11}}(H^2_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,11}p_{n_2,21}S_{\sigma_{21}}(H^2_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,21}p_{n_2,11}S_{\sigma_{31}}(H^2_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,21}p_{n_2,21}S_{\sigma_{41}}(H^2_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,11}p_{n_2,12}S_{\sigma_{12}}(H^2_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,11}p_{n_2,22}S_{\sigma_{22}}(H^2_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,21}p_{n_2,12}S_{\sigma_{32}}(H^2_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,21}p_{n_2,22}S_{\sigma_{42}}(H^2_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,12}p_{n_2,11}S_{\sigma_{13}}(H^2_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,12}p_{n_2,21}S_{\sigma_{23}}(H^2_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,22}p_{n_2,11}S_{\sigma_{33}}(H^2_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,22}p_{n_2,21}S_{\sigma_{43}}(H^2_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,12}p_{n_2,12}S_{\sigma_{14}}(H^2_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,12}p_{n_2,22}S_{\sigma_{24}}(H^2_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,22}p_{n_2,12}S_{\sigma_{34}}(H^2_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,22}p_{n_2,22}S_{\sigma_{44}}(H^2_{4,(k_1-n_1,k_2-n_2)})
\end{pmatrix}
$$

$$
+ \sum_{n_1,n_2\in\mathbb{Z}} \begin{pmatrix}
q_{n_1,11}q_{n_2,11}S_{\gamma_{11}}(H^3_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,11}q_{n_2,21}S_{\gamma_{21}}(H^3_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,21}q_{n_2,11}S_{\gamma_{31}}(H^3_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,21}q_{n_2,21}S_{\gamma_{41}}(H^3_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,11}q_{n_2,12}S_{\gamma_{12}}(H^3_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,11}q_{n_2,22}S_{\gamma_{22}}(H^3_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,21}q_{n_2,12}S_{\gamma_{32}}(H^3_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,21}q_{n_2,22}S_{\gamma_{42}}(H^3_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,12}q_{n_2,11}S_{\gamma_{13}}(H^3_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,12}q_{n_2,21}S_{\gamma_{23}}(H^3_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,22}q_{n_2,11}S_{\gamma 33}(H^3_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,22}q_{n_2,21}S_{\gamma_{43}}(H^3_{4,(k_1-n_1,k_2-n_2)}) \\
\\
q_{n_1,12}q_{n_2,12}S_{\gamma_{14}}(H^3_{1,(k_1-n_1,k_2-n_2)}) + q_{n_1,12}q_{n_2,22}S_{\gamma_{24}}(H^3_{2,(k_1-n_1,k_2-n_2)}) \\
+q_{n_1,22}q_{n_2,12}S_{\gamma_{34}}(H^3_{3,(k_1-n_1,k_2-n_2)}) + q_{n_1,22}q_{n_2,22}S_{\gamma_{44}}(H^3_{4,(k_1-n_1,k_2-n_2)})
\end{pmatrix} \tag{7.1}
$$

7.2 Equivalence between non-linear diffusion and multiwavelet shrinkage in two-dimension

Suppose the multifilters P, Q are given by:

$$P(w) = \begin{bmatrix} a(w) \\ b(w) \end{bmatrix} \; Q(w) = \begin{bmatrix} e(w) \\ f(w) \end{bmatrix}$$

then

$$P \otimes Q = \begin{bmatrix} a(w_1)e(w_2) \\ a(w_1)f(w_2) \\ b(w_1)e(w_2) \\ b(w_1)f(w_2) \end{bmatrix}, \; Q \otimes P = \begin{bmatrix} e(w_1)a(w_2) \\ e(w_1)b(w_2) \\ f(w_1)a(w_2) \\ f(w_1)b(w_2) \end{bmatrix}, \; Q \otimes Q = \begin{bmatrix} e(w_1)e(w_2) \\ e(w_1)f(w_2) \\ f(w_1)e(w_2) \\ f(w_1)f(w_2) \end{bmatrix} \tag{7.2}$$

where

$$a(w) = \sum_{k=0}^{N} (p_{k,11} e^{-2ikw} + p_{k,12} e^{-i(2k+1)w})$$

$$b(w) = \sum_{k=0}^{N} (p_{k+1,21} e^{-i(2k+1)w} + p_{k,22} e^{-2ikw})$$

$$e(w) = \sum_{k=0}^{N} (q_{k,11} e^{-2ikw} + q_{k,12} e^{-i(2k+1)w})$$

$$f(w) = \sum_{k=0}^{N} (q_{k+1,21} e^{-i(2k+1)w} + q_{k,22} e^{-2ikw})$$

For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ and $w \in \mathbb{R}^2$, denote

$$\frac{\partial^{\alpha_1+\alpha_2}}{\partial w_2^{\alpha_2} \partial w_1^{\alpha_1}}, \quad |\alpha| = \alpha_1 + \alpha_2, \; and \; \; \alpha! = \alpha_1! \alpha_2!$$

The highpass filter q has vanishing moment of order $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, if

$$\sum_{k \in \mathbb{Z}^2} k^\alpha q_k = i^{|\alpha|} \frac{\partial^\alpha}{\partial w^\alpha} \hat{q}(w)|_{w=0} = 0$$

provided that:

$$C_\alpha = \frac{i^{|\alpha|}}{\alpha!} \frac{\partial^\alpha}{\partial w^\alpha} q(w)|_{w=0}$$

The corresponding nonlinear diffusion equation to the two-dimensional highpass multiwavelet-filters Eq.(7.2) is given by:

$$u_t = \frac{\partial}{\partial x_1}[g((\frac{\partial u}{\partial x_1})^2)\frac{\partial u}{\partial x_1}] + \frac{\partial}{\partial x_2}[g((\frac{\partial u}{\partial x_2})^2)\frac{\partial u}{\partial x_2}] - \frac{\partial^2}{\partial x_1 \partial x_2}[g((\frac{\partial^2 u}{\partial x_1 \partial x_2})^2)\frac{\partial^2 u}{\partial x_1 \partial x_2}] \tag{7.3}$$

The approximation of $\frac{\partial}{\partial x_1}u(x_1,x_2,0)$, $\frac{\partial}{\partial x_2}u(x_1,x_2,0)$, and $\frac{\partial^2}{\partial x_1 \partial x_2}u(x_1,x_2,0)$ at $(2(k_1,k_2)h)$, $([2k_1, 2k_2+1]h)$, $([2k_1+1, 2k_2]h)$, *and* $([2k_1+1, 2k_2+1]h)$ are defined respectively by:

$$\frac{\partial}{\partial x_1}[u(2k_1h, 2k_2h)] \approx \frac{1}{E_1h}\sum_{k_1,k_2}(a *_{k_2} e *_{k_1} C)_{(2k_1,2k_2)} = \frac{H^2_{1,(k_1,k_2)}}{E_1h};$$

$$\frac{\partial}{\partial x_1}[u([2k_1, 2k_2+1]h)] \approx \frac{1}{E_1h}\sum_{k_1,k_2}(b *_{k_2} e *_{k_1} C)_{(2k_1,2k_2+1)} = \frac{H^2_{2,(k_1,k_2)}}{E_1h}$$

$$\frac{\partial}{\partial x_1}[u([2k_1+1, 2k_2]h)] \approx \frac{1}{F_1h}\sum_{k_1,k_2}(a *_{k_2} f *_{k_1} C)_{(2k_1+1,2k_2)} = \frac{H^2_{3,(k_1,k_2)}}{F_1h}$$

$$\frac{\partial}{\partial x_1}[u([2k_1+1, 2k_2+1]h)] \approx \frac{1}{F_1h}\sum_{k_1,k_2}(b *_{k_2} f *_{k_1} C)_{(2k_1+1,2k_2+1)} = \frac{H^2_{4,(k_1,k_2)}}{F_1h}$$

$$\frac{\partial}{\partial x_2}[u(2k_1h, 2k_2h)] \approx \frac{1}{E_2h}\sum_{k_1,k_2}(e *_{k_2} a *_{k_1} C)_{(2k_1,2k_2)} = \frac{H^1_{1,(k_1,k_2)}}{E_2h}$$

$$\frac{\partial}{\partial x_2}[u([2k_1, 2k_2+1]h)] \approx \frac{1}{F_2h}\sum_{k_1,k_2}(f *_{k_2} a *_{k_1} C)_{(2k_1,2k_2+1)} = \frac{H^1_{2,(k_1,k_2)}}{F_2h}$$

$$\frac{\partial}{\partial x_2}[u([2k_1+1, 2k_2]h)] \approx \frac{1}{E_2h}\sum_{k_1,k_2}(e *_{k_2} b *_{k_1} C)_{(2k_1+1,2k_2)} = \frac{H^1_{3,(k_1,k_2)}}{E_2h}$$

$$\frac{\partial}{\partial x_2}[u([2k_1+1, 2k_2+1]h)] \approx \frac{1}{F_2h}\sum_{k_1,k_2}(f *_{k_2} b *_{k_1} C)_{(2k_1+1,2k_2+1)} = \frac{H^1_{4,(k_1,k_2)}}{F_2h}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}[u(2k_1h, 2k_2h)] \approx \frac{1}{E_1E_2h^2}\sum_{k_1,k_2}(e *_{k_2} e *_{k_1} C)_{(2k_1,2k_2)} = \frac{H^3_{1,(k_1,k_2)}}{E_1E_2h^2}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}[u([2k_1, 2k_2+1]h)] \approx \frac{1}{E_1F_2h^2}\sum_{k_1,k_2}(f *_{k_2} e *_{k_1} C)_{(2k_1,2k_2+1)} = \frac{H^3_{2,(k_1,k_2)}}{E_1F_2h^2}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}[u([2k_1+1, 2k_2]h)] \approx \frac{1}{F_1E_2h^2}\sum_{k_1,k_2}(e *_{k_2} f *_{k_1} C)_{(2k_1+1,2k_2)} = \frac{H^3_{3,(k_1,k_2)}}{F_1E_2h^2}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}[u([2k_1+1, 2k_2+1]h)] \approx \frac{1}{F_1F_2h^2}\sum_{k_1,k_2}(f *_{k_2} f *_{k_1} C)_{(2k_1+1,2k_2+1)} = \frac{H^3_{4,(k_1,k_2)}}{F_1F_2h}$$

and the approximating partial derivatives of $\frac{\partial}{\partial x_1}G_1(x_1,x_2,t)$, $\frac{\partial}{\partial x_2}G_2(x_1,x_2,t)$, and $\frac{\partial}{\partial x_1 \partial x_2}G_3(x_1,x_2,t)$ where

$$\begin{aligned} G_1(x_1,x_2,t) &:= g[(\frac{\partial u}{\partial x_1})^2]\frac{\partial u}{\partial x_1} \\ G_2(x_1,x_2,t) &:= g[(\frac{\partial u}{\partial x_2})^2]\frac{\partial u}{\partial x_2} \\ G_3(x_1,x_2,t) &:= g[(\frac{\partial^2 u}{\partial x_1 \partial x_2})^2]\frac{\partial^2 u}{\partial x_1 \partial x_2} \end{aligned}$$

at

$(2(k_1,k_2)h, j\tau)$, $([2k_1, 2k_2+1]h, j\tau), ([2k_1+1, 2k_2]h, j\tau)$ *and* $([2k_1+1, 2k_2+1]h, j\tau)$

are defined respectively as:

$$\frac{\partial}{\partial x_1}\begin{bmatrix} G_1(2k_1h, 2k_2h, j\tau) \\ G_1([2k_1, 2k_2+1]h, j\tau) \\ G_1([2k_1+1, 2k_2]h, j\tau) \\ G_1([2k_1+1, 2k_2+1]h, j\tau) \end{bmatrix} = \begin{bmatrix} \frac{-1}{E_1h} & 0 & 0 & 0 \\ 0 & \frac{-1}{E_1h} & 0 & 0 \\ 0 & 0 & \frac{-1}{F_1h} & 0 \\ 0 & 0 & 0 & \frac{-1}{F_1h} \end{bmatrix} \sum_{k_1,k_2}(Q_{m_1}\otimes P_{m_2})^T \times \begin{pmatrix} G_1([2(k_1-m_1), 2(k_2-m_2)]h, j\tau) \\ G_1([2(k_1-m_1), 2(k_2-m_2)+1]h, j\tau) \\ G_1([2(k_1-m_1)+1, 2(k_2-m_2)]h, j\tau) \\ G_1([2(k_1-m_1)+1, 2(k_2-m_2)+1]h, j\tau) \end{pmatrix}$$

$$\frac{\partial}{\partial x_2}\begin{bmatrix} G_2(2k_1h, 2k_2h, j\tau) \\ G_2([2k_1, 2k_2+1]h, j\tau) \\ G_2([2k_1+1, 2k_2]h, j\tau) \\ G_2([2k_1+1, 2k_2+1]h, j\tau) \end{bmatrix} = \begin{bmatrix} \frac{-1}{E_2h} & 0 & 0 & 0 \\ 0 & \frac{-1}{F_2h} & 0 & 0 \\ 0 & 0 & \frac{-1}{E_2h} & 0 \\ 0 & 0 & 0 & \frac{-1}{F_2h} \end{bmatrix} \times \sum_{k_1,k_2}(P_{m_1}\otimes Q_{m_2})^T \begin{pmatrix} G_2([2(k_1-m_1), 2(k_2-m_2)]h, j\tau) \\ G_2([2(k_1-m_1), 2(k_2-m_2)+1]h, j\tau) \\ G_2([2(k_1-m_1)+1, 2(k_2-m_2)]h, j\tau) \\ G_2([2(k_1-m_1)+1, 2(k_2-m_2)+1]h, j\tau) \end{pmatrix}$$

$$\frac{\partial^2}{\partial x_1 \partial x_2}\begin{bmatrix} G_3(2k_1h, 2k_2h, j\tau) \\ G_3([2k_1, 2k_2+1]h, j\tau) \\ G_3([2k_1+1, 2k_2]h, j\tau) \\ G_3([2k_1+1, 2k_2+1]h, j\tau) \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1E_2h^2} & 0 & 0 & 0 \\ 0 & \frac{1}{E_1F_2h^2} & 0 & 0 \\ 0 & 0 & \frac{1}{F_1E_2h^2} & 0 \\ 0 & 0 & 0 & \frac{1}{F_1F_2h^2} \end{bmatrix} \times \sum_{k_1,k_2}(Q_{m_1}\otimes Q_{m_2})^T \begin{pmatrix} G_3([2(k_1-m_1), 2(k_2-m_2)]h, j\tau) \\ G_3([2(k_1-m_1), 2(k_2-m_2)+1]h, j\tau) \\ G_3([2(k_1-m_1)+1, 2(k_2-m_2)]h, j\tau) \\ G_3([2(k_1-m_1)+1, 2(k_2-m_2)+1]h, j\tau) \end{pmatrix}$$

Then when $j = 0$, the nonlinear diffusion equation can be discretized as:

$$\begin{bmatrix} u^1_{2k_1,2k_2} \\ u^1_{2k_1,2k_2+1} \\ u^1_{2k_1+1,2k_2} \\ u^1_{2k_1+1,2k_2+1} \end{bmatrix} = \begin{bmatrix} u^0_{2k_1,2k_2} \\ u^0_{2k_1,2k_2+1} \\ u^0_{2k_1+1,2k_2} \\ u^0_{2k_1+1,2k_2+1} \end{bmatrix} - \begin{bmatrix} \frac{\tau}{E_1 h} & 0 & 0 & 0 \\ 0 & \frac{\tau}{E_1 h} & 0 & 0 \\ 0 & 0 & \frac{\tau}{F_1 h} & 0 \\ 0 & 0 & 0 & \frac{\tau}{F_1 h} \end{bmatrix} \sum_{m_1,m_2} (Q_{m_1} \otimes P_{m_2})^T$$

$$\times \begin{bmatrix} \frac{1}{E_1 h} g(\frac{H^2_{1,(k_1-m_1,k_2-m_2)}}{E_1 h})^2 H^2_{1,(k_1-m_1,k_2-m_2)} \\ \frac{1}{E_1 h} g(\frac{H^2_{2,(k_1-m_1,k_2-m_2)}}{E_1 h})^2 H^2_{2,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_1 h} g(\frac{H^2_{3,(k_1-m_1,k_2-m_2)}}{F_1 h})^2 H^2_{3,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_1 h} g(\frac{H^2_{4,(k_1-m_1,k_2-m_2)}}{F_1 h})^2 H^2_{4,(k_1-m_1,k_2-m_2)} \end{bmatrix} - \begin{bmatrix} \frac{\tau}{E_2 h} & 0 & 0 & 0 \\ 0 & \frac{\tau}{F_2 h} & 0 & 0 \\ 0 & 0 & \frac{\tau}{E_2 h} & 0 \\ 0 & 0 & 0 & \frac{\tau}{F_2 h} \end{bmatrix}$$

$$\times \sum_{m_1,m_2} (P_{m_1} \otimes Q_{m_2})^T \begin{bmatrix} \frac{1}{E_2 h} g(\frac{H^1_{1,(k_1-m_1,k_2-m_2)}}{E_2 h})^2 H^1_{1,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_2 h} g(\frac{H^1_{2,(k_1-m_1,k_2-m_2)}}{F_2 h})^2 H^1_{2,(k_1-m_1,k_2-m_2)} \\ \frac{1}{E_2 h} g(\frac{H^1_{3,(k_1-m_1,k_2-m_2)}}{E_2 h})^2 H^1_{3,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_2 h} g(\frac{H^1_{4,(k_1-m_1,k_2-m_2)}}{F_2 h})^2 H^1_{4,(k_1-m_1,k_2-m_2)} \end{bmatrix}$$

$$- \begin{bmatrix} \frac{\tau}{E_1 E_2 h^2} & 0 & 0 & 0 \\ 0 & \frac{\tau}{E_1 F_2 h^2} & 0 & 0 \\ 0 & 0 & \frac{\tau}{F_1 E_2 h^2} & 0 \\ 0 & 0 & 0 & \frac{\tau}{F_1 F_2 h^2} \end{bmatrix} \sum_{m_1,m_2} (Q_{m_1} \otimes Q_{m_2})^T$$

$$\times \begin{bmatrix} \frac{1}{E_1 E_2 h^2} g(\frac{H^3_{1,(k_1-m_1,k_2-m_2)}}{E_1 E_2 h^2})^2 H^3_{1,(k_1-m_1,k_2-m_2)} \\ \frac{1}{E_1 F_2 h^2} g(\frac{H^3_{2,(k_1-m_1,k_2-m_2)}}{E_1 F_2 h^2})^2 H^3_{2,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_1 E_2 h^2} g(\frac{H^3_{3,(k_1-m_1,k_2-m_2)}}{F_1 E_2 h^2})^2 H^3_{3,(k_1-m_1,k_2-m_2)} \\ \frac{1}{F_1 F_2 h^2} g(\frac{H^3_{4,(k_1-m_1,k_2-m_2)}}{F_1 F_2 h^2})^2 H^3_{4,(k_1-m_1,k_2-m_2)} \end{bmatrix}$$

then

$$\begin{bmatrix} u^1_{2k_1,2k_2} \\ u^1_{2k_1,2k_2+1} \\ u^1_{2k_1+1,2k_2} \\ u^1_{2k_1+1,2k_2+1} \end{bmatrix} = \begin{bmatrix} u^0_{2k_1,2k_2} \\ u^0_{2k_1,2k_2+1} \\ u^0_{2k_1+1,2k_2} \\ u^0_{2k_1+1,2k_2+1} \end{bmatrix} - \begin{bmatrix} \begin{array}{c} q_{m_1,11}p_{m_2,11}\frac{\tau}{E_1^2h^2}g(\frac{H^2_{1,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{1,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,11}p_{m_2,21}\frac{\tau}{E_1^2h^2}g(\frac{H^2_{2,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{2,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,21}p_{m_2,11}\frac{\tau}{E_1F_1h^2}g(\frac{H^2_{3,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{3,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,21}p_{m_2,21}\frac{\tau}{E_1F_1h^2}g(\frac{H^2_{4,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{4,(k_1-m_1,k_2-m_2)} \end{array} \\ \\ \begin{array}{c} q_{m_1,11}p_{m_2,12}\frac{\tau}{E_1^2h^2}g(\frac{H^2_{1,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{1,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,11}p_{m_2,22}\frac{\tau}{E_1^2h^2}g(\frac{H^2_{2,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{2,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,21}p_{m_2,12}\frac{\tau}{E_1F_1h^2}g(\frac{H^2_{3,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{3,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,21}p_{m_2,22}\frac{\tau}{E_1F_1h^2}g(\frac{H^2_{4,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{4,(k_1-m_1,k_2-m_2)} \end{array} \\ \\ \begin{array}{c} q_{m_1,12}p_{m_2,11}\frac{\tau}{F_1E_1h^2}g(\frac{H^2_{1,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{1,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,12}p_{m_2,21}\frac{\tau}{F_1E_1h^2}g(\frac{H^2_{2,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{2,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,22}p_{m_2,11}\frac{\tau}{F_1^2h^2}g(\frac{H^2_{3,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{3,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,22}p_{m_2,21}\frac{\tau}{F_1^2h^2}g(\frac{H^2_{4,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{4,(k_1-m_1,k_2-m_2)} \end{array} \\ \\ \begin{array}{c} q_{m_1,12}p_{m_2,12}\frac{\tau}{F_1E_1h^2}g(\frac{H^2_{1,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{1,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,12}p_{m_2,22}\frac{\tau}{F_1E_1h^2}g(\frac{H^2_{2,(k_1-m_1,k_2-m_2)}}{E_1h})^2H^2_{2,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,22}p_{m_2,12}\frac{\tau}{F_1^2h^2}g(\frac{H^2_{3,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{3,(k_1-m_1,k_2-m_2)} \\ +q_{m_1,22}p_{m_2,22}\frac{\tau}{F_1^2h^2}g(\frac{H^2_{4,(k_1-m_1,k_2-m_2)}}{F_1h})^2H^2_{4,(k_1-m_1,k_2-m_2)} \end{array} \end{bmatrix}$$

$$
-\left[\begin{array}{c}
p_{m_1,11}q_{m_2,11}\frac{\tau}{E_2^2h^2}g(\frac{H^1_{1,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{1,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,11}q_{m_2,21}\frac{\tau}{E_2F_2h^2}g(\frac{H^1_{2,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{2,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,21}q_{m_2,11}\frac{\tau}{E_2^2h^2}g(\frac{H^1_{3,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{3,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,21}q_{m_2,21}\frac{\tau}{E_2F_2h^2}g(\frac{H^1_{4,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{4,(k_1-m_1,k_2-m_2)}\\
\\
p_{m_1,11}q_{m_2,12}\frac{\tau}{F_2E_2h^2}g(\frac{H^1_{1,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{1,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,11}q_{m_2,22}\frac{\tau}{F_2^2h^2}g(\frac{H^1_{2,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{2,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,21}q_{m_2,12}\frac{\tau}{F_2E_2h^2}g(\frac{H^1_{3,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{3,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,21}q_{m_2,22}\frac{\tau}{F_2^2h^2}g(\frac{H^1_{4,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{4,(k_1-m_1,k_2-m_2)}\\
\\
p_{m_1,12}q_{m_2,11}\frac{\tau}{E_2^2h^2}g(\frac{H^1_{1,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{1,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,12}q_{m_2,21}\frac{\tau}{E_2F_2h^2}g(\frac{H^1_{2,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{2,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,22}q_{m_2,11}\frac{\tau}{E_2^2h^2}g(\frac{H^1_{3,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{3,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,22}q_{m_2,21}\frac{\tau}{E_2F_2h^2}g(\frac{H^1_{4,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{4,(k_1-m_1,k_2-m_2)}\\
\\
p_{m_1,12}q_{m_2,12}\frac{\tau}{F_2E_2h^2}g(\frac{H^1_{1,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{1,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,12}q_{m_2,22}\frac{\tau}{F_2^2h^2}g(\frac{H^1_{2,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{2,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,22}q_{m_2,12}\frac{\tau}{F_2E_2h^2}g(\frac{H^1_{3,(k_1-m_1,k_2-m_2)}}{E_2h})^2H^1_{3,(k_1-m_1,k_2-m_2)}\\
+p_{m_1,22}q_{m_2,22}\frac{\tau}{F_2^2h^2}g(\frac{H^1_{4,(k_1-m_1,k_2-m_2)}}{F_2h})^2H^1_{4,(k_1-m_1,k_2-m_2)}
\end{array}\right]
$$

$$
-\begin{bmatrix}
\begin{array}{c}
q_{m_1,11}q_{m_2,11}\frac{\tau}{E_1^2E_2^2h^2}g\left(\frac{H^3_{1,(k_1-m_1,k_2-m_2)}}{E_1E_2h^2}\right)^2H^3_{1,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,11}q_{m_2,21}\frac{\tau}{E_1^2E_2F_2h^2}g\left(\frac{H^3_{2,(k_1-m_1,k_2-m_2)}}{E_1F_2h^2}\right)^2H^3_{2,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,21}q_{m_2,11}\frac{\tau}{E_1F_1E_2^2h^2}g\left(\frac{H^3_{3,(k_1-m_1,k_2-m_2)}}{F_1E_2h^2}\right)^2H^3_{3,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,21}q_{m_2,21}\frac{\tau}{E_1E_2F_1F_2h^2}g\left(\frac{H^3_{4,(k_1-m_1,k_2-m_2)}}{F_1F_2h^2}\right)^2H^3_{4,(k_1-m_1,k_2-m_2)}
\end{array}\\
\\
\begin{array}{c}
q_{m_1,11}q_{m_2,12}\frac{\tau}{F_2E_1^2E_2h^2}g\left(\frac{H^3_{1,(k_1-m_1,k_2-m_2)}}{E_1E_2h^2}\right)^2H^3_{1,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,11}q_{m_2,22}\frac{\tau}{E_1^2F_2^2h^2}g\left(\frac{H^3_{2,(k_1-m_1,k_2-m_2)}}{E_1F_2h^2}\right)^2H^3_{2,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,21}q_{m_2,12}\frac{\tau}{E_1F_2F_1E_2h^2}g\left(\frac{H^3_{3,(k_1-m_1,k_2-m_2)}}{F_1E_2h^2}\right)^2H^3_{3,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,21}q_{m_2,22}\frac{\tau}{E_1F_1F_2^2h^2}g\left(\frac{H^3_{4,(k_1-m_1,k_2-m_2)}}{F_1F_2h^2}\right)^2H^3_{4,(k_1-m_1,k_2-m_2)}
\end{array}\\
\\
\begin{array}{c}
q_{m_1,12}q_{m_2,11}\frac{\tau}{F_1E_1E_2^2h^2}g\left(\frac{H^3_{1,(k_1-m_1,k_2-m_2)}}{E_1E_2h^2}\right)^2H^3_{1,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,12}q_{m_2,21}\frac{\tau}{F_1E_2E_1F_2h^2}g\left(\frac{H^3_{2,(k_1-m_1,k_2-m_2)}}{E_1F_2h^2}\right)^2H^3_{2,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,22}q_{m_2,11}\frac{\tau}{F_1^2E_2^2h^2}g\left(\frac{H^3_{3,(k_1-m_1,k_2-m_2)}}{F_1E_2h^2}\right)^2H^3_{3,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,22}q_{m_2,21}\frac{\tau}{E_2F_1^2F_2h^2}g\left(\frac{H^3_{4,(k_1-m_1,k_2-m_2)}}{F_1F_2h^2}\right)^2H^3_{4,(k_1-m_1,k_2-m_2)}
\end{array}\\
\\
\begin{array}{c}
q_{m_1,12}q_{m_2,12}\frac{\tau}{F_1F_2E_1E_2h^2}g\left(\frac{H^3_{1,(k_1-m_1,k_2-m_2)}}{E_1E_2h^2}\right)^2H^3_{1,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,12}q_{m_2,22}\frac{\tau}{F_1E_1F_2^2h^2}g\left(\frac{H^3_{2,(k_1-m_1,k_2-m_2)}}{E_1F_2h^2}\right)^2H^3_{2,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,22}q_{m_2,12}\frac{\tau}{F_1^2F_2E_2h^2}g\left(\frac{H^3_{3,(k_1-m_1,k_2-m_2)}}{F_1E_2h^2}\right)^2H^3_{3,(k_1-m_1,k_2-m_2)}\\
+q_{m_1,22}q_{m_2,22}\frac{\tau}{F_1^2F_2^2h^2}g\left(\frac{H^3_{4,(k_1-m_1,k_2-m_2)}}{F_1F_2h^2}\right)^2H^3_{4,(k_1-m_1,k_2-m_2)}
\end{array}
\end{bmatrix}
\tag{7.4}
$$

Theorem (7.1): Suppose $[u_{1,(k_1,k_2)}\quad u_{2,(k_1,k_2)}\quad u_{3,(k_1,k_2)}\quad u_{4,(k_1,k_2)}]^T$ in Eq.(7.1) is the denoised signal after 1-step multiwavelet shrinking with $\underline{c}^0_{k_1,k_2} = [c_{2k_1,2k_2}\ c_{2k_1,2k_2+1}\ c_{2k_1+1,2k_2}\ c_{2k_1+1,2k_2+1}]^T = f(k_1h, k_2h)$, $(k_1,k_2)\in\mathbb{Z}^2$ as the original input, and $[u^1_{2k_1,2k_2}\quad u^1_{2k_1,2k_2+1}\quad u^1_{2k_1+1,2k_2}\quad u^1_{2k_1+1,2k_2+1}]^T$ in Eq.(7.4) be the signal after 1-step diffusing with original input $\underline{u}^0_{k_1,k_2} = [u^0_{2k_1,2k_2}\quad u^0_{2k_1,2k_2+1}\quad u^0_{2k_1+1,2k_2}\quad u^0_{2k_1+1,2k_2+1}]^T = f(k_1h, k_2h)$. Then

$u_{1,(k_1,k_2)} = u_{2k_1,2k_2}$ if:

$$S_{\theta_{11}}(x) = S_{\theta_{31}}(x) = x[1 - \frac{\tau}{E_2^2h^2}g(\frac{x^2}{E_2^2h^2})], \quad S_{\theta_{21}}(x) = S_{\theta_{41}}(x) = x[1 - \frac{\tau}{E_2F_2h^2}g(\frac{x^2}{F_2^2h^2})]$$
$$S_{\sigma_{11}}(x) = S_{\sigma_{21}}(x) = x[1 - \frac{\tau}{E_1^2h^2}g(\frac{x^2}{E_1^2h^2})], \quad S_{\sigma_{31}}(x) = S_{\sigma_{41}}(x) = x[1 - \frac{\tau}{E_1F_1h^2}g(\frac{x^2}{F_1^2h^2})]$$
$$S_{\gamma_{11}}(x) = x[1 - \frac{\tau}{E_1^2E_2^2h^4}g(\frac{x^2}{E_1^2E_2^2h^4})], \quad S_{\gamma_{21}}(x) = x[1 - \frac{\tau}{E_1^2E_2F_2h^4}g(\frac{x^2}{E_1^2F_2^2h^4})]$$
$$S_{\gamma_{31}}(x) = x[1 - \frac{\tau}{E_1F_1E_2^2h^4}g(\frac{x^2}{F_1^2E_2^2h^4})], \quad S_{\gamma_{41}}(x) = x[1 - \frac{\tau}{E_1E_2F_1F_2h^4}g(\frac{x^2}{F_1^2F_2^2h^4})],$$

$u_{2,(k_1,k_2)} = u_{2k_1,2k_2+1}$ if

$$S_{\theta_{12}}(x) = S_{\theta_{32}}(x) = x[1 - \frac{\tau}{F_2E_2h^2}g(\frac{x^2}{E_2^2h^2})], \quad S_{\theta_{22}}(x) = S_{\theta_{42}}(x) = x[1 - \frac{\tau}{F_2^2h^2}g(\frac{x^2}{F_2^2h^2})]$$
$$S_{\sigma_{12}}(x) = S_{\sigma_{22}}(x) = x[1 - \frac{\tau}{E_1^2h^2}g(\frac{x^2}{E_1^2h^2})], \quad S_{\sigma_{32}}(x) = S_{\sigma_{42}}(x) = x[1 - \frac{\tau}{E_1F_1h^2}g(\frac{x^2}{F_1^2h^2})]$$
$$S_{\gamma_{12}}(x) = x[1 - \frac{\tau}{F_2E_1^2E_2h^4}g(\frac{x^2}{E_1^2E_2^2h^4})]|, \quad S_{\gamma_{22}}(x) = x[1 - \frac{\tau}{E_1^2F_2^2h^4}g(\frac{x^2}{E_1^2F_2^2h^4})]$$
$$S_{\gamma_{32}}(x) = x[1 - \frac{\tau}{E_1F_2F_1E_2h^4}g(\frac{x^2}{F_1^2E_2^2h^4})], \quad S_{\gamma_{42}}(x) = x[1 - \frac{\tau}{E_1F_1F_2^2h^4}g(\frac{x^2}{F_1^2F_2^2h^4})],$$

$u_{3,(k_1,k_2)} = u_{2k_1+1,2k_2}$ if

$$S_{\theta_{13}}(x) = S_{\theta_{33}}(x) = x[1 - \frac{\tau}{E_2^2h^2}g(\frac{x^2}{E_2^2h^2})], \quad S_{\theta_{23}}(x) = S_{\theta_{43}}(x) = x[1 - \frac{\tau}{E_2F_2h^2}g(\frac{x^2}{F_2^2h^2})]$$
$$S_{\sigma_{13}}(x) = S_{\sigma_{23}}(x) = x[1 - \frac{\tau}{F_1E_1h^2}g(\frac{x^2}{E_1^2h^2})], \quad S_{\sigma_{33}}(x) = S_{\sigma_{43}}(x) = x[1 - \frac{\tau}{F_1^2h^2}g(\frac{x^2}{F_1^2h^2})]$$
$$S_{\gamma_{13}}(x) = x[1 - \frac{\tau}{F_1E_1E_2^2h^4}g(\frac{x^2}{E_1^2E_2^2h^4})], \quad S_{\gamma_{23}}(x) = x[1 - \frac{\tau}{F_1E_2E_1F_2h^4}g(\frac{x^2}{E_1^2F_2^2h^4})]$$
$$S_{\gamma_{33}}(x) = x[1 - \frac{\tau}{F_1^2E_2^2h^4}g(\frac{x^2}{F_1^2E_2^2h^4})], \quad S_{\gamma_{43}}(x) = x[1 - \frac{\tau}{E_2F_1^2F_2h^4}g(\frac{x^2}{F_1^2F_2^2h^4})]$$

and $u_{4,(k_1,k_2)} = u_{2k_1+1,2k_2+1}$ if

$$S_{\theta_{14}}(x) = S_{\theta_{34}}(x) = x[1 - \frac{\tau}{F_2E_2h^2}g(\frac{x^2}{E_2^2h^2})], \quad S_{\theta_{24}}(x) = S_{\theta_{44}}(x) = x[1 - \frac{\tau}{F_2^2h^2}g(\frac{x^2}{F_2^2h^2})]$$
$$S_{\sigma_{14}}(x) = S_{\sigma_{24}}(x) = x[1 - \frac{\tau}{F_1E_1h^2}g(\frac{x^2}{E_1^2h^2})], \quad S_{\sigma_{34}}(x) = S_{\sigma_{44}}(x) = x[1 - \frac{\tau}{F_1^2h^2}g(\frac{x^2}{F_1^2h^2})]$$
$$S_{\gamma_{14}}(x) = x[1 - \frac{\tau}{F_1F_2E_1E_2h^4}g(\frac{x^2}{E_1^2E_2^2h^4})], \quad S_{\gamma_{24}}(x) = x[1 - \frac{\tau}{F_1E_1F_2^2h^4}g(\frac{x^2}{E_1^2F_2^2h^4})]$$
$$S_{\gamma_{34}}(x) = x[1 - \frac{\tau}{F_1^2F_2E_2h^4}g(\frac{x^2}{F_1^2E_2^2h^4})], \quad S_{\gamma_{44}}(x) = x[1 - \frac{\tau}{F_1^2F_2^2h^4}g(\frac{x^2}{F_1^2F_2^2h^4})]. \quad (7.5)$$

7.3 Non-linear diffusion derived from two-dimensional multiple frame Shrinkage

Suppose $A, B^{(1)}$ and $B^{(2)}$ are B-spline tight multiple frame filter banks that defines as follow:

$$A(w) = \begin{pmatrix} \frac{1}{2} & \frac{1}{4}(e^{iw} + e^{-iw}) \\ \frac{1}{2}e^{-iw} & \frac{1}{4}(1 + e^{-2iw}) \end{pmatrix}, \quad B^{(1)}(w) = \begin{pmatrix} 0 & \frac{\sqrt{2}}{4}(e^{iw} - e^{-iw}) \\ 0 & \frac{\sqrt{2}}{4}(1 - e^{-2iw}) \end{pmatrix}$$

$$B^{(2)}(w) = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4}(e^{iw} + e^{-iw}) \\ \frac{1}{2}e^{-iw} & -\frac{1}{4}(1 + e^{-2iw}) \end{pmatrix}$$

Assume the 2D B-spline tight multiple frame filter banks with $w = (w_1, w_2)$ are difened as:

$$\begin{aligned} P(w) &= A(w_1) \otimes A(w_2), & Q^{(1)}(w) &= B^{(1)}(w_1) \otimes A(w_2) \\ Q^{(2)}(w) &= A(w_1) \otimes B^{(1)}(w_2), & Q^{(3)}(w) &= B^{(2)}(w_1) \otimes A(w_2) \\ Q^{(4)}(w) &= B^{(1)}(w_1) \otimes B^{(1)}(w_2), & Q^{(5)}(w) &= A(w_1) \otimes B^{(2)}(w_2) \\ Q^{(6)}(w) &= B^{(2)}(w_1) \otimes B^{(1)}(w_2), & Q^{(7)}(w) &= B^{(1)}(w_1) \otimes B^{(2)}(w_2) \\ Q^{(8)}(w) &= B^{(2)}(w_1) \otimes B^{(2)}(w_2). \end{aligned} \tag{7.6}$$

The highpass multifilters $\{Q^{(1)}, ..., Q^{(8)}\}$ have vanishing moment of orders:

$$\beta_1 = (1,0), \beta_2 = (0,1), \beta_3 = (2,0), \beta_4 = (1,1), \beta_5 = (0,2), \\ \beta_6 = (2,1), \beta_7 = (1,2), \beta_8 = (2,2)$$

Then the corresponding nonlinear diffusion equation to the 2D B-Spline tight multiple frame filter banks $\{Q^{(1)}, ..., Q^{(8)}\}$ is given by:

$$\begin{aligned} u_t = & \frac{\partial}{\partial x_1}[g_1((\frac{\partial u}{\partial x_1})^2)\frac{\partial u}{\partial x_1}] + \frac{\partial}{\partial x_2}[g_2((\frac{\partial u}{\partial x_2})^2)\frac{\partial u}{\partial x_2}] - \frac{\partial^2}{\partial {x_1}^2}[g_3((\frac{\partial^2 u}{\partial {x_1}^2})^2) \\ & \frac{\partial^2 u}{\partial {x_1}^2}] - \frac{\partial^2}{\partial x_1 x_2}[g_4((\frac{\partial^2 u}{\partial x_1 x_2})^2)\frac{\partial^2 u}{\partial x_1 x_2}] - \frac{\partial^2}{\partial x_2^2}[g_5((\frac{\partial^2 u}{\partial x_2^2})^2)\frac{\partial^2 u}{\partial x_2^2}] \\ & + \frac{\partial^3}{\partial x_1^2 \partial x_2}[g_6((\frac{\partial^3 u}{\partial x_1^2 \partial x_2})^2)\frac{\partial^3 u}{\partial x_1^2 \partial x_2}] + \frac{\partial^3}{\partial x_1 x_2^2}[g_7((\frac{\partial^3 u}{\partial x_1 x_2^2})^2)\frac{\partial^3 u}{\partial x_1 x_2^2}] \\ & - \frac{\partial^4}{\partial x_1^2 x_2^2}[g_8((\frac{\partial^4 u}{\partial x_1^2 x_2^2})^2)\frac{\partial^4 u}{\partial x_1^2 x_2^2}] \end{aligned} \tag{7.7}$$

with initial condition $u(x, 0) = f(x), x \in \mathbb{R}^2$,

$$C^{(1)}_{\beta_1} = C^{(2)}_{\beta_2} = -\frac{\sqrt{2}}{2}, C^{(3)}_{\beta_3} = -\frac{1}{4}, C^{(4)}_{\beta_4} = \frac{1}{2}, \\ C^{(5)}_{\beta_5} = -\frac{1}{4}, C^{(6)}_{\beta_6} = C^{(7)}_{\beta_7} = \frac{\sqrt{2}}{8}, C^{(8)}_{\beta_8} = \frac{1}{16}$$

where C_α defines as:

$$C_\alpha = \frac{i^{|\alpha|}}{\alpha!}\frac{\partial^\alpha}{\partial w^\alpha}q(w)|_{w=0}; \quad \frac{\partial^{\alpha_1+\alpha_2}}{\partial w_2^{\alpha_2}\partial w_1^{\alpha_1}}, \quad |\alpha| = \alpha_1 + \alpha_2, \ and \ \ \alpha! = \alpha_1!\alpha_2!$$

Approximation solution of Eq.(7.7) in discrete setting provides that $u_{1,(k_1,k_2)} = u^1_{2k_1,2k_2}$ if:

$$\begin{aligned}
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{2\tau}{h^2}g_l(\frac{2x^2}{h^2})], \ l = 1,2 \\
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{16\tau}{h^4}g_l(\frac{16x^2}{h^4})], \ l = 3,5 \\
S^{(4)}_{\theta^l_{11}}(x) = S^{(4)}_{\theta^l_{21}}(x) = S^{(4)}_{\theta^l_{31}}(x) = S^{(4)}_{\theta^l_{41}}(x) &= x[1 - \frac{4\tau}{h^4}g_4(\frac{4x^2}{h^4})], \\
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{32\tau}{h^6}g_l(\frac{32x^2}{h^6})], \ l = 6,7 \\
S^{(8)}_{\theta^l_{11}}(x) = S^{(8)}_{\theta^l_{21}}(x) = S^{(8)}_{\theta^l_{31}}(x) = S^{(8)}_{\theta^l_{41}}(x) &= x[1 - \frac{256\tau}{h^8}g_8(\frac{256x^2}{h^8})]. \qquad (7.8)
\end{aligned}$$

$u_{2,(k_1,k_2)} = u_{2k_1+1,2k_2}$, $u_{3,(k_1,k_2)} = u_{2k_1,2k_2+1}$ and $u_{4,(k_1,k_2)} = u_{2k_1+1,2k_2+1}$ if the shrinkage function and diffusivity are also satisfying Eq.(7.8).

Theorem (7.2): Let $[u_{1,(k_1,k_2)} \ \ u_{2,(k_1,k_2)} \ \ u_{3,(k_1,k_2)} \ \ u_{4,(k_1,k_2)}]^T$ be the denoised signal after 1-step multiple frame shrinking with $\underline{c}^0_{k_1,k_2} = [c_{2k_1,2k_2} \ c_{2k_1,2k_2+1} \ c_{2k_1+1,2k_2} \ c_{2k_1+1,2k_2+1}]^T = f(k_1h, k_2h), \ (k_1,k_2) \in \mathbb{Z}^2$ as the original input, and using multiple frame filter banks $\{P, Q^{(1)}, ..., Q^{(8)}\}$ defined in Eq(7.6). Let $[u^1_{2k_1,2k_2} \ \ u^1_{2k_1,2k_2+1} \ \ u^1_{2k_1+1,2k_2} \ \ u^1_{2k_1+1,2k_2+1}]^T$ be the signal after 1-step diffusing with original data $\underline{u}^0_{k_1,k_2} = [u^0_{2k_1,2k_2} \ \ u^0_{2k_1,2k_2+1} \ \ u^0_{2k_1+1,2k_2} \ \ u^0_{2k_1+1,2k_2+1}]^T = f(k_1h, k_2h)$. Then the solution of nonlinear diffusion Eq(7.7) provided that the shrinkage functions of the multiple frame shrinking algorithm are chosen as:

$$\begin{aligned}
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{2\tau}{h^2}g_l(\frac{2x^2}{h^2})], \ l = 1,2 \\
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{16\tau}{h^4}g_l(\frac{16x^2}{h^4})], \ l = 3,5 \\
S^{(4)}_{\theta^l_{11}}(x) = S^{(4)}_{\theta^l_{21}}(x) = S^{(4)}_{\theta^l_{31}}(x) = S^{(4)}_{\theta^l_{41}}(x) &= x[1 - \frac{4\tau}{h^4}g_4(\frac{4x^2}{h^4})], \\
S^{(l)}_{\theta^l_{11}}(x) = S^{(l)}_{\theta^l_{21}}(x) = S^{(l)}_{\theta^l_{31}}(x) = S^{(l)}_{\theta^l_{41}}(x) &= x[1 - \frac{32\tau}{h^6}g_l(\frac{32x^2}{h^6})], \ l = 6,7 \\
S^{(8)}_{\theta^l_{11}}(x) = S^{(8)}_{\theta^l_{21}}(x) = S^{(8)}_{\theta^l_{31}}(x) = S^{(8)}_{\theta^l_{41}}(x) &= x[1 - \frac{256\tau}{h^8}g_8(\frac{256x^2}{h^8})].
\end{aligned}$$

Assume Perona-Malik diffusivity is given by:

$$g(x^2) = \frac{c}{1 + (\frac{x}{\lambda})^2}$$

where c is a constant, then the corresponding shrinkage functions are

$$\begin{cases} S^{(l)}_{\theta^{(l)}}(x) = x[1 - \frac{2\tau c_1}{[1+[\sqrt{2}x/\theta^{(l)}]^2]}]; \ l = 1,2 \\ S^{(4)}_{\theta^{(4)}}(x) = x[1 - \frac{4\tau c_1}{[1+[2x/\theta^{(4)}]^2]}] \\ S^{(6)}_{\theta^{(6)}}(x) = x[1 - \frac{32\tau c_1}{[1+[4\sqrt{2}x/\theta^{(6)}]^2]}] \end{cases} \tag{7.9}$$

$$\begin{cases} S^{(l)}_{\theta^{(l)}}(x) = x[1 - \frac{16\tau c_2}{[1+[4x/\theta^{(l)}]^2]}]; \ l = 3,5 \\ S^{(7)}_{\theta^{(l)}}(x) = x[1 - \frac{32\tau c_2}{[1+[4\sqrt{2}x/\theta^{(7)}]^2]}] \\ S^{(8)}_{\theta^{(8)}}(x) = x[1 - \frac{256\tau c_2}{[1+[16x/\theta^{(8)}]^2]}] \end{cases} \tag{7.10}$$

where the spatial step size h=1. If the Weickert diffusivity g is defined as:

$$g(x^2) = \begin{cases} 1 & if \quad x = 0 \\ 1 - exp(-3.31488\lambda^8/x^8) & if \quad x \neq 0, \end{cases}$$

then the corresponding multiple frame shrinkage functions are given by:

$$S^{(l)}_{\theta^{(l)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 2\tau[1 - exp(-3.31488\ (\theta^{(l)})^8/(\sqrt{2}x)^8)\ ]) & if \quad x \neq 0\ ; \quad l = 1,2 \end{cases}$$

$$S^{(4)}_{\theta^{(4)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 4\tau[1 - exp(-3.31488\ (\theta^{(4)})^8/(2x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

$$S^{(6)}_{\theta^{(6)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 32\tau[1 - exp(-3.31488\ (\theta^{(6)})^8/(4\sqrt{2}x)^8)\ ]) & if \quad x \neq 0 \end{cases} \tag{7.11}$$

$$S^{(l)}_{\theta^{(l)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 16\tau[1 - exp(-3.31488\ (\theta^{(l)})^8/(4x)^8)\ ]) & if \quad x \neq 0\ ; \quad l = 3,5 \end{cases}$$

$$S^{(7)}_{\theta^{(7)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 32\tau[1 - exp(-3.31488\ (\theta^{(7)})^8/(4\sqrt{2}x)^8)\ ]) & if \quad x \neq 0 \end{cases}$$

$$S^{(8)}_{\theta^{(8)}}(x) = \begin{cases} 0 & if \quad x = 0 \\ x(1 - 256\tau[1 - exp(-3.31488\ (\theta^{(8)})^8/(16x)^8)\ ]) & if \quad x \neq 0 \end{cases} \tag{7.12}$$

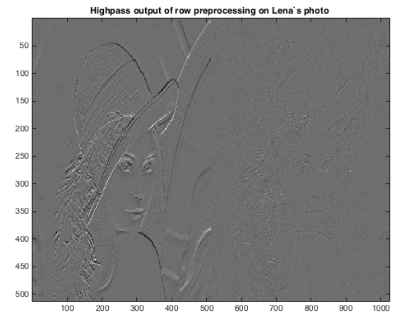
The Hard shrinkage functions are defined as

$$S^{(l)}_{\theta^{(l)}}(x) = \begin{cases} 0 & if \quad |x| \leq \theta^{(l)} \\ x & if \quad |x| > \theta^{(l)} \end{cases} \qquad l = 1, 2..., 8,$$

and the Soft shrinkage functions are given by

$$S^{(l)}_{\theta^{(l)}}(x) = \begin{cases} 0 & if \quad |x| \leq \theta^{(l)} \\ x - \theta^{(l)} sgn(x) & if \quad |x| > \theta^{(l)} \end{cases} \qquad l = 1, 2..., 8,$$

(a)



(b)

Figure 7.1: The result of row preprocessing on Lena image

7.4 Denoising of images

The performance of the image denoising based on B-Spline multiple frame filter bank was tested on Lena and Barbara images with various shrinkage functions.

For Lena image, we added Gaussian noise to the original Lena image, then we applied B-Spline multiple frame shrinking iteratively 6 times to the noised image with PSNR=17. However, for Barbara image, we applied B-Spline multiple frame shrinking iteratively 4 times to the noised image with PSNR=20.
Table 7.1 presents the Peak SNRs of the image denoising results with different shrinkage functions that define as [29]:

$$PSNR = 10.\ log_{10}(\frac{MAX_I^2}{MSE})$$

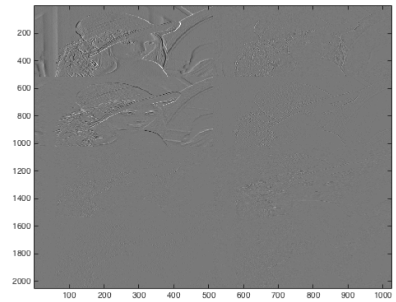
where MAX and MSE are the maximum possible pixel value of the image and mean squared error,respectively:

$$MSE = \frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[I(i,j) - K(i,j)]^2$$

We set $h = 1, \tau = \frac{1}{4}, c_1 = 1, c_2 = \frac{1}{8}$, if we are applying Perona-Malik(PM) diffusivity functions Eq.(7.9) and Eq.(7.10). However, we set $h = 1, \tau = \frac{1}{4}$ if we are applying Weickert diffusivity-based Eq.(7.11), and $h = 1, \tau = \frac{1}{16}$ if Eq.(7.12) is applying. The parameters $\theta^{(l)}$ are chosen such that the PSNRs of the denoised image are as big as possible.

Next, we proceed to compare image denoising results using diffusivity-inspired multiple frame shrinkage with image denoising results using diffusion in terms of wavelet-shrinkage. In figure 7.3 and 7.4, (c) and (d) provide better results compared to (a) and (b).

(a) Lowpass output



(b) Highpass output

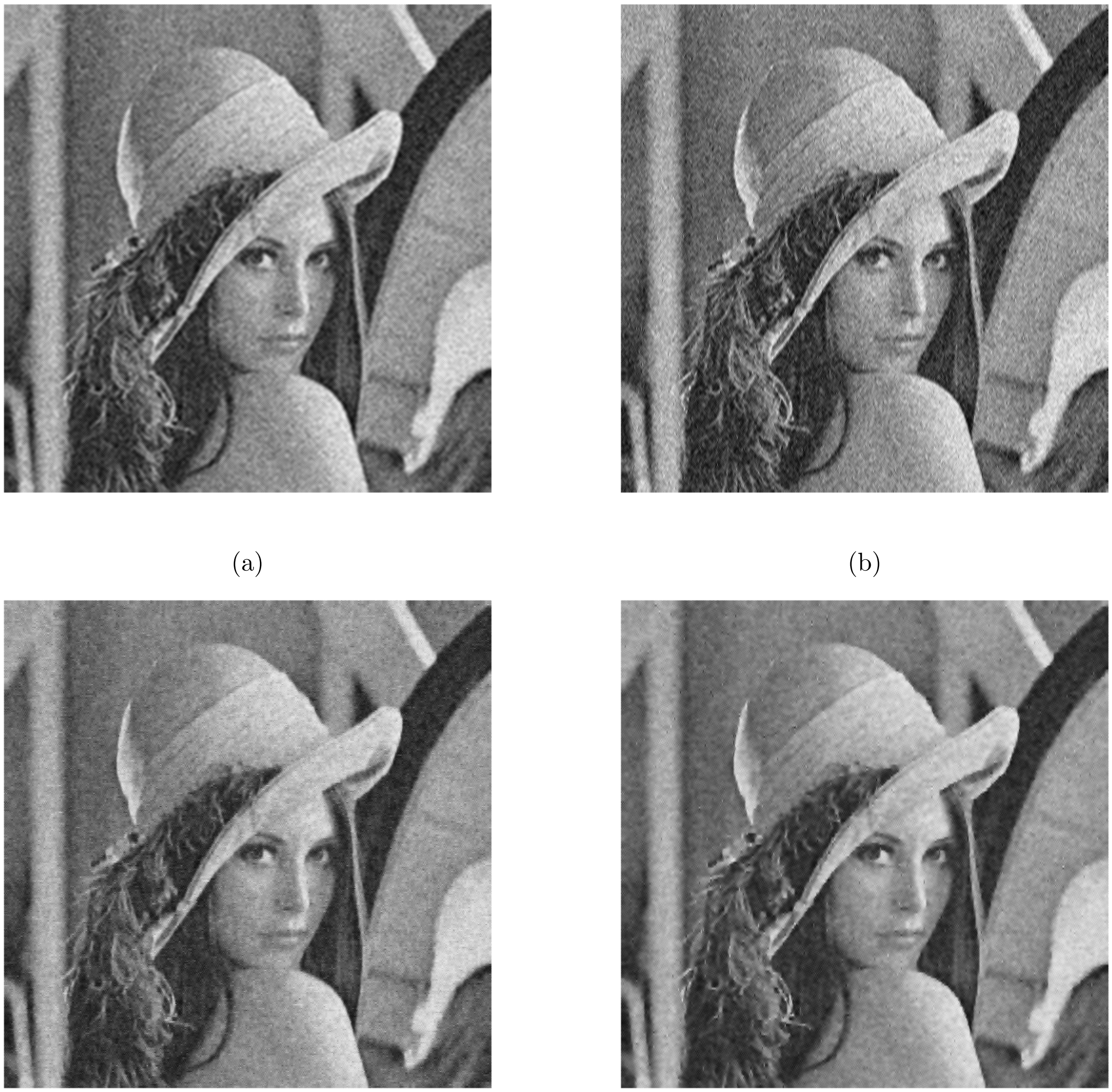
Figure 7.2: The result of column preprocesing on figure 7.1

Table 7.1: Image denoising results using diffusion-inspired multiple frame shrinkage functions.

Shrinkage Method	$S_{\theta^l}^{(l)} = PM$	$S_{\theta^l}^{(l)} = Weickert$	$S_{\theta^l}^{(l)} = Hard$	$S_{\theta^l}^{(l)} = Soft$
PSNR for Lena image	25.9752	25.6317	24.4472	26.3126
PSNR for Barbara image	24.8184	23.9868	24.1845	23.5858

Table 7.2: Image denoising results using diffusion-inspired wavelet shrinkage functions.

Shrinkage Method	$S_\theta = PM$	$S_\theta = Weickert$	$S_\theta = Hard$	$S_\theta = Soft$
PSNR for Lena image	23.4431	23.6149	23.1068	23.4021
PSNR for Barbara image	23.2059	21.6554	23.1648	21.5039



(a) (b)

(c) (d)

Figure 7.3: (a),(b):Denoised Lena image with Perona-Malik shrinkage and Weickert shrinkage, respectively, using wavelet filter banks, (c),(d):Denoised Lena image with Perona-Malik shrinkage and Weickert shrinkage, respectively, using multiple frame filter banks.

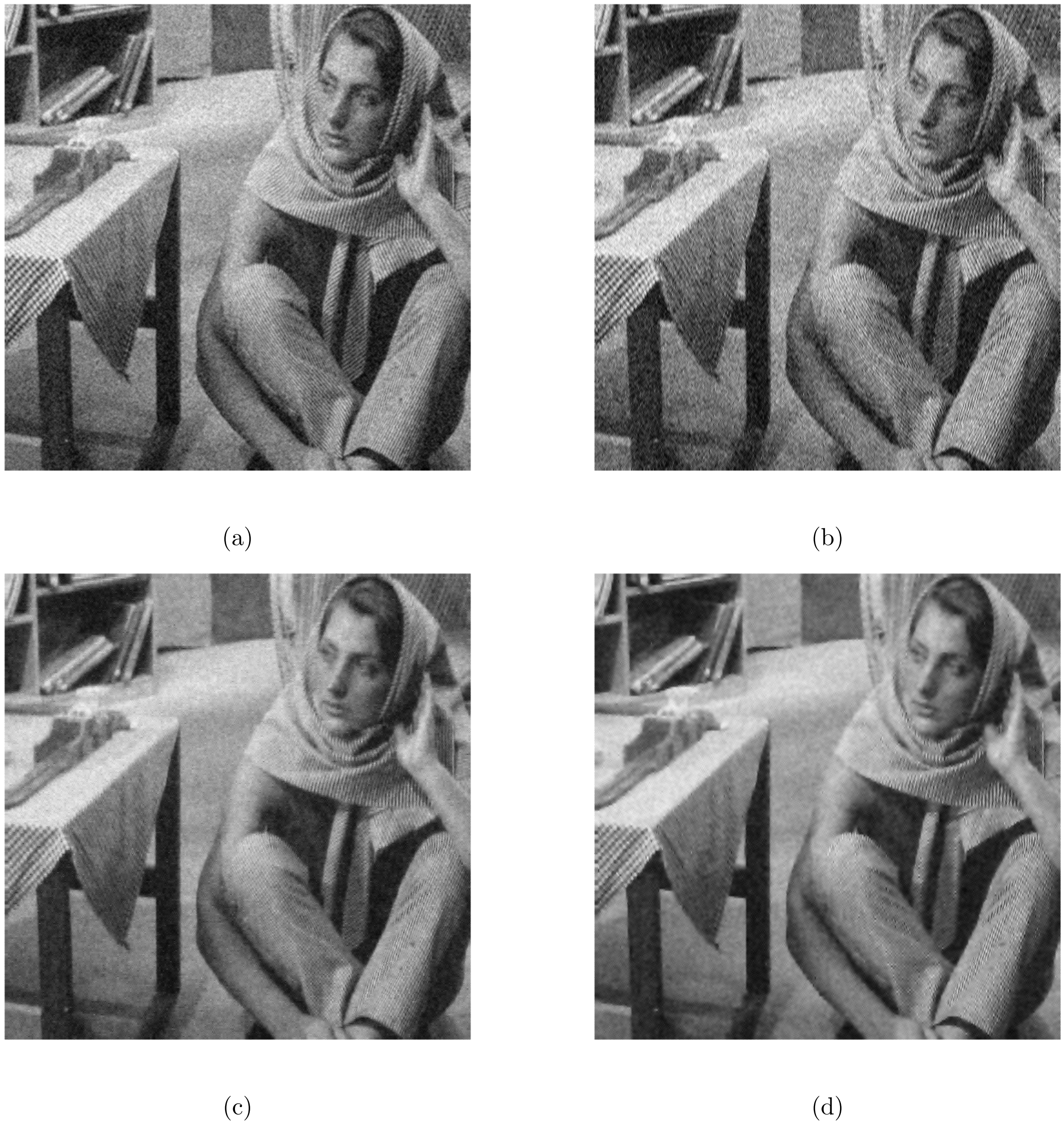


Figure 7.4: (a),(b): Denoised Barbara image with Perona-Malik shrinkage and Weickert shrinkage, respectively, using D_4 wavelet filter banks, (c),(d): Denoised Barbara image with Perona-Malik shrinkage and Weickert shrinkage, respectively, using B-Spline multiple frame filter banks.

Chapter 8

Conclusion and Future Work

This chapter presents the primary conclusion arising from this dissertation.

First, the correspondence between one-dimensional Haar, CL(2), DGHM multiwavelet shrinkage and second-order nonlinear diffusion is formulated and discussed. Also, we formulated one-dimensional high-order nonlinear diffusion derived from multiwavelet shrinkage. The experiment results on CL(2) multiwavelet shrinkage is presented. Further, we compared the results with D_4 wavelet shrinkage. Equivalence between one-dimensional B-Spline multiple frame shrinkage and nonlinear diffusion equation are provided. We tested this approach to different noised signals and with different shrinkage functions. In addition, we compared the result with the original scheme that presents nonlinear diffusion equation in terms of Ron-shen frame shrinkage. According to the results, this scheme provided better results than the original approach.

Second, we show how two-dimensional multiwavelet shrinkage corresponds to two-dimensional nonlinear diffusion equation. Also, we provided new algorithms that present correspondence between two-dimensional B-Spline multiple frame shrinkage and nonlinear diffusion equation. We examined this scheme to different noised images and with different shrinkage functions. Furthermore, we compared the results with two-dimensional D_4 wavelet shrinkage.

Future work can investigate the association between channel mixed multiple frame shrinkage and channel mixed nonlinear diffusion. Two dimension B-spline multiple frame shrinkage can be generalized easily to other multiple frame system and high-order diffusion can be derived similary. Also, equivalence between non-linear diffusion and multiwavelet shrinkage/ multiple frame shrinkage can be easily extended to multi-level setting.

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