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Jawaher Alzahrani
University of Missouri-St. Louis, jha2qc@mail.umsl.edu

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Jawaher Hamoud Alzahrani,
M.Sc. in Mathematics, University of Western Ontario,Canada, 2011,
B.Sc. in Mathematics, Taif University, Saudi Arabia, 2004

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Advisory Committee:
Qingtang Jiang, Ph.D.(Chair)
Haiyan Cai, Ph.D.
Wenjie He, Ph.D.
Yuefeng Wu, Ph.D.

Abstract

The prevalence of the separation of multicomponent non-stationary signals across many elds of research makes this concept an important subject of study. The synchrosqueezing transform (SST) is a particular type of reassignment method. It aims to separate and recover the components of a multicomponent non-stationary signal. The short time Fourier transform (STFT)-based SST (FSST) and the continuous wavelet transform (CWT)-based SST (WSST) have been used in engineering and medical data analysis applications. The current study introduces the dierent versions of FSST and WSST to estimate instantaneous frequency (IF) and to recover components. It has a good concentration and reconstruction for a wide variety of amplitude and frequency modulated multicomponent signals. Earlier studies have improved existing FSSTs by computing more accurate estimates of the IFs of the modes making up the signal. The higher order approximations for both the amplitude and phase were used. Therefore, there is a better concentration and reconstruction for a wider variety of AM-FM modes than what was possible with current synchrosqueezing techniques. In this study, we propose to improve the adaptive FSST, the adaptive WSST, and to introduce a new type of 2nd-order FSST with a new phase transformation. We use higher order approximations for both the amplitude and phase function. We study the higher order adaptive FSST and adaptive WSST. The result shows an even better concentration and reconstruction for a wider variety of AM-FM modes with the higher order adaptive SSTs. We also study the theoretical analysis of the 2nd-order FSST with a new phase transformation. The new phase transformation introduced by us is much simpler than the convectional one, while the performance in IF estimation and component recovery of the new 2nd-order FSST is comparable with, and even better in some cases than, that of the conventional 2nd-order FSST.

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Chapter 1

Introduction

Non-stationary signals can be modeled as superpositions of band limited, amplitude and frequency modulated (AM-FM) sub-signals. Non-stationary signals can be shown as,

$$x(t) = A_0(t) + \sum_{k=1}^{K} A_k(t) cos(2\pi\phi_k(t)) \tag{1.0.1}$$

with $A_k(t)$ and $\phi_k'(t) > 0$, where $A_k(t)$ is the instantaneous ampllitude (IA), and $\phi_k'(t)$ is the instantaneous frequenc (IF) of $x_k(t)$. The study of separating multicomponent non-stationary signals is a significant research topic in many different fields such as engineering and medical data analysis applications.

In non-stationary signal analysis [1, 2, 3], one of the most important tools is the time-frequency analysis (TFA). The continuous wavelet transform (CWT) and the short time Fourier transform (STFT) are the most common methods in TFA. CWT and STFT are linear time-frequency representations and applicable to component reconstruction. CWT and STFT have suffered because of the uncertainly principle that imposes an unavoidable tradeoff between time and frequency resolutions (see e.g. [3, 4, 5]).

The synchrosqueezing transform (SST) is a particular type of reassignment method on the CWT and STFT used to sharpen the time-frequency representation of signals and recover the components of a multicomponent. In previous research, the SST has introduced and further developed (see [6, 7]). Using the CWT-based SST(WSST) [7, 8], Thakur and Wu proposed an extention of the SST given by the STFT-based SST (FSST) [9]. Then, researchers investigated the SST to greater lengths. The FSST and the WSST were also topics of study for Daubechies, Meignen, Wu, Iatsenko [7, 10, 11, 12, 13]. The 2nd-order SST is proposed in [10, 11]. Likewise, the adaptive FSST, the adaptive 2nd-order FSST with a time-varying, the adaptive WSST, and the adaptive 2nd-order WSST with time-varying have been studied [14, 15]. Recently, Sheu, Hsu, Chou, and Wu introduced a method to select the time-varying window width for sharp SST representation by minimizing the Renyi entropy [16]. The SST is a useful tool in engineering and medical data application. This includes radar, sonar, anesthesia evaluation, and heartbeat classification. In 2017, Pham and Meignen improved existing STFT-based SSTs by computing more accurate estimates of the instantaneous frequencies (IFs) of the modes making up the signal [17]. They used higher order approximations both for the amplitude and phase. They concluded that there is a better concentration and reconstruction for a wider variety of AM-FM modes than what was possible with current synchrosqueezing techniques. In this study, we used the same technique. We propose to improve the WSST, the adaptive WSST,the adaptive FSST, and the FSST with a new phase transformation by using higher order approximations both for the amplitude and phase. The result also shows an ideal concentration and reconstruction for a wider variety of AM-FM modes.

The theorretical analysis of the 2nd-order FSST was proposed in [18]. Very recently, the theoretical analysis of the FSST obtains the error bounds for the instantaneous frequency (IF) estimation and component recovery with the conventional 2nd-order FSST as was introduced in [16, 19]. We study the theoretical analysis of the 2nd-order FSST with a new phase transformation.

The organization of the remainder of this dissertation is as follows. First, we start with an overview of CWT-based SST and the adaptive CWT-based SST with a time-varying parameter as shown in Chapter 2. The STFT-based SST and the adaptive STFT-based SST with a time-varying parameter are also derived in Chapter 2. In Chapter 3, we propose the higher-order WSS and FSST and the new higher order FSST with the new phase transformation. We address the numerical simulation in Chapter 4. The theoretical analysis of the new formulation of the FSST is described in Chapter 5. Finally, we present the conclusion and future work in Chapter 6.

Chapter 2

Preliminaries

In this chapter, some of the basic concepts are presented. The continuous wavelet transform-based synchrosqueezed transform (WSST) are discussed in Section 2.1. WSST with a time varying parameter is studied in Section 2.2. In addition, Section 2.3 is devoted to the short-time Fourier transform-based synchrosqueezed transform (FSST). FSST with a time varying parameter are discussed in Section 2.4. The analysis of FSST and adaptive FSST are provided in Section 2.5 and 2.6.

2.1 Continuous wavelet transform-based synchrosqueezed transform (WSST)

A funcation $\psi(t) \in L_2(\mathbb{R})$ is called a continuous wavelet if it satisfies (see e.g. [20, 21]) the following condition

$$C_\psi := \int_{-\infty}^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{|\xi|} < \infty \text{ and } \psi_{a,b}(t) = \frac{1}{a}\psi(\frac{t-b}{a})$$

where $\hat{\psi}$ is the Fourier transform of a signal $\psi(t)$ is defined by

$$\hat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(t) e^{-i2\pi\xi t} dt.$$

Definition 2.1.1. We can define the continuous wavelet transform (CWT) of a signal $x(t) \in L_2(\mathbb{R})$ with a continuous wavelet $\psi(t)$ as

$$W_x(a,b) = \int_{-\infty}^{\infty} x(t)\frac{1}{a}\overline{\psi(\frac{t-b}{a})}dt = \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi. \qquad (2.1.1)$$

In this instance, a is the scale variable, and b is the time variable.

Note that Fourier transform $\hat{\psi}_{a,b}(\xi)$ of $\psi_{a,b}(t)$ is $\hat{\psi}(a\xi)e^{-i2\pi b\xi}$. For more details see [7].

proposition: A function $x(t)$ is an analytic signal if it satisfies $\hat{x}(\xi) = 0$ for $\xi < 0$. This is defined by

$$W_x(a,b) = \int_{0}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi. \qquad (2.1.2)$$

Proof. If a and b are two real numbers, then the Fourier transform is

$$\hat{\psi}_{a,b}(\xi) = \int_{-\infty}^{\infty} \psi_{a,b}(t)e^{-2i\pi b\xi t}dt = \int_{-\infty}^{\infty} \frac{1}{a}\psi(\frac{t-b}{a})e^{-2i\pi b\xi t}dt$$

A change of variable $y = \frac{t-b}{a}$, which implies

$$\begin{aligned}\hat{\psi}_{a,b}(\xi) &= \int_{-\infty}^{\infty} \psi(y)e^{-2i\pi(ay+b)\xi}dy \\ &= e^{-2i\pi b\xi}\int_{-\infty}^{\infty} \psi(y)e^{-2i\pi ay\xi}dy = e^{-2i\pi b\xi}\hat{\psi}(a\xi)\end{aligned}$$

Then we have

$$\begin{aligned}W_x(a,b) &= < x, \psi_{a,b} > = < \hat{x}, \hat{\psi}_{a,b} > \\ &= \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)e^{-2i\pi b\xi}}d\xi \\ &= \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi\end{aligned}$$

If $x(t)$ is analytic or $\psi(t)$ is analytic, then for $a > 0$ and $\xi < 0$, $\hat{\psi}(a\xi) = 0$.

$$\begin{aligned} W_x(a,b) &= \int_{-\infty}^{0} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi + \int_{0}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi \\ &= \int_{0}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi. \end{aligned}$$

$\square$

properties(Fourier Transform). Let $x, y \in L_2(\mathbb{R})$ are two signals, then we have

$$\begin{aligned} \int_{-\infty}^{\infty} x(\xi)\overline{y(\xi)}d\xi &= \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{y}(\xi)}d\xi \\ \int_{-\infty}^{\infty} \hat{x}(\xi)y(\xi)d\xi &= \int_{-\infty}^{\infty} x(\xi)\hat{y}(\xi)d\xi \end{aligned}$$

Example: Let a be a positive real. We define the function $f_a(x) = e^{-ax^2}$, then we have

$$\hat{f}_a(\xi) = \int_{-\infty}^{\infty} f(t)e^{-2i\pi\xi t}dt = \int_{-\infty}^{\infty} e^{-2i\pi\xi t}e^{-at^2}dt$$

Consider the function

$$g(y) = \int_{-\infty}^{\infty} e^{-at^2+yt}dt,\ for\ y \in \mathbb{R}$$

Completing squares, we have

$$\begin{aligned} g(y) &= \int_{-\infty}^{\infty} e^{-a(t-\frac{y}{2a})^2+\frac{y^2}{4a}}dt = \frac{1}{\sqrt{a}}e^{\frac{y^2}{4a}}\int_{-\infty}^{\infty} e^{-x^2}dx \\ &= \frac{1}{\sqrt{a}}e^{\frac{y^2}{4a}}\sqrt{\pi} = \sqrt{\frac{\pi}{a}}e^{\frac{y^2}{4a}} \end{aligned}$$

Let $h(y) = \sqrt{\frac{\pi}{a}}e^{\frac{y^2}{4a}}$ be extended to be entire analytic functions, and since they agree on $\mathbb{R}$ as shown, they must agree on the complex plane $\mathbb{C}$. In particular, by setting $y = -2i\pi\xi$, we have

$$
\begin{aligned}
\hat{f}_a(\xi) &= \int_{-\infty}^{\infty} e^{-2i\pi\xi t} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} e^{\frac{(-2i\pi\xi)^2}{4a}} \\
&= \sqrt{\frac{\pi}{a}} e^{\frac{-4\pi^2\xi^2}{4a}}.
\end{aligned} \tag{2.1.3}
$$

Examples of Continuous Wavelet Transform

Bump Wavelet: The Bump wavelet is defined by

$$
\hat{\psi}(\xi) = e^{1-\frac{1}{1-\sigma^2(\xi-\mu)^2}} \chi_{(\mu-\frac{1}{\sigma},\mu+\frac{1}{\sigma})} \tag{2.1.4}
$$

where $\sigma, \mu > 0$, such that $\sigma\mu > 1$.

Morlet Wavelet: Morlet wavelet ψ_σ is defined by the function

$$
\psi_\sigma(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{t^2}{2\sigma^2}} (e^{i2\pi\mu t} - e^{-2\pi^2\mu^2\sigma^2}) \tag{2.1.5}
$$

where $\sigma > 0$, μ is a constant.

Lemma 2.1.2. *Let $\sigma > 0$ and μ two constant, then the Fourier transform of Mortlet wavelet given by*

$$
\hat{\psi}_\sigma(\xi) = e^{-2\pi^2\sigma^2(\xi-\mu)^2} - e^{-2\pi^2\sigma^2(\xi^2+\mu^2)}.
$$

Proof. Let ψ_σ Morlet's complex wavelet, the Fourier transform of $\hat{\psi}_\sigma$ is defined by

$$
\begin{aligned}
\hat{\psi}_\sigma(\xi) &= \tfrac{1}{\sigma\sqrt{2\pi}}\Big(\int_{-\infty}^{\infty} e^{-2i\pi\xi t} e^{-\frac{t^2}{2\sigma^2}} e^{2i\pi\mu t} dt - e^{-2\pi^2\mu^2\sigma^2} \int_{-\infty}^{\infty} e^{-2i\pi\xi t} e^{-\frac{t^2}{2\sigma^2}} dt\Big) \\
&= \tfrac{1}{\sigma\sqrt{2\pi}}\Big(\int_{-\infty}^{\infty} e^{-2i\pi(\xi-\mu)} e^{-\frac{t^2}{2\sigma^2}} dt - e^{-2\pi^2\mu^2\sigma^2} \int_{-\infty}^{\infty} e^{-2i\pi\xi t} e^{-\frac{t^2}{2\sigma^2}} dt\Big)
\end{aligned}
$$

According to the example of Fourier transform (2.1.3)

$$\begin{aligned}
\hat{\psi}_\sigma(\xi) &= \frac{1}{\sigma\sqrt{2\pi}}\Big(\hat{f}_{\frac{1}{2\sigma^2}}(\xi-\mu) - e^{-2\pi^2\mu^2\sigma^2}\hat{f}_{\frac{1}{2}\sigma^2}(\xi)\Big)\\
&= \frac{1}{\sigma\sqrt{2\pi}}\Big(\sqrt{\pi 2\sigma^2}e^{-2\pi^2\sigma^2(\xi-\mu)} - e^{-2\pi^2\mu^2\sigma^2}\sqrt{2\pi\sigma^2}e^{-2\pi^2\xi^2}\Big)\\
&= \Big(e^{-2\pi^2\sigma^2(\xi-\mu)} - e^{-2\pi^2\sigma^2(\xi^2+\mu^2)}\Big).
\end{aligned}$$

□

Maxican hat Wavelet:
Let $\sigma > 0$, then the Maxican hat wavelet is defined by

$$\psi_\sigma(t) = \big(1 - \frac{t^2}{\sigma^2}\big)e^{-\frac{t^2}{2\sigma^2}} \tag{2.1.6}$$

Lemma 2.1.3. *The Fourier Transform of Maxican hat wavelet is given by*

$$\hat{\psi}_\sigma(\xi) = \begin{cases} (2\pi\sigma\xi)^2\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2} & \textit{for } \ \xi > 0 \\ 0 & \textit{for } \ \xi \leq 0 \end{cases} \tag{2.1.7}$$

Proof. The Fourier Transform of Maxican hat wavelet function ψ_σ

$$\begin{aligned}
\hat{\psi}_\sigma(\xi) &= \int_{-\infty}^{\infty}(1-\frac{t^2}{\sigma^2})e^{-\frac{t^2}{2\sigma^2}}e^{-2i\pi t\xi}dt\\
&= \underbrace{\int_{-\infty}^{\infty}e^{-\frac{t^2}{2\sigma^2}}e^{-2i\pi t\xi}dt}_{I_1} + \underbrace{\int_{-\infty}^{\infty}\frac{t^2}{\sigma^2}e^{-\frac{t^2}{2\sigma^2}}e^{-2i\pi t\xi}dt}_{I_2}.
\end{aligned}$$

Calculation of the integral $I_1 = \mathcal{F}(f_{\frac{1}{2\sigma^2}})(\xi) = \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}$.

Calculation of the integral $I_2 = \mathcal{F}(t^2e^{-\frac{t^2}{2\sigma^2}})(\xi)$. Using the Fourier transform derivation formula $\mathcal{F}((-2i\pi t)^m f(t))(\xi) = (\mathcal{F}(f))^{(m)}(\xi)$

$$\begin{aligned}
\mathcal{F}((-2i\pi t)^2 f_{\frac{1}{2\sigma^2}}(t))(\xi) &= (\mathcal{F}(f_{\frac{1}{2\sigma}})(\xi))^{(2)} = \frac{d^2}{d^2\xi}\Big(\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}\Big)\\
&= \frac{d}{d\xi}\Big(\sqrt{2\pi\sigma^2}(-4\pi^2\sigma^2\xi)e^{-2\pi^2\sigma^2\xi^2}\Big)\\
&= -4\pi^{-2}\sigma^2\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi} + 4^2\pi^4\sigma^4\xi^2\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}.
\end{aligned}$$

This result in

$$
\begin{aligned}
\mathcal{F}(\tfrac{t^2}{\sigma^2} f_{\frac{1}{2\sigma}}(t))(\xi) &= \frac{1}{-4\pi^2\sigma^2}(\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2})(-4\pi^2\sigma^2 + 4^2\pi^4\sigma^4\xi^2) \\
&= \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}(1 - \pi^2\sigma^2\xi^2)
\end{aligned}
$$

We group the integrals calculation above

$$
\begin{aligned}
\hat{\psi}(\xi) &= \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2} - \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}(1 - 4\pi^2\sigma^2\xi^2) \\
&= \sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}(1 - 1^2 + 4\pi^2\sigma^2\xi^2) \\
&= (2\pi\sigma\xi)^2\sqrt{2\pi\sigma^2}e^{-2\pi^2\sigma^2\xi^2}.
\end{aligned}
$$

□

Inverse Continuous Wavelet Transform

Theorem 2.1.4. *(Inverse CWT)*
Let $x(t) \in L_2(\mathbb{R})$ and $\psi(t)$ is continuous wavelet, the inverse wavelet transform can be recover the signal $x(t)$ (see e.g. [20, 22, 23, 24])

$$x(t) = \frac{1}{C_\psi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_x(a,b)\psi_{a,b}(t)db\frac{da}{|a|} \tag{2.1.8}$$

where $C_\psi = \int_{-\infty}^{\infty}\left|\hat{\psi}(\xi)\right|^2 \frac{d\xi}{|\xi|} < \infty$

If $x(t)$ is analytic or $\psi(t)$ is real, then:

$$x(t) = \frac{1}{\tilde{C}_\psi}\int_{0}^{\infty}\int_{-\infty}^{\infty} W_x(a,b)\psi_{a,b}(t)db\frac{da}{a} \tag{2.1.9}$$

where $\tilde{C}_\psi = \int_{0}^{\infty}\left|\hat{\psi}(\xi)\right|^2 \frac{d\xi}{|\xi|} < \infty.$

Proof. If $x(t) \in L_2(\mathbb{R})$ and $\psi(t)$ a continuous wavelet, then we have

$$\int_0^\infty\int_{-\infty}^{\infty} W_x(a,b)\psi_{a,b}(t)db\frac{da}{a} = \int_0^\infty\int_{-\infty}^{\infty}\int_0^\infty \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{i2\pi b\xi}d\xi\psi_{a,b}(t)db\frac{da}{a}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \hat{x}(\xi)\overline{\hat{\psi}(a\xi)} \int_{-\infty}^\infty \psi_{a,b} e^{i2\pi b\xi}(t) db \frac{da}{a} d\xi \\
&= \int_0^\infty \int_0^\infty \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}\hat{\psi}(a\xi) e^{i2\pi t\xi} db \frac{da}{a} d\xi \\
&= \int_0^\infty \hat{x}(\xi) e^{i2\pi t\xi} \int_0^\infty \left|\hat{\psi}(a\xi)\right|^2 db \frac{da}{a} d\xi \\
&= \tilde{C}_\psi \int_0^\infty \hat{x}(\xi) e^{i2\pi t\xi} d\xi = \tilde{C}_\psi x(t)
\end{aligned}
\tag{2.1.10}
$$

□

note we assume that a continuous wavelet $\psi(t)$ also satisfies

$$
0 \neq c_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)} \frac{d\xi}{\xi} < \infty. \tag{2.1.11}
$$

Theorem 2.1.5. *(Inverse CWT involving a for analytic signal)*
Suppose $x(t) \in L_2(\mathbb{R})$ and $\psi(t)$ is a continuous wavelet. If $x(t)$ is analytic, then

$$
x(b) = \frac{1}{c_\psi} \int_0^\infty W_x(a,b) \frac{da}{a} \tag{2.1.12}
$$

where C_ψ is defined $0 \neq C_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)} \frac{d\xi}{\xi} < \infty.$

Proof. From (2.1.2), we have

$$
\begin{aligned}
\int_0^\infty W_x(a,b) \frac{da}{a} &= \int_0^\infty \int_0^\infty \hat{x}(\xi)\overline{\hat{\psi}(a\xi)} e^{2i\pi b\xi} d\xi \frac{da}{a} \\
&= \int_0^\infty \hat{x}(\xi) e^{2i\pi b\xi} \int_0^\infty \overline{\hat{\psi}(a\xi)} \frac{da}{a} d\xi \\
&= \int_0^\infty \hat{x}(\xi) e^{2i\pi b\xi} d\xi \int_0^\infty \overline{\hat{\psi}(a)} \frac{da}{a} \\
&= c_\psi \int_0^\infty \hat{x}(\xi) e^{2i\pi b\xi} d\xi = c_\psi x(b)
\end{aligned}
$$

$\square$

Furthermore, a real signal $x(t) \in L_2(\mathbb{R})$ can be recovered from its CWT with an analytic continuous wavelet by the following formula which does not involve the time variable b either.

Theorem 2.1.6. *(Inverse CWT involving a for a real signal (refer to [7])) Suppose $x(t) \in L_2(\mathbb{R})$ and $\psi(t)$ is a continuous wavelet. If $x(t)$ is real and ψ is analytic, then*

$$x(b) = Re\big(\frac{2}{c_\psi}\int_0^\infty W_x(a,b)\frac{da}{a}\big) \tag{2.1.13}$$

where C_ψ is defined $0 \neq C_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)}\frac{d\xi}{\xi} < \infty$.

Proof. When $x(t)$ is real, we have $\overline{\hat{x}(\xi)} = \hat{x}(-\xi)$, thus,

$$\int_{-\infty}^0 \hat{x}(\xi)e^{i2\pi b\xi}d\xi = \int_0^\infty \hat{x}(-\xi)e^{-i2\pi b\xi}d\xi = \overline{\int_0^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi}$$

and hence

$$\begin{aligned} x(b) &= \int_{-\infty}^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi = \int_{-\infty}^0 \hat{x}(\xi)e^{i2\pi b\xi}d\xi + \int_0^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi \\ &= 2Re\big(\int_0^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi\big) \end{aligned} \tag{2.1.14}$$

From (2.1.1) and the proof of theorem 2.1.4, we have

$$\int_0^\infty W_x(a,b)\frac{da}{a} = c_\psi \int_0^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi$$

Therefore,

$$x(b) = 2Re\big(\int_0^\infty \hat{x}(\xi)e^{i2\pi b\xi}d\xi\big) = Re\big(\frac{2}{c_\psi}\int_0^\infty W_x(a,b)\frac{da}{a}\big). \tag{2.1.15}$$

$\square$

2.1.1 1st-order WSST

The 1st-order WSST was introuced in [14]. The WSST is designed to reassign the scale variable a to the frequency variable. Let's start with the most classic example. The CWT of $x(t) = A\cos(2\pi ct)$ where c is a positive constant. Then the Fourier transform of x is defined by

$$\hat{x}(\xi) = A\int_{-\infty}^{\infty} \cos(2\pi ct)e^{-2i\pi\xi t}dt$$

We use the fact that $\cos(2\pi ct) = \frac{e^{2i\pi ct}+e^{-2i\pi ct}}{2}$

$$\begin{aligned} \hat{x}(\xi) &= \frac{A}{2}\Big(\int_{-\infty}^{\infty} e^{2i\pi ct}e^{-2i\pi\xi t}dt + \int_{-\infty}^{\infty} e^{-2i\pi ct}e^{-2i\pi\xi t}dt\Big) \\ &= \frac{A}{2}\Big(e^{-2it\pi(\xi-c)} + e^{-2it\pi(\xi+c)}\Big) = \frac{A}{2}\Big(\delta_c(\xi) + \delta_{-c}(\xi)\Big) \end{aligned} \tag{2.1.1}$$

Thus for $a > 0$

$$W_x(a,b) = \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}(a\xi)}e^{2i\pi b\xi}d\xi = \frac{1}{2}A\overline{\hat{\psi}(ac)}e^{2i\pi bc}$$

The instantaneous frequency (IF) of $x(t)$ is represented as c

$$\frac{\partial}{\partial b}W_x(a,b) = \frac{1}{2}A\hat{\psi}(ac)e^{i2\pi bc} = 2i\pi cW_x(a,b).$$

This implies that

$$\frac{\frac{\partial}{\partial b}W_x(a,b)}{2i\pi W_x(a,b)} = c$$

For a general $x(t)$, at (a,b) for which $W_x(a,b) \neq 0$, a good approximation for the phase transformation (also called theinstantaneous frequency (IF)) of x is ω_x^{1st}

$$\omega_x^{1st}(a,b) = \frac{\frac{\partial}{\partial b} W_x(a,b)}{2i\pi W_x(a,b)}, \quad for \quad W_x(a,b) \neq 0.$$

The 1st-order WSST of a signal $x(t)$ is defined by

$$T_x^{1st}(\xi,b) = \int_{a\in\mathbb{R}_+ : W_x(a,b)\neq 0} W_x(a,b)\delta(\omega_x^{1st}(a,b) - \xi)\frac{da}{a}, \tag{2.1.2}$$

where ξ is the frequency variable.

The input signal $x(t)$ can be recovered from its WSST. For the analytic $x(t) \in L_2(\mathbb{R})$, we have

$$x(b) = \frac{1}{c_\psi}\int_0^\infty T_x^{1st}(\xi,b)d\xi;$$

and for a real-valued $x(t) \in L_2(\mathbb{R})$

$$x(b) = Re\Big(\frac{2}{c_\psi}\int_0^\infty T_x^{1st}(\xi,b)d\xi\Big),$$

where C_ψ is defined $0 \neq C_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)}\frac{d\xi}{\xi} < \infty$.

For a multicomponent signal $x(t) = A_0(t) + \sum_{k=1}^K x_k(t),\ x_k(t) = A_k(t)cos(r\pi\phi_k(t))$ with $A_0(t) = 0$, each component $x_k(b)$ can be recovered from WSST:

$$x(b) \approx Re\Big(\frac{2}{c_\psi}\int_{|\xi-\phi_k'(b)|<\Gamma} T_x^{1st}(\xi,b)d\xi\Big),$$

for certain $\Gamma > 0$.

2.1.2 2nd-order WSST

The 2nd-order WSST was proposed in [14]. This defines a new phase transformation more precise than that of 1st-order ω_x^{2nd}. This 1st-order is associated with the 2nd order partial derivatives of the CWT of $x(t)$; when $x(t)$ is a linear frequency modulation signal (linear chirp), then ω_x^{2nd} is exactly the IF of $x(t)$. Therefore, $x(t)$ is a linear frequency modulation (LFM) signal if

$$x(t) = A(t)e^{i2\pi\phi(t)} = A(t)e^{pt+\frac{q}{2}t^2}e^{i2\pi(ct+\frac{1}{2}rt^2)}$$

with phase function $\phi(t) = ct+\frac{1}{2}rt^2$, IF $\phi'(t) = c+rt$ and chirp rate $\phi''(t) = r$, IA $A(t) = Ae^{pt+\frac{q}{2}t^2}$. In the following we derive the phase transformation ω_x^{2nd}. For a given wavelet ψ, $W_x(b,a)$ is the CWT of $x(t)$ with ψ defined by (2.1.1). For $\psi_1(t) = t\psi(t)$, $W_x^{\psi_1}(b,a)$ denotes the CWT of $x(t)$ with $\psi_1(t)$ and represents the integral on the right-hand side of (2.1.1) with $\psi(t)$ replaced by $\psi_1(t)$. The derivative of the signal x is given by

$$x'(t) = (p + qt + i2\pi(c + rt))x(t)$$

$$W_x(b,a) = \int_{-\infty}^{\infty} x(b + at)\overline{\psi(t)}dt.$$

The derivative of the CWT with respect to the variable b is

$$\begin{aligned}
\partial_b W_x^{(}b,a) &= \int_{-\infty}^{\infty} x'(b + at)\overline{\psi(t)}dt \\
&= \int_{-\infty}^{\infty} (p + q(b + at) + i2\pi(c + rb + rat))x(b + at))\overline{\psi(t)}dt \\
&= (p + qb + i2\pi(c + rb))W_x(b,a) + (q + i2\pi r)aW_x^{\psi_1}(b,a).
\end{aligned} \tag{2.1.1}$$

At (a,b), on which $W_x(a,b) \neq 0$, we then denote that

$$\frac{\partial_b W_x(b,a)}{W_x(b,a)} = p + qb + i2\pi(c + rb) + (q + i2\pi r)a\frac{W^{\psi_1}(b,a)}{W_x(b,a)}$$

$$\frac{\partial}{\partial a}\Big(\frac{\partial_b W_x(b,a)}{W_x(b,a)}\Big) = (q + i2\pi r)U(a,b)$$

where

$$U(a,b) = \frac{\partial}{\partial a}\Big(\frac{aW_x^{\psi_1}(b,a)}{W_x(b,a)}\Big) = \frac{W_x^{\psi_1}(b,a)}{W_x(b,a)} + a\frac{\partial}{\partial a}\Big(\frac{W_x^{\psi_1}(b,a)}{W_x(b,a)}\Big).$$

If $U(a,b) \neq 0$, then

$$q + i2\pi r = \frac{1}{U(a,b)}\frac{\partial}{\partial a}\Big(\frac{\partial_b W_x(b,a)}{W_x(b,a)}\Big).$$

Then, we have

$$\frac{\partial_b W_x(b,a)}{W_x(b,a)} = p + qb + i2\pi(c + rb) + a\frac{W_x^{\psi_1}(b,a)}{W_x(b,a)U(a,b)}\frac{\partial}{\partial a}\Big(\frac{\partial_b W_x(b,a)}{W_x(b,a)}\Big).$$

Thus,

$$\phi'(b) = c + rb = Re\Big\{\frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}\Big\} - aRe\Big\{\frac{W_x^{\psi_1}(b,a)}{W_x(b,a)U(a,b)}\frac{\partial}{\partial a}\Big(\frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}\Big)\Big\}.$$

Therefore, we define the second-order phase transformation ω_x^{2nd} by the formula

$$\omega_x^{2nd}(a,b) = \begin{cases} Re\{\frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}\} - aRe\{\frac{W^{\psi_1}(b,a)}{W_x(b,a)U(a,b)}\frac{\partial}{\partial a}\Big(\frac{\partial_b W(b,a)}{i2\pi W_x(b,a)}\Big)\}, & \text{if} \quad U(a,b) \neq 0, W_x(b,a) \neq 0 \\ Re\{\frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}\}, & \text{if} \quad U(a,b) = 0, W_x(b,a) \neq 0. \end{cases}$$

We define the 2nd-order WSST of a signal $x(t)$ as

$$T_x^{2nd}(\xi,b) = \int_{a\in\mathbb{R}_+:W_x\neq 0} W_x(a,b)\delta(\omega_x^{2nd}(a,b) - \xi)\frac{da}{a}$$

where ξ is the frequency variable.

To recover the input signal $x(t)$ from its WSST for analytic $x(t) \in L_2(\mathbb{R})$, we say that

$$x(b) = \frac{1}{C_\psi(b)} \int_0^\infty T_x^{2nd}(\xi, b) d\xi$$

and for a real valued $x(t) \in L_2(\mathbb{R})$,

$$x(b) = Re(\frac{2}{C_\psi(b)} \int_0^\infty T_x^{2nd}(\xi, b) d\xi)$$

where $c_\psi(b)$ is defined by

$$0 \neq C_\psi = \int_0^\infty \overline{\hat{\psi}(\xi)} \frac{d\xi}{\xi} < \infty.$$

For a multicomponent signal $x(t)$, we can recover each component $x_k(b)$ from the WSST:

$$x_k(b) = Re(\frac{2}{C_\psi(b)} \int_{|\xi - \phi'(b)| < \Gamma} T_x^{2nd}(\xi, b) d\xi)$$

for certain $\Gamma > 0$.

2.2 WSST with a time varying parameter

Continuous wavelets ψ_σ are dependent on the function σ defined by

$$\psi_\sigma(t) = \frac{1}{\sigma} g(\frac{t}{\sigma}) e^{i2\pi\mu t}.$$

The CWT of a signal $x(t)$ with a time-varying parameter is defined by

$$\widetilde{W}_x(a,b) = \int_{-\infty}^{\infty} x(t)\frac{1}{a}\overline{\psi_{\sigma(b)}(\frac{t-b}{a})}dt = \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}_{\sigma(b)}(a\xi)}e^{i2\pi b\xi}d\xi.$$

We call $\widetilde{W}_x(a,b)$ the adaptive CWT of $x(t)$ with ψ_σ. If ψ_σ is an analytic wavelet or $x(t)$ is analytic, then we have $a > 0$

$$\widetilde{W}_x(a,b) = \int_{0}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}_{\sigma(b)}(a\xi)}e^{i2\pi b\xi}d\xi.$$

An analytic signal $x(t)$ can be recovered back from its CWT

$$x(b) = \frac{1}{c_{\psi(b)}}\int_{0}^{\infty} \widetilde{W}_x(a,b)\frac{da}{a}$$

where $C_\psi(b)$ is defined by

$$C_\psi(b) = \int_{0}^{\infty} \overline{\hat{\psi}_{\sigma(b)}(\xi)}\frac{d\xi}{\xi} = \int_{0}^{\infty} \overline{\hat{g}(\xi - \sigma(b)\mu)}\frac{d\xi}{\xi}.$$

If $x(t)$ is a real signal, then we have

$$x(b) = Re(\frac{2}{C_\psi(b)}\int_{0}^{\infty} \widetilde{W}_x(a,b)\frac{da}{a}).$$

2.2.1 Adaptive 1st-order WSST

The adaptive 1st-order WSST was introduced in [14]. In order to define the adaptive CWT, we must first define the $\omega_x^{1st,adp}$. Let $\psi_\sigma(t)$ and $\psi_\sigma^2(t) = \frac{t}{\sigma^2}g'(\frac{t}{\sigma})e^{i2\pi\mu t}$ be the continuous wavelet. Denote $\widetilde{W}_x^{\psi^2}(a,b)$ defined by

$$\widetilde{W}_x^{\psi^2}(a,b) = \int_{-\infty}^{\infty} x(t)\frac{1}{a}\overline{\psi_{\sigma(b)}^2(\frac{t-b}{a})}dt = \int_{-\infty}^{\infty} x(b+at)\frac{t}{\sigma^2(b)}\overline{g'(\frac{t}{\sigma})}e^{-i2\pi\mu t}dt.$$

One can obtain that

$$\hat{\psi}^2_\sigma(\xi) = -\hat{g}(\sigma(\xi-\mu)) - \sigma(\xi-\mu)(\hat{g})'(\sigma(\xi-\mu)).$$

To define the phase transformation $\omega_x^{adp}(a,b)$, we consider $x(t) = Ae^{i2\pi ct}$. From

$$\widetilde{W}_x(a,b) = \int_{-\infty}^{\infty} x(b+at)\overline{\psi_{\sigma(b)}(t)}dt = A\int_{-\infty}^{\infty} e^{i2\pi c(b+at)}\frac{1}{\sigma(b)}g\overline{\frac{t}{\sigma(b)}}e^{-i2\pi\mu t}dt$$

We take the partial derivative $\frac{\partial}{\partial b}$ to both sides, and we have

$$\begin{aligned}
\partial_b \widetilde{W}_x(a,b) &= A\int_{-\infty}^{\infty}(i2\pi c)e^{i2\pi c(b+at)}\frac{1}{\sigma(b)}\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt \\
&+ A\int_{-\infty}^{\infty} e^{i2\pi c(b+at)}(\frac{-\sigma'(b)}{\sigma(b)^2})\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt \\
&+ A\int_{-\infty}^{\infty} e^{i2\pi c(b+at)}(\frac{-\sigma'(b)}{\sigma(b)^3})\overline{g'(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt \\
&= i2\pi c\widetilde{W}_x(a,b) - \frac{\sigma'(b)}{\sigma(b)}\widetilde{W}_x(a,b) - \frac{\sigma'(b)}{\sigma(b)}\widetilde{W}_x^{\psi^2}(a,b).
\end{aligned}$$

If $\widetilde{W}_x(a,b) \neq 0$, then

$$\frac{\frac{\partial}{\partial b}\widetilde{W}_x(a,b)}{i2\pi\widetilde{W}_x(a,b)} = c - \frac{\sigma'(b)}{i2\pi\sigma(b)} - \frac{\sigma'(b)}{\sigma(b)}\frac{\widetilde{W}_x^{\psi^2}(a,b)}{i2\pi\widetilde{W}_x(a,b)}.$$

Thus, the instantaneous frequency (IF) c of $x(t)$ can be denoted by

$$c = Re\Big\{\frac{\frac{\partial}{\partial b}\widetilde{W}_x(a,b)}{i2\pi\widetilde{W}_x(a,b)}\Big\} + \frac{\sigma'(b)}{\sigma(b)}Re\Big\{\frac{\widetilde{W}_x^{\psi^2}(a,b)}{i2\pi\widetilde{W}_x(a,b)}\Big\}.$$

For a general signal $x(t)$, at (a,b) for which $\widetilde{W}_x(a,b) \neq 0$, denote

$$\widetilde{\omega}_x^{1st,adp}(a,b) = Re\Big\{\frac{\frac{\partial}{\partial b}\widetilde{W}_x(a,b)}{i2\pi\widetilde{W}_x(a,b)}\Big\} + \frac{\sigma'(b)}{\sigma(b)}Re\Big\{\frac{\widetilde{W}_x^{\psi^2}(a,b)}{i2\pi\widetilde{W}_x(a,b)}\Big\}.$$

The quantity $\omega_x^{1st,adp}(a,b)$ is called the "phase transformation." The WSST with a time-varying parameter (also called the adaptive WSST) of $x(t)$ is defined by

$$T_x^{1st,adp}(\xi,b) = \int_{\{a\in\mathbb{R}_+:\widetilde{W}_x(a,b)\neq 0\}} \widetilde{W}_x(a,b)\delta(\widetilde{\omega}_x^{1st,adp}(a,b)-\xi)\frac{da}{a} \qquad (2.2.1)$$

where ξ is the frequency variable.

The input signal $x(t)$ can be recovered from its adaptive WSST for the analytic $x(t) \in L_2(\mathbb{R})$. We have

$$x(b) = \frac{1}{C_\psi(b)}\int_0^\infty T_x^{1st,adp}(\xi,b)d\xi$$

and for a real valued $x(t) \in L_2(\mathbb{R})$

$$x(b) = Re(\frac{2}{C_\psi(b)}\int_0^\infty T_x^{1st,adp}(\xi,b)d\xi)$$

where $c_\psi(b)$ is defined by

$$C_\psi(b) = \int_0^\infty \overline{\hat{\psi}_{\sigma(b)}(\xi)}\frac{d\xi}{\xi} = \int_0^\infty \overline{\hat{g}(\xi-\sigma(b)\mu)}\frac{d\xi}{\xi}.$$

The following formula can be used to recover the k-th component $x_k(b)$ of a multicomponent signal from the adaptive WSST

$$x_k(b) = Re(\frac{2}{C_\psi(b)}\int_{|\xi-\phi'(b)|<\Gamma_1} T_x^{1st,adp}(\xi,b)d\xi)$$

for certain $\Gamma_1 > 0$.

Remark: Here, we may say that if ψ_σ is the simplified version of Morlet's wavelet given by

$$\hat{\psi}_\sigma(\xi) = e^{-2\pi^2\sigma^2(\xi-\mu)^2} \text{ and } \hat{\psi}^2_\sigma(\xi) = (4\pi^2\sigma^2(\xi-\mu)^2 - 1)e^{-2\pi^2\sigma^2(\xi-\mu)^2}.$$

Next, we calculate:

$$\begin{aligned}
\widetilde{W}_x^{\psi^2}(a,b) &= \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{\psi}^2_{\sigma(b)}(a\xi)}e^{i2\pi b\xi}d\xi = A\hat{\psi}^2_{\sigma(b)}(ac)e^{i2\pi bc} \\
&= A(4\pi^2\sigma^2(b)(ac-\mu)^2 - 1)e^{i2\pi^2\sigma^2(b)(\xi-\mu)^2} \\
&= (4\pi^2\sigma^2(b)(ac-\mu)^2 - 1)\widetilde{W}_x(a,b).
\end{aligned}$$

Thus,

$$\frac{\widetilde{W}_x^{\psi^2}(a,b)}{i2\pi\widetilde{W}_x(a,b)} = \frac{1}{i2\pi}(4\pi^2\sigma^2(b)(ac-\mu)^2 - 1).$$

In this case the phase transformation ω_x of the Morlet's wavelet is defined by

$$\widetilde{\omega}_x^{1st,adp}(a,b) = Re\Big\{\frac{\partial_b\widetilde{W}_x(a,b)}{i2\pi\widetilde{W}_x(a,b)}\Big\}.$$

2.2.2 Adaptive 2nd-order WSST

The adaptive 2nd-order WSST is proposed in [14]. To find the adaptive 2nd-order WSST, the CWT is a time-varying parameter. Let $\psi_\sigma(t)$ of the continuous wavelet be defined as

$$\psi_\sigma(t) = \frac{1}{\sigma}g(\frac{t}{\sigma})e^{i2\pi\mu t}.$$

Denote $\psi^2_\sigma(t) = \frac{t}{\sigma^2}g'(\frac{t}{\sigma})e^{i2\pi\mu t}$, and let $\widetilde{W}_x^{\psi^2}(a,b)$ denote the CWT. We define

$$\psi_\sigma^1(t) = \frac{t}{\sigma}\psi_\sigma(t) = \frac{t}{\sigma^2}g(\frac{t}{\sigma})e^{i2\pi\mu t}.$$

Let $W_x^{\psi_1}(a,b)$ denote the CWT of $x(t)$ with $\psi_\sigma^1(t)$

$$\widetilde{W}_x^{\psi^1}(a,b) = \int_{-\infty}^{\infty} x(t)\frac{1}{a}\overline{\psi_\sigma^1(t)(\frac{t-b}{a})}dt = \int_{-\infty}^{\infty} x(b+at)\frac{t}{\sigma^2(b)}\overline{g(\frac{t}{\sigma(b)})e^{-i2\pi\mu t}}dt \tag{2.2.1}$$

$$\hat{\psi}_\sigma^1(\xi) = \frac{i}{2\pi}(\hat{g})'(\sigma(\xi-\mu)).$$

If ψ_σ is Morlet's wavelet defined by

$$\hat{\psi}_\sigma(\xi) = e^{-2\pi^2\sigma^2(\xi-\mu)^2}$$

then

$$\hat{\psi}_\sigma^1(\xi) = -i2\pi\sigma(\xi-\mu)e^{-2\pi^2\sigma^2(\xi-\mu)^2}.$$

For a similar calculation presented in the case of 2nd-order WSST non adaptive, the phase transformation $\widetilde{\omega}^{2nd,adp}$ is defined by

$$\widetilde{\omega}_x^{2nd,adp} = \begin{cases} Re\Big\{\frac{\partial_b\widetilde{W}_x(b,a)}{i2\pi\widetilde{W}_x(b,a)}\Big\} + \frac{\sigma'(b)}{\sigma(b)}Re\Big\{\frac{\widetilde{W}_x^{\psi^2}(b,a)}{i2\pi\widetilde{W}_x(b,a)}\Big\} - aRe\Big\{\frac{\widetilde{W}_x^{\psi^1}(b,a)}{i2\pi\widetilde{W}_x(b,a)}R_0(a,b)\Big\}, \\ \qquad \text{if } \frac{\partial}{\partial a}\Big(a\frac{\widetilde{W}^{\psi^1}(b,a)}{\widetilde{W}_x(b,a)}\Big) \neq 0, \widetilde{W}_x(b,a) \neq 0 \\ Re\Big\{\frac{\partial_b\widetilde{W}_x(b,a)}{i2\pi\widetilde{W}_x(b,a)}\Big\} + \frac{\sigma'(b)}{\sigma(b)}Re\Big\{\frac{\widetilde{W}_x^{\psi^2}(b,a)}{i2\pi\widetilde{W}_x(b,a)}\Big\}, \quad \text{if } \frac{\partial}{\partial a}\Big(a\frac{\widetilde{W}_x^{\psi^1}(b,a)}{\widetilde{W}_x(b,a)}\Big) = 0, \widetilde{W}_x(b,a) \neq 0. \end{cases}$$

where

$$R_0(a,b) = \frac{1}{\frac{\partial}{\partial a}\Big(a\frac{\widetilde{W}^{\psi^1}(b,a)}{\widetilde{W}_x(b,a)}\Big)}\Bigg(\frac{\partial}{\partial a}\Big(\frac{\frac{\partial}{\partial b}\widetilde{W}_x(b,a)}{\widetilde{W}_x(b,a)}\Big) + \frac{\sigma'(b)}{\sigma(b)}\frac{\partial}{\partial a}\Big(\frac{\widetilde{W}_x^{\psi^2}(b,a)}{\widetilde{W}_x(b,a)}\Big)\Bigg).$$

The 2nd-order adaptive WSST of a signal $x(t)$ is defined by

$$T_s^{2nd,adp}(\xi, b) = \int_{\{a \in \mathbb{R}_+ : \widetilde{W}_s \neq 0\}} \widetilde{W}_x(a, b)\delta(\widetilde{\omega}_x^{2nd,adp}(a, b) - \xi)\frac{da}{a} \qquad (2.2.2)$$

where ξ is the frequency variable.

The input signal $x(t)$ can be recovered from its adaptive WSST for analytic $x(t) \in L_2(\mathbb{R})$, and we have

$$x(b) = \frac{1}{C_\psi(b)} \int_0^\infty T_x^{2nd,adp}(\xi, b)d\xi$$

and for a real valued $x(t) \in L_2(\mathbb{R})$

$$x(b) = Re(\frac{2}{C_\psi(b)} \int_0^\infty T_x^{2nd,adp}(\xi, b)d\xi)$$

where $c_\psi(b)$ is defined by

$$C_\psi(b) = \int_0^\infty \overline{\hat{\psi}_{\sigma(b)}(\xi)}\frac{d\xi}{\xi} = \int_0^\infty \overline{\hat{g}(\xi - \sigma(b)\mu)}\frac{d\xi}{\xi}.$$

For a multicomponent signal $x(t)$, each component $x_k(b)$ can be recovered from adaptive WSST

$$x_k(b) = Re(\frac{2}{C_\psi(b)} \int_{|\xi - \phi'(b)| < \Gamma_1} T_x^{2nd,adp}(\xi, b)d\xi) \qquad (2.2.3)$$

for certain $\Gamma_1 > 0$.

2.3 Short-time Fourier transform-based synchrosqueezed transform (FSST)

Definition 2.3.1. The short-time Fourier transform (STFT) of $x(t) \in L_2(\mathbb{R})$ with a window function $g(t) \in L_2(\mathbb{R})$ is defined by

$$
\begin{aligned}
V_x(t,\eta) &= \int_{-\infty}^{\infty} x(\tau)g(\tau - t)e^{-i2\pi\eta(\tau-t)}d\tau && (2.3.1)\\
&= \int_{-\infty}^{\infty} (x+\tau)g(\tau)e^{-i2\pi\eta\tau}d\tau, && (2.3.2)
\end{aligned}
$$

where t is the time variable and η is the frequency variable.

The STFT is written as follows

$$
V_x(t,\eta) = \int_{-\infty}^{\infty} x(\tau)g(\tau - t)e^{-i2\pi\eta(\tau-t)}d\tau = \int_{-\infty}^{\infty} \hat{x}(\xi)\hat{g}(\eta - \xi)e^{i2\pi t\xi}d\xi \quad (2.3.3)
$$

To prove the result in equation (2.3.3) we start writing the short-time Fourier transform V_x in the form

$$
V_x(t,\eta) =< x, y_{t,\eta} >_{L_2(\mathbb{R})} \quad \text{with} \quad y_{t,\eta}(\tau) = g(\tau - t)e^{-i2\pi\eta(\tau-t)}
$$

which implies

$$
V_x(t,\eta) =< \hat{x}, \hat{y}_{t,\eta} >_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} \hat{x}(\xi)\overline{\hat{y}_{t,\eta}(\xi)}d\xi \quad (2.3.4)
$$

and the calculation $\hat{y}_{t,\eta}(\xi)$ denotes

$$
\hat{y}_{t,\eta}(\xi) = \int_{-\infty}^{\infty} g(\tau - t)e^{i2\pi\eta(\tau-t)}e^{-i2\pi\xi\tau}d\tau \quad (2.3.5)
$$

$$
\hat{y}_{t,\eta}(\xi) = \int_{-\infty}^{\infty} g(y)e^{i2\pi\eta y}e^{-i2\pi\xi(y-t)}dy = \underbrace{\int_{-\infty}^{\infty} g(y)e^{i2\pi(\eta-\xi)}dy}_{\hat{g}(\eta-\xi)}\, e^{i2\pi\xi t} = \hat{g}(\eta - \xi)e^{i2\pi\xi t}.
$$

Therefore, we have

$$V_x(t,\eta) = <\hat{x}, \hat{y}_{t,\eta}>_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} \hat{x}(\xi)\hat{g}(\eta-\xi)e^{i2\pi\xi t}d\xi. \tag{2.3.6}$$

We can recover the original signal $x(t)$ from its STFT:

$$x(t) = \frac{1}{\| g \|_2^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} V_x(t,\eta)\overline{g(t-\tau)}e^{-i2\pi\eta(\tau-t)}d\tau d\eta. \tag{2.3.7}$$

Theorem 2.3.2. *(Inverse STFT).*
Involving ξ suppose $g(t) \in L_2(\mathbb{R})$ with $g(0) \neq 0$ Then for $x(t) \in L_2(\mathbb{R})$, we have

$$x(t) = \frac{1}{g(0)} \int_{-\infty}^{\infty} V_x(t,\eta)d\eta \tag{2.3.8}$$

Proof. From (2.3.3), we have

$$\begin{aligned}
\int_{-\infty}^{\infty} V_x(t,\eta)d\eta &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{x}(\xi)\hat{g}(\eta-\xi)e^{i2\pi t\xi}d\xi d\eta \\
&= \int_{-\infty}^{\infty} \hat{x}(\xi)e^{i2\pi t\xi}\int_{-\infty}^{\infty} \hat{g}(\eta)d\eta d\xi \\
&= \int_{-\infty}^{\infty} \hat{g}(\eta)e^{i2\pi\times 0\times\eta}d\eta \int_{-\infty}^{\infty} \hat{x}(\xi)e^{i2\pi t\xi}d\xi \\
&= g(0)x(t).
\end{aligned}$$

□

Theorem 2.3.3. *(Inverse STFT).*
Involving η for real signal. Suppose the window function $g(t) \in L_2(\mathbb{R})$ is real with $g(0) \neq 0$. Then for a real-valued $x(t) \in L_2(\mathbb{R})$, we have

$$x(t) = \frac{2}{g(0)} Re(\int_{0}^{\infty} V_x(t,\eta)d\eta). \tag{2.3.9}$$

Proof. Let $g(t)$ and $x(t)$ be two real functions, we have

$$
\begin{aligned}
V_x(t,-\eta) &= \int_{-\infty}^{\infty} x(\tau)g(\tau-t)e^{i2\pi\eta(\tau-t)}d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)g(\tau-t)\overline{e^{-i2\pi\eta(\tau-t)}}d\tau = \overline{V_x(t,\eta)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
x(t) &= \frac{1}{g(0)}\int_0^{\infty} V_x(t,\eta)d\eta + \frac{1}{g(0)}\int_{-\infty}^{0} V_x(t,\eta)d\eta \\
&= \frac{1}{g(0)}\int_0^{\infty} V_x(t,\eta)d\eta + \frac{1}{g(0)}\int_{0}^{\infty} V_x(t,-\eta)d\eta \\
&= \frac{1}{g(0)}\int_0^{\infty} V_x(t,\eta)d\eta + \frac{1}{g(0)}\overline{\int_{0}^{\infty} V_x(t,\eta)d\eta} = \frac{2}{g(0)}Re\Big\{\int_0^{\infty} V_x(t,\eta)d\eta\Big\}.
\end{aligned}
$$

□

2.3.1 1st-order FSST

The 1st-order FSST was proposed in [12, 15]. We derive the FSST through the STFT. We begin with the STFT of $x(t) = Ae^{i2\pi ct}$ (for more detail, see [12]) where A and c are constants and $c > 0$. We have

$$
\begin{aligned}
V_x(t,\eta) &= \int_{-\infty}^{\infty} Ae^{i2\pi c(t+\tau)}g(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= Ae^{i2\pi tc}\hat{g}(\eta-c).
\end{aligned}
$$

Thus, we can obtain the instantaneous frequency (IF), defined here as c, of $x(t)$ by

$$
\frac{\frac{\partial}{\partial t}V_x(t,\eta)}{2i\pi V_x(t,\eta)} = c.
$$

For a general signal $x(t)$, at (t,η) for which $V_x(t,\eta) \neq 0$ and where $c = \omega_x(t,\eta)$, a good candidate for the instantaneous frequency (IF) of $x(t)$ is

$$\omega_x^{1st}(t,\eta) = \frac{\frac{\partial}{\partial t}V_x(t,\eta)}{2i\pi V_x(t,\eta)}.$$

This is also called the "reference IF function" or the "phase transformation."

The FSST reassigns the frequency variable ξ by transforming the STFT $V_x(t,\xi)$ of $x(t)$ to a quantity, denoted by $R_x^{1st}(t,\eta)$, on the time-frequency plane:

$$R_x^{1st}(t,\xi) = \int_{\{\zeta : V_x(t,\zeta)\neq 0\}} V_x(t,\zeta)\delta(\omega_x^{1st}(t,\zeta)-\xi)d\zeta,$$

where ξ is the frequency variable.

We can recover the input signal $x(t)$ from its FSST. If we say $g(t) \in L_2(\mathbb{R})$ with a window function where $g(0) \neq 0$, then for $x(t) \in L_2(\mathbb{R})$,

$$x(t) = \frac{1}{g(0)}\int_{-\infty}^{\infty} R_x^{1st}(t,\xi)d\xi. \tag{2.3.1}$$

If $g(t)$ and $x(t)$ are also real-valued, then

$$x(t) = \frac{2}{g(0)}Re\Big(\int_0^{\infty} R_x^{1st}(t,\xi)d\xi\Big). \tag{2.3.2}$$

We denote a multicomponent signal $x(t)$ as

$$x(t) = \sum_{k=1}^{K} x_k(t) = \sum_{k=1}^{K} A_k(t)e^{i2\pi\phi_k(t)}.$$

when $A_k(t)$ and $\phi_k(t)$ satisfy certain conditions, each component $x_k(t)$ can be recovered from its FSST (see e.g. [12])

$$x_k(t) \approx \frac{1}{g(0)}\int_{|\xi-\phi_k'(t)|<\Gamma} R_x^{1st}(t,\xi)d\xi.$$

for certain $\Gamma > 0$.

2.3.2 2nd-order FSST

Daubechies, Lu, and Wu introduce the 2nd-order SST in [7]. The focus of this paper to define a new phase transformation ω_x^{2nd}. This new phase transformation is associated with 2nd order derivatives of the STFT of $x(t)$ such that when $x(t)$ is a linear frequncy modulation (LFM) signal (linear chirp), then ω_x^{2nd} is exactly the IF of $x(t)$. We say $x(t)$ is a LFM signal if

$$x(t) = A(t)e^{i2\pi\phi(t)} = Ae^{pt+\frac{q}{2}t^2}e^{i2\pi(ct+\frac{1}{2}rt^2)}$$

with phase function $\phi(t) = ct+\frac{1}{2}rt^2$, IF $\phi'(t) = c+rt$ and chirp rate $\phi''(t) = r$, and IA $A(t) = Ae^{pt+\frac{q}{2}t^2}$, where p, q are eral numbers and $|p|$, $|q|$ are much smaller than c. For $u_1(t) = tg(t)$, $V_x^{u_1}(t, \eta)$ denotes the STFT of $x(t)$ with $u_1(t)$ (see e.g. [15])

$$x'(t) = \big(p + qt + i2\pi(c + rt)\big)x(t).$$

From

$$V_x(t, \eta) = \int_{-\infty}^{\infty} x(t + \tau)g(\tau)e^{-i2\pi\eta\tau}d\tau,$$

we can say

$$\begin{aligned}
\tfrac{\partial}{\partial t}V_x(t, \eta) &= \int_{-\infty}^{\infty} x'(t + \tau)g(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= \int_{-\infty}^{\infty} \big(p + q(t + \tau) + i2\pi(c + rt + t\tau)\big)x(t + \tau)g(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= (p + qt + i2\pi(c + rt))V_x(t, \eta) + (q + i2\pi r)V_x^{u_1}(t, \eta).
\end{aligned}$$

Hence at (t, η) on which $V_x(t, \eta) \neq 0$, we have

$$\frac{\frac{\partial}{\partial t}V_x(t, \eta)}{V_x(t, \eta)} = p + qt + i2\pi(c + rt) + (q + i2\pi r)\frac{V_x^{u_1}(t, \eta)(t, \eta)}{V_x(t, \eta)}$$

$$\frac{\partial}{\partial\eta}\Big(\frac{\frac{\partial}{\partial t}V_x(t,\eta)}{V_x(t,\eta)}\Big) = (q + i2\pi r)U(t,\eta)$$

where we use $U(t,\eta)$ to denote

$$U(t,\eta) = \frac{\partial}{\partial\eta}\Big(\frac{V_x^{u_1}(t,\eta)(t,\eta)}{V_x(t,\eta)}\Big).$$

Thus if $U(t,\eta) \neq 0$, then

$$q + i2\pi r = \frac{1}{U(t,\eta)}\frac{\partial}{\partial\eta}\Big(\frac{V_x^{u_1}(t,\eta)}{V_x(t,\eta)}\Big)$$

$$\frac{\partial_t V_x(t,\eta)}{V_x(t,\eta)} = (p + qt) + i2\pi(c + rt) + \frac{1}{U(t,\eta)}\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x(t,\eta)}\Big)\Big(\frac{V_x^{u_1}(t,\eta)}{V_x(t,\eta)}\Big).$$

This implies

$$c + rt = \frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)} - \frac{p + qt}{i2\pi} - \frac{V_x^{u_1}(t,\eta)}{V_x(t,\eta)U(t,\eta)}.\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big),$$

So we have

$$\phi'(t) = c + rt = Re\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\} - Re\Big\{\frac{V_x^{u_1}(t,\eta)}{V_x(t,\eta)U(t,\eta)}.\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big)\Big\}.$$

Hence, the phase transformation ω_x^{2nd} is defined as

$$\omega_x^{2nd}(t,\eta) = \begin{cases} Re\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\} - Re\Big\{\frac{V_x^{u_1}(t,\eta)}{V_x(t,\eta)U(t,\eta)}.\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big)\Big\}. & \text{if } U(t,\eta) \neq 0,\ V_x(t,\eta) \neq 0 \\ Re\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\}, & \text{if } U(t,\eta) = 0,\ V_x(t,\eta) \neq 0 \end{cases}$$

The FSST reassigns the frequency variable ξ by transforming the STFT

$V_x(t,\eta)$ of $x(t)$ to a quantity, denoted by by $R_x(t,\xi)$, on the time-frequency plane:

$$R_x^{2nd}(t,\xi) = \int_{\{\zeta : V_x(t,\zeta)\neq 0\}} V_x(t,\zeta)\delta(\omega_x^{2nd}(t,\zeta)-\xi)d\zeta$$

where ξ is the frequency variable.

We can recover the input signal $x(t)$ from its FSST. If $g(t)\in L_2(\mathbb{R})$ and is window function with $g(0)\neq 0$. Then for $x(t)\in L_2(\mathbb{R})$,

$$x(t) = \frac{1}{g(0)}\int_{-\infty}^{\infty} R_x^{2nd}(t,\xi)d\xi. \tag{2.3.1}$$

If in addition, $g(t)$ and $x(t)$ are real-valued, then

$$x(t) = \frac{2}{g(0)}Re\Big(\int_0^{\infty} R_x^{2nd}(t,\xi)d\xi\Big). \tag{2.3.2}$$

For a multicomponent signal $x(t)$ given by

$$x(t) = \sum_{k=1}^{K} x_k(t) = \sum_{k=1}^{K} A_k(t)e^{i2\pi\phi_k(t)},$$

each component $x_k(t)$ can be recovered from its FSST

$$x_k(t) \approx \frac{1}{g(0)}\int_{\left|\xi-\phi_k'(t)\right|<\Gamma} R_x^{2nd}(t,\xi)d\xi.$$

for certain $\Gamma > 0$.

2.4 FSST with a time varying parameter

the window function given by $g_{\sigma(t)}(t) = \frac{1}{\sigma(t)} g\big(\frac{\tau}{\sigma(t)}\big)$ where $\sigma > 0$ is a parameter, and $g \in L^2(\mathbb{R})$ is a function with $g(0) \neq 0$. If $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$, the $g_{\sigma(t)}(\tau)$ is the Gaussion window function. (see e.g. [15, 13])

For a signal $x(t)$, we define the STFT of $x(t)$ with a $g_{\sigma(t)}$ parameter as

$$\begin{aligned}
\widetilde{V}_x(t,\eta) &= \int_{-\infty}^{\infty} x(\tau) g_{\sigma(t)}(\tau - t) e^{-i2\pi\eta(\tau-t)} d\tau \\
&= \int_{-\infty}^{\infty} x(\tau + t) \frac{1}{\sigma(t)} g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta(\tau)} d\tau \\
&= \int_{-\infty}^{\infty} \hat{x}(\xi) \hat{g}_{\sigma(t)}(x - \xi) e^{i2\pi t\xi} d\xi \\
&= \int_{-\infty}^{\infty} \hat{x}(\xi) \hat{g}(\sigma(t)(\eta - \xi)) e^{i2\pi t\xi} d\xi.
\end{aligned}$$

A signal $x(t)$ can be recovered from its adaptive STFT

$$x(t) = \frac{\sigma(t)}{g(0)} \int_{-\infty}^{\infty} \widetilde{V}_x(t,\eta) d\eta.$$

If in addition $g(t)$ is real-valued, then for real-valued $x(t)$, we have

$$x(t) = \frac{2\sigma(t)}{g(0)} Re\Big(\int_{-\infty}^{\infty} \widetilde{V}_x(t,\eta) d\eta\Big).$$

2.4.1 Adaptive 1st-order FSST

In order to define the adaptive 1st-order FSST introduced in [14], we must define the phase transformation ω_x^{adp} associated with the adaptive STFT. Let $g_\sigma(t)$ and $g_\sigma^2(t) = \frac{t}{\sigma^2} g'$ be the continuous wavelet. $\widetilde{V}_x^{g^2}(t,\eta)$ is defined by

$$\widetilde{V}_x^{g^2}(t,\eta) = \int_{-\infty}^{\infty} x(\tau + t) \frac{\tau}{\sigma^2(t)} g'\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau.$$

To define the phase transformation ω_x^{adp}, we consider $x(t) = Ae^{i2\pi ct}$, so that

$$\widetilde{V}_x(t,\eta)=\int_{-\infty}^{\infty}x(t+\tau)g_{\sigma(t)}(\tau)e^{-i2\pi\eta\tau}d\tau=A\int_{-\infty}^{\infty}e^{i2\pi c(t+\tau)}\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau.$$

therefore, we have

$$\begin{aligned}
\frac{\partial\widetilde{V}_x}{\partial t}(t,\eta) &= A\int_{-\infty}^{\infty}(i2\pi c)e^{i2\pi c(t+\tau)}\frac{1}{\sigma(t)}g\Big(\frac{\tau}{\sigma(t)}\Big)e^{-i2\pi\eta\tau}d\tau \\
&+ A\int_{-\infty}^{\infty}e^{i2\pi c(t+\tau)}\Big(\frac{-\sigma'(t)}{\sigma^2(t)}\Big)g\Big(\frac{\tau}{\sigma(t)}\Big)e^{-i2\pi\eta\tau}d\tau \\
&+ A\int_{-\infty}^{\infty}e^{i2\pi c(t+\tau)}\Big(\frac{-\sigma'(t)}{(\sigma(t))^3}\tau\Big)g'\Big(\frac{\tau}{\sigma(t)}\Big)e^{-i2\pi\eta\tau}d\tau \\
&= i2\pi c\widetilde{V}_x(t,\eta)-\frac{\sigma'(t)}{\sigma(t)}\widetilde{V}_x(t,\eta)-\frac{\sigma'(t)}{\sigma(t)}\widetilde{V}_x^{g^2}(t,\eta).
\end{aligned}$$

If $\widetilde{V}_x(t,\eta)\neq 0$, we then have

$$\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}=c-\frac{\sigma'(t)}{2i\pi\sigma(t)}-\frac{\sigma'(t)}{\sigma(t)}.\frac{\widetilde{V}_x^{g^2}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}.$$

The IF of $x(t)$ can therefore be obtained by

$$c=Re\Big\{\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}+\frac{\sigma'(t)}{2i\pi\sigma(t)}+\frac{\sigma'(t)}{\sigma(t)}.\frac{\widetilde{V}_x^{g^2}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\}.$$

Moreover, if σ is real function $Re\big\{\frac{\sigma'(t)}{2i\pi\sigma(t)}\big\}=0$, we have

$$c=Re\Big\{\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\}+\frac{\sigma'(t)}{\sigma(t)}Re\Big\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\}.$$

This quantity is also called the phase transformation or the reference IF function, and we denote it by $\tilde{\omega}_x(t,\eta)$:

$$\tilde{\omega}_x^{1st,adp}(t,\eta)=Re\Big\{\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\}+\frac{\sigma'(t)}{\sigma(t)}Re\Big\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\},\ \text{ if }\ \widetilde{V}_x(t,\eta)\neq 0.$$

The adaptive FSST of $x(t)$with a $\sigma(t)$ parameter is defined by

$$R_x^{1st,adp}(t,\xi) = \int_{\{\eta\in\mathbb{R}:\widetilde{V}_x(t,\eta)\neq 0\}} \widetilde{V}_x(t,\eta)\delta(\tilde{\omega}_x^{1st,adp}(t,\eta)-\xi)d\eta \qquad (2.4.1)$$

where ξ is the frequency variable.

The input signal $x(t)$ can be recovered from its adaptive FSST:

$$x(t) = \frac{\sigma(t)}{g(0)}\int_{-\infty}^{\infty} R_x^{1st,adp}(t,\xi)d\xi.$$

If $x(t)$ is also real-valued, then for real-valued $x(t)$, we denote that

$$x(t) = \frac{2\sigma(t)}{g(0)}Re\Big(\int_{-\infty}^{\infty} R_x^{1st,adp}(t,\xi)d\xi\Big).$$

We may use the following formula to recover the k-th component $x_k(b)$ of a multicomponent signal from the adaptive FSST

$$x_k(b) = \frac{2\sigma(t)}{g(0)}Re\Big(\int_{|\xi-\phi'(t)|<\Gamma} R_x^{1st,adp}(\xi,b)d\eta\Big)$$

for certain $\Gamma > 0$.

Remark:

If g is the Gaussion function defined by $g(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}$ for a signal $x(t) = Ae^{i2\pi ct}$, we have (refer to [15])

$$\begin{aligned}
\widetilde{V}_x(t,\eta) &= \int_{-\infty}^{\infty} Ae^{i2\pi c(t+\tau)}g_{\sigma(t)}(\tau)e^{-i2\pi\eta\tau}d\tau \\
&= Ae^{i2\pi ct}\hat{g}_{\sigma(t)}(\eta-c) \\
&= Ae^{i2\pi ct}e^{-2\pi^2\sigma(t)^2(\eta-c)^2}.
\end{aligned}$$

$$\begin{aligned}\partial_t \widetilde{V}_x(t,\eta) &= i2\pi c\ Ae^{i2\pi ct}e^{-2\pi^2\sigma(t)^2(\eta-c)^2} + Ae^{i2\pi ct}e^{-2\pi^2\sigma(t)^2(\eta-c)^2}(-4\pi^2)(\eta-c)^2\sigma(t)\sigma'(t)\\ &= i2\pi c\widetilde{V}_x(t,\eta) + \widetilde{V}_x(t,\eta)(-4\pi^2)(\eta-c)^2\sigma(t)\sigma'(t).\end{aligned}$$

Thus, we have

$$\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)} = c + i2\pi(\eta-c)^2\sigma(t)\sigma'(t).$$

Both c and $2\pi(\eta-c)^2\sigma(t)\sigma'(t)$ are real, so the phase transformation of $x(t)$ can be obtained by

$$c = Re\left\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)}\right\}.$$

Thus if g is the Gaussian function for a general signal $x(t)$, we may write the phase transformation as

$$\widetilde{\omega}_x^{1st,adp} = Re\left\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)},\right\},\quad for\ \widetilde{V}_x(t,\eta)\neq 0.$$

$$\widetilde{\omega}_x^{1st,adp}(t,\eta) = \begin{cases} Re\left\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)}\right\} & \text{if g (Gaussion function)}\\ Re\left\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)}\right\} + \frac{\sigma'(t)}{\sigma(t)}Re\left\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{i2\pi \widetilde{V}_x(t,\eta)}\right\}, & \text{otherwise}\end{cases}$$

2.4.2 Adaptive 2nd-order FSST

In order to define the adaptive 2nd-order FSST, we apply a timevarying parameter to the STFT. Let $g_{\sigma(t)}(t) = \frac{1}{\sigma(t)}g\left(\frac{\tau}{\sigma(t)}\right)$ be the window function. We define $g^1_{\sigma(t)}(t) = \frac{t}{\sigma^2(t)}\ g\left(\frac{t}{\sigma(t)}\right)$ and $g^2_{\sigma(t)}(t) = \frac{t}{\sigma^2(t)}g'\left(\frac{t}{\sigma(t)}\right)$ (see e.g. [10]).

We use $\widetilde{V}_x^{g^1}(t,\eta)$ to denote the STFT defined by

$$\widetilde{V}_x^{g^1}(t,\eta) = \int_{-\infty}^{\infty} x(\tau) g'_{\sigma(t)}(\tau - t) e^{-i2\pi\eta(\tau-t)} d\tau = \int_{-\infty}^{\infty} x(t+\tau) \frac{\tau}{\sigma^2(t)} g(\frac{\tau}{\sigma(t)}) e^{-i2\pi\eta(\tau-t)} d\tau$$

and $\widetilde{V}_x^{g^2}(t,\eta)$ to denote the STFT defined by

$$\widetilde{V}_x^{g^2}(t,\eta) = \int_{-\infty}^{\infty} x(\tau+t) \frac{\tau}{\sigma^2(t)} g'\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau,$$

we then can note that

$$\begin{aligned}
\partial_t \widetilde{V}_x(t,\eta) &= \int_{-\infty}^{\infty} x'(t+\tau) \frac{1}{\sigma(t)} g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau + \int_{-\infty}^{\infty} x(t+\tau)(-\frac{\sigma'(t)}{\sigma(t)^2}) g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau \\
&+ \int_{-\infty}^{\infty} x(t+\tau)\big(-\frac{\sigma'(t)}{\sigma(t)^3}\big) g'\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau \\
&= \big(p + qt + i2\pi(c+rt)\big)\widetilde{V}_x(t,\eta) + (q + i2\pi r)\int_{-\infty}^{\infty} \tau x(t+\tau) \frac{1}{\sigma(t)} g(\frac{\tau}{\sigma(t)}) e^{-i2\pi\eta\tau} d\tau \\
&- \frac{\sigma'(t)}{\sigma(t)} \widetilde{V}_x(t,\eta) - \frac{\sigma'(t)}{\sigma(t)} \widetilde{V}_x^{g^2}(t,\eta) \\
&= \big(p + qt + i2\pi(c+rt) - \frac{\sigma'(t)}{\sigma(t)}\big)\widetilde{V}_x(t,\eta) + (q + i2\pi r)\sigma(t)\widetilde{V}_x^{g^2}(t,\eta) - \frac{\sigma'(t)}{\sigma(t)} \widetilde{V}_x^{g^2}(t,\eta),
\end{aligned}$$

and if $\widetilde{V}_x(t,\eta) \neq 0$, we have

$$\frac{\partial_t \widetilde{V}_x(t,\eta)}{\widetilde{V}_x(t,\eta)} = p + qt + \frac{\sigma'(t)}{\sigma(t)} + i2\pi(c+rt) + (q+i2\pi r)\sigma(t)\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)} - \frac{\sigma'(t)}{\sigma(t)} \frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}$$

$$\frac{\partial}{\partial\eta}\Big(\frac{\partial_t \widetilde{V}_x(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big) = (q+i2\pi r)\sigma(t)\frac{\partial}{\partial\eta}\Big(\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big) - \frac{\sigma'(t)}{\sigma(t)}\Big(\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big).$$

Therefore, if in addition, $\frac{\partial}{\partial\eta}\Big(\frac{\widetilde{V}_s^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big) \neq 0$, then $(q + i2\pi r)\sigma(t) = R_0(t,\eta)$ with

$$R_0(t,\eta) = \frac{\frac{\partial}{\partial\eta}\left(\frac{\partial_t \widetilde{V}_x(t,\eta)}{\widetilde{V}_x(t,\eta)}\right) + \frac{\sigma'(t)}{\sigma(t)}\frac{\partial}{\partial\eta}\left(\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\right)}{\frac{\partial}{\partial\eta}\left(\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)}\right)} \tag{2.4.1}$$

$$q + i2\pi r = \frac{R_0(t,\eta)}{\sigma(t)}$$

$$\frac{\partial_t \widetilde{V}_x(t,\eta)}{\widetilde{V}_x(t,\eta)} = p + qt - \frac{\sigma'(t)}{\sigma(t)} + i2\pi(c+rt) + R_0(t,\eta)\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)} - \frac{\sigma'(t)}{\sigma(t)}\Big(\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big).$$

Thus

$$\phi'(t) = c + rt = Re\Big\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\} - Re\Big\{\frac{\widetilde{V}_x^{g^1}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}R_0(t,\eta)\Big\} + \frac{\sigma'(t)}{\sigma(t)}Re\Big\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big\}.$$

For a signal $x(t)$ in the following, we define the phase transformation $\widetilde{\omega}_x^{2nd,adp}$ as

$$\widetilde{\omega}_x^{2nd,adp} = \begin{cases} Re\Big\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\} - Re\Big\{\frac{\widetilde{V}_x^{g^1}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}R_0(t,\eta)\Big\} + \frac{\sigma'(t)}{\sigma(t)}Re\Big\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big\}, & \text{if } \frac{\partial}{\partial\eta}\left(\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)}\right) \neq 0 \\ Re\Big\{\frac{\partial_t \widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\Big\} + \frac{\sigma'(t)}{\sigma(t)}Re\Big\{\frac{\widetilde{V}_x^{g^2}(t,\eta)}{\widetilde{V}_x(t,\eta)}\Big\}, & \text{if } \frac{\partial}{\partial\eta}\left(\frac{\widetilde{V}_x^{g^1}(t,\eta)}{\widetilde{V}_x(t,\eta)}\right) = 0. \end{cases}$$

The adaptive 2nd-orded FSST of a signal $x(t)$ is defined by

$$R_x^{2nd,adp}(t,\xi) = \int_{\{\eta\in\mathbb{R}:\widetilde{V}_x(t,\eta)\neq 0\}} \widetilde{V}_x(t,\eta)\delta(\tilde{\omega}_x^{2nd,adp}(t,\eta) - \xi)d\eta, \tag{2.4.2}$$

where ξ is the frequency variable.

To recover the input signal $x(t)$ from its adaptive FSST, we have

$$x(t) = \frac{\sigma(t)}{g(0)}\int_{-\infty}^{\infty} R_x^{2nd,adp}(t,\xi)d\xi.$$

If in addition $x(t)$ is real-valued, then for real-valued $x(t)$, we have

$$x(t) = \frac{2\sigma(t)}{g(0)} Re\Big(\int_{-\infty}^{\infty} R_x^{2nd,adp}(t,\xi) d\xi \Big).$$

We may use the following formula to recover the k-th component $x_k(b)$ of a multicomponent signal from the adaptive FSST

$$x_k(b) = \frac{2\sigma(t)}{g(0)} Re\Big(\int_{|\xi-\phi'(t)|<\Gamma} R_x^{2nd,adp}(\xi,b) d\eta \Big)$$

for certain $\Gamma > 0$.

2.5 Analysis of FSST

In this section, many pre-established concepts are reiterated and then expanded upon. Refrences to these materials can be found in [19].

2.5.1 STFT-based synchrosqueezing transform

The short-time Fourier transform (STFT) of $x(t) \in L_2(\mathbb{R})$ with a window function $g(t) \in L_2(\mathbb{R})$ given by (2.3.3). For a signal $x(t)$, at (t,η) for which $V_x(t,\eta) \neq 0$, note that

$$\omega_x(t,\eta) = Re\Big(\frac{\partial_t V_x(t,\eta)}{2i\pi V_x(t,\eta)} \Big).$$

The FSST reassigns the frequency variable τ by transforming STFT $V_x(t,\eta)$ of $x(t)$ to a quantity, denoted by

$$R_{x,\gamma}(t,\xi) = \int_{|V_x(t,\eta)|\neq\gamma} V_x(t,\eta) \frac{1}{\lambda} h\Big(\frac{\xi - \omega_x(t,\eta))}{\lambda} \Big) d\eta \tag{2.5.1}$$

where $h(t)$ is a compactly supported function with certain smoothness and

$\int_{\mathbb{R}} h(t)dt = 1$. $\int_{|V_x(t,\eta)|>\gamma}$ means the integral $\int_{\{\eta:|V_x(t,\eta)|>\gamma\}}$ with η over the set $\{\eta : |V_x(t,\eta)| > \gamma\}$. We consider multicomponent signals $x(t)$, defined as follows

$$x(t) = \sum_{k=1}^{K} x_k(t), \quad \text{with} \quad x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}, \tag{2.5.2}$$

where $A_k(t)$ and $\phi_k(t)$ satisfy

$$A_k(t) \in C^1(\mathbb{R}) \cap L_\infty(\mathbb{R}), \ \phi_k(t) \in C^2(\mathbb{R}), \tag{2.5.3}$$

$$A_k(t) > 0, \ \inf_{t\in R} \phi_k'(t) > 0, \ \sup_{t\in R} \phi_k'(t) < \infty. \tag{2.5.4}$$

Let $\varepsilon > 0$ and $\Delta > 0$, and let $\beta_{\varepsilon,\Delta}$ denote the set of multicomponent signals satisfying (2.5.3), (2.5.4), and the following condition:

$$|A_k(t)| \le \varepsilon\phi_k'(t), |\phi_k(t)| \le \varepsilon\phi_k''(t), t \in \mathbb{R}, M_k'' = \sup_{t\in\mathbb{R}} |\phi_k(t)| < \infty, \tag{2.5.5}$$

$$\phi_k'(t) - \phi_{k-1}'(t) \ge 2\Delta, \quad 2 \le k \le K, t \in \mathbb{R}. \tag{2.5.6}$$

The condition (2.5.6) is called the well-separated condition with resolution Δ. For the well-separated condition, [9] uses a stronger condition than that in (2.5.6):

$$\inf_{t\in\mathbb{R}} \phi_k'(t) - \sup_{t\in\mathbb{R}} \phi_{k-1}'(t) \ge 2\Delta, \quad 2 \le k \le K. \tag{2.5.7}$$

The condition (2.5.5) considered in [12] implies that $A_k(t)$ and IF $\phi_k'(t)$ change slowly as compared with $\phi_k'(t)$. The Fourier-based synchrosqueezing transform, [25], uses another condition for the change of $A_k(t)$ and IF $\phi_k'(t)$:

$$|A_k'(t)| \le \varepsilon, \quad \text{and} \quad |\phi_k'(t)| \le \varepsilon, \quad \text{for} \quad t \in \mathbb{R}. \tag{2.5.8}$$

Let $\beta_{\varepsilon,\Delta}$ denote the set of multicomponent signals that satisfy (2.5.3), (2.5.4), (2.5.8), and well-separated condition (2.5.6). For $1 \le k \le K$ let

$$Z_k = \{\eta : |\eta - \phi_k'(t)| < \Delta\}. \tag{2.5.9}$$

Hence, the well-separated condition (2.5.6) implies that Z_k are not overlapping. We denote

$$\Gamma_0(t) = MI_1 + \pi I_2 \sum_{k=1}^{K} A_k(t), \quad \text{and} \quad \widetilde{\Gamma_0}(t) = M\widetilde{I_1} + \pi \widetilde{I_2} \sum_{k=1}^{K} A_k(t), \tag{2.5.10}$$

where

$$I_n = \int_{\mathbb{R}} |\tau^n g(\tau)|\, d\tau, \quad \text{and} \quad \widetilde{I_n} = \int_{\mathbb{R}} |\tau^n g'(\tau)|\, d\tau, \quad \text{for} \quad n = 1, 2, \ldots \tag{2.5.11}$$

Theorem 2.5.1. *As stated in [19], let $x(t) \in B_{\varepsilon,\Delta}$ and g be a function in the Schwartz class with supp($\hat{g}$) $\subseteq [-\Delta, \Delta]$. Let $\Gamma_0(t), \widetilde{\Gamma_0}(t)$ be defined by (2.5.10). Then we have the following.*

(a) Suppose $\widetilde{\varepsilon}$ satisfies $\widetilde{\varepsilon} \geq \varepsilon\Gamma_0(t)$. Then for any η with $|V_x(t,\eta)| > \widetilde{\varepsilon}$, there exists a unique $k \in \{1, 2, \ldots, K\}$ such that $(t, \eta) \in Z_k$.

(b) Suppose (t, η) satisfies $|V_x(t,\eta)| > \widetilde{\varepsilon}$ and $(t,\eta) \in Z_k$. Then

$$|\omega_x(t,\eta) - \phi_k'(t)| < \frac{\varepsilon}{\widetilde{\varepsilon}}(\Gamma_0(t)\Delta + \frac{1}{2\pi}\widetilde{\Gamma_0}(t)) \tag{2.5.12}$$

(c) Suppose that $\widetilde{\varepsilon}$ satisfies $(\Gamma_0(t)\Delta + \frac{1}{2\pi}\widetilde{\Gamma_0}(t))\frac{\varepsilon}{\widetilde{\varepsilon}} \leq \widetilde{\varepsilon}_3 \leq \Delta$, we have

$$\left| \lim_{\lambda \to 0} \frac{1}{g(0)} \int_{|\varepsilon - \phi_k'| < \widetilde{\varepsilon}_3} R^{\lambda}_{x,\widetilde{\varepsilon}}(t,\xi) d\xi - x_k(t) \right| \leq \frac{2\Delta(\varepsilon\Gamma_0(t) + \widetilde{\varepsilon})}{|g(0)|}. \tag{2.5.13}$$

(d) If $x(t) \in B_{\varepsilon,\Delta}$, then the above statements (a)-(c) hold with $\Gamma_0(t)$ and $\widetilde{\Gamma_0}(t)$ in (2.5.10) replaced by

$$\Gamma_0(t) = \sum_{k=1}^{K}\Big\{\phi_k'(t)I_1 + \frac{1}{2}M_k''I_2 + \pi A_k(t)(\phi_k'(t)I_2 + \frac{1}{3}M_k''I_3)\Big\} \quad (2.5.14)$$

$$\widetilde{\Gamma_0}(t) = \sum_{k=1}^{K}\Big\{\phi_k'(t)\widetilde{I}_1 + \frac{1}{2}M_k'\widetilde{I}_2 + \pi A_k(t)(\phi_k'(t)\widetilde{I}_2 + \frac{1}{3}M_k'\widetilde{I}_3)\Big\} \quad (2.5.15)$$

we may note that $\widetilde{\varepsilon}$ *and* $\widetilde{\varepsilon}_3$ *in Theorem (2.5.1) could be a function of* t. *If we choose* $\widetilde{\varepsilon} = \varepsilon^{\frac{1}{3}}$, *and if* ε *is small enough, such that*

$$\widetilde{\varepsilon} \leq min\Big\{\Delta, \frac{1}{||\Gamma_0(t)\Delta + \frac{1}{2\pi}\widetilde{\Gamma_0}(t)||_\infty}\Big\}, \quad (2.5.16)$$

then $\widetilde{\varepsilon}(\Gamma_0(t)\Delta + \frac{1}{2\pi}\widetilde{\Gamma_0}(t)) \leq 1$. *Hence,*

$$(\Gamma_0(t)\Delta + \frac{1}{2\pi}\widetilde{\Gamma_0}(t))\frac{\varepsilon}{\widetilde{\varepsilon}} \leq \widetilde{\varepsilon} \leq \Delta. \quad (2.5.17)$$

Thus, the conditions in the Theorem (2.5.1) are satisfied, and Theorem (2.5.1) (with $\widetilde{\varepsilon}_3 = \widetilde{\varepsilon}$) can be defined in theorem (2.5.2).

Theorem 2.5.2. *As stated in [12, 19, 25], let* $x(t) \in B_{\varepsilon,\Delta}$, *and* $\widetilde{\varepsilon} = \varepsilon^{\frac{1}{3}}$. *Let* g *be a function in he Schwartz class with* $supp(\hat{g}) \subseteq [-\Delta, \Delta]$. *If* ε *is small enough, then the following statements hold.*

(a) For (t,η) *satisfying* $|V_x(t,\eta)| > \widetilde{\varepsilon}$, *there exists a unique* $k \in \{1, 2, \ldots, K\}$ *such that* $(t,\eta) \in Z_k$

(b) Suppose (t,η) *satisfies* $|V_x(t,\eta)| > \widetilde{\varepsilon}$ *and* $(t,\eta) \in Z_k$ *Then*

$$\Big|\omega_x(t,\eta) - \phi_k'(t)\Big| < \widetilde{\varepsilon}. \quad (2.5.18)$$

(c) For any $k \in \{1, 2, \ldots, K\}$,

$$\Big|\lim_{\lambda\to 0}\frac{1}{g(0)}\int_{|\varepsilon-\phi_k'|<\widetilde{\varepsilon}} R^{\lambda}_{x,\widetilde{\varepsilon}}(t,\xi)d\xi - x_k(t)\Big| \leq \frac{4\Delta}{g(0)}\widetilde{\varepsilon}. \quad (2.5.19)$$

"ε is small enough" in Theorem (2.5.2) implies that $\widetilde{\varepsilon}$ defined by $\widetilde{\varepsilon} = \varepsilon^{\frac{1}{3}}$ satisfies some inequalities like (2.5.16). Most theorems on the WSST and FSST analysis are stated in the form of Theorem 2.5.2, see e.g. [7, 9, 12, 18, 25]. Part (b) and part (c) in theorem 2.5.1 denote more direct bounds of the estimates. The quantity on the left-hand side (LHS) of (2.5.12) the IF estimate error. Additionally, we call that on LHS of (2.5.13) the error of component recovery or component separation. The statements in Theorem 2.5.1 can be found in [12, 18, 25] with some different IF estimate errors. For example,[18, 25] gave IF estimate error $\frac{\varepsilon}{\widetilde{\varepsilon}}(\Gamma_0(t)(\Delta+2\phi_k'(t))+\frac{1}{2\pi}\widetilde{\Gamma_0}(t))$ instead of $\frac{\varepsilon}{\widetilde{\varepsilon}}(\Gamma_0(t)\Delta+\frac{1}{2\pi}\widetilde{\Gamma_0}(t))$ in (2.5.12). One can also find that Theorem (2.5.1) is a special case of Theorem (2.6.1).

Observe that the condition (2.5.5) or (2.5.8) requires the slow change of the IF $\phi_k'(t)$ of each component $x_k(t)$. There is no mathematical guarantee that the IF estimate and the component separation for a multicomponent signal $x(t)$ with a component $x_k(t)$ will have a fast-changing frequency. For example, the changing rate of IF of $x_k(t)$ is not very small in the second derivative $\phi_k''(t)$. To address this, the 2nd-order FSST was introduced in [10], and the 2nd-order WSST was proposed in [11]. The theoretical analysis of the 2nd-order FSST is established in [18].

If $V_x(t,\eta) \neq 0$ and $\partial_t\big(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\big) \neq i2\pi$, then

$$\tilde{q}(t,\eta) = \frac{\partial_t\big(\frac{\partial_t V_x(t,\eta)}{V_x(t,\eta)}\big)}{i2\pi - \partial_t\big(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}\big)}. \tag{2.5.20}$$

The 2nd-order FSST in [18] is defined as

$$R_{x,\gamma}^{2nd,\lambda}(t,\xi) = \int_{|V_x(t,\eta)|>\gamma} V_x(t,\eta)\frac{1}{\lambda}h(\frac{\xi-\omega_x^{2nd}(t,\eta)}{\lambda})d\eta, \tag{2.5.21}$$

where $\omega_x^{2nd}(t,\eta)$ is the phase transformation for the 2nd-order FSST. For (t,η) with $V_x(t,\eta)\neq 0$,

$$\omega_x^{2nd}(t,\eta) = \begin{cases} Re\{\frac{\partial_t V_x(t,\eta)}{2\pi i V_x(t,\eta)}\} + Re\{\tilde{q}(t,\eta)\partial_t(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)})\} & \text{if} \quad \partial_t(\frac{\partial_\eta V_x(t,\eta)}{V_x(t,\eta)}) \neq i2\pi \\ Re\{\frac{\partial_t V_x(t,\eta)}{2\pi i V_x(t,\eta)}\} & \text{Otherwise.} \end{cases} \tag{2.5.22}$$

Let $\varepsilon > 0$ and $\Delta > 0$. $B^{(2)}_{\varepsilon,\Delta}$ is the set of multicomponent signals satisfying (2.5.4), the well-separated condition (2.5.6), and the condition:

$$A_k(t) \in C^2(\mathbb{R}) \cap L_\infty(\mathbb{R}), \phi_k(t) \in C^3(\mathbb{R}), \phi'_k(t) \in L_\infty(\mathbb{R}), \tag{2.5.23}$$

$$|A'_k(t)| \le \varepsilon, |A''_k(t)| \le \varepsilon, \left|\phi^{(3)}_k(t)\right| \le \varepsilon, t \in (\mathbb{R}). \tag{2.5.24}$$

Then, when $x(t) \in B^{(2)}_{\varepsilon,\Delta}$, statements for the 2nd-order FSST that are similar to (2.5.2) hold under certain conditions that are more complicated than (2.5.16) because no hand-limited window functions and the 2nd-order phase transformation $\omega^{2nd}_x(t,\eta)$ are involved. The deffinition of $B^{(2)}_{\varepsilon,\Delta}$ has no direct boundedness restriction on $\phi''_k(t)$. See [18] for the details.

2.6 Analysis of Adaptive FSST

In this section, many pre-established concepts are reiterated and then expanded upon. Refrences to these materials can be found in [19].

2.6.1 Adaptive FSST with a time-varying parameter

We begin with the window function given by

$$g_\sigma(t) = \frac{1}{\sigma} g(\frac{t}{\sigma}), \tag{2.6.1}$$

where $\sigma > 0$ is a parameter and $g(t) \in L_2(\mathbb{R})$ with $g(0) \neq 0$. In addition, $g(t)$ has a decaying order as $t \to \infty$. If

$$g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}, \tag{2.6.2}$$

then $g_\sigma(t)$ is the Gaussian window function. The parameter σ is also the window width in the time-domain of the window function $g_\sigma(t)$ because the time duration $\triangle_{g_\sigma}$ of $g_\sigma(t)$ is σ (up to a constant). $\triangle_{g_\sigma} = \sigma\triangle_g$, where $\triangle_g$ is the time duration of g.

For a signal $x(t)$, the STDT of $x(t)$ with a time-varying parameter, as defined in [15] is

$$\widetilde{V_x}(t,\eta) = \int_{\mathbb{R}} x(\tau)g_{\sigma(t)}(\tau - t)e^{-i2\pi\eta(\tau-t)}d\tau = \int_{\mathbb{R}} x(t+\tau)\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \quad (2.6.3)$$

where $\sigma = \sigma(t)$ is positive function of t. $\widetilde{V_x}(t,\eta)$ is the adaptive STFT of $x(t)$ with g_σ.

Before review the SST associated with the adaptive STFT, we must introduce some notations used in this and the next two subsections. Note that

$$g_1(\tau) = \tau g(\tau), g_2(\tau) = \tau^2 g(\tau), g_3(\tau) = \tau g'(\tau).$$

Therefore,

$$g_{1,\sigma}(\tau) = \frac{\tau}{\sigma^2}g(\frac{\tau}{\sigma}), g_{2,\sigma}(\tau) = \frac{\tau^2}{\sigma^3}g(\frac{\tau}{\sigma}), g_{3,\sigma}(\tau) = \frac{\tau}{\sigma^2}g'(\frac{\tau}{\sigma}).$$

We use $\widetilde{V}_x^{g_i}(t,\eta)$ and $\widetilde{V}_x^{g'}(t,\eta)$ to denote the adaptive STFT is defined by (2.6.3). $g_{j,\sigma}$ replaces g_σ and $g'_\sigma(\tau) = \frac{1}{\sigma}g'(\frac{\tau}{\sigma})$ where $1 \leq j \leq 3$.
For $x(t) = Ae^{i2\pi ct}$, we can prove that, if $\widetilde{V}_x(t,\eta)$, then

$$\omega_x^{adp,c}(t,\eta) = \frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)} + \frac{\sigma'(t)}{i2\pi\sigma(t)} + \frac{\sigma'(t)}{\sigma(t)}\frac{\widetilde{V}_x^{g_3}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}. \quad (2.6.4)$$

Furthermore, $\omega_x^{adp,c}(t,\eta)$ is c, the IF of $x(t)$. For a general $x(t)$ at (t,η), we must then define the real part of the quantity of $\omega_x^{adp,c}(t,\eta)$ in the above equation. This real part denoted by $\omega_x^{adp}(t,\eta)$ and as the phase transformation of the adaptive FSST (see[15]):

$$\omega_x^{adp}(t,\eta) = Re\{\frac{\partial_t(\widetilde{V}_x(t,\eta))}{i2\pi\widetilde{V}_x(t,\eta)}\} + \frac{\sigma'(t)}{\sigma(t)}Re\{\frac{\widetilde{V}_x^{g_3}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\}, \quad for \quad \widetilde{V}_x(t,\eta) \neq 0 \tag{2.6.5}$$

Then, the 1st-order adaptive FSST, or $R_{x,\gamma}^{adp,\lambda}$, is defined by

$$R_{x,\gamma}^{adp,\lambda}(t,\xi) = \int_{|\widetilde{V}_x(t,\eta)|>\gamma} \widetilde{V}_x(t,\eta)\frac{1}{\lambda}h(\frac{\xi - \omega_x^{adp}(t,\eta)}{\lambda})d\eta. \tag{2.6.6}$$

We must now consider the 2nd-order adaptive FSST. For a linear chirp signal,

$$x(t) = Ae^{i2\pi\phi(t)} = Ae^{i2\pi(ct+\frac{1}{2}rt^2)}. \tag{2.6.7}$$

Adaptive short-time Fourier transform and synchrosqueezing transform for non-stationary signal separation [15] shows $\omega_x^{adp,2nd,c}$, defined below as $c+rt$, as the IF of $x(t)$:

$$\omega_x^{adp,2nd,c} = \frac{\sigma'(t)}{i2\pi\sigma(t)} + \frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)} - \frac{\widetilde{V}_x^{g_1}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}P_0(t,\eta) + \frac{\sigma'(t)}{\sigma(t)}\frac{\widetilde{V}_x^{g_3}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)} \tag{2.6.8}$$

for (t,η) satisfying $\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1(t,\eta)}}{\widetilde{V}_x(t,\eta)}) \neq 0$ and $\widetilde{V}_x(t,\eta) \neq 0$, where

$$P_0(t,\eta) = \frac{1}{\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta}{\widetilde{V}_x(t,\eta)})}\{\frac{\partial}{\partial\eta}(\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{\widetilde{V}_x(t,\eta)}) + \frac{\sigma'(t)}{\sigma(t)}\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_3}(t,\eta)}{\widetilde{V}_x(t,\eta)})\}. \tag{2.6.9}$$

Li, Cai, Han, Jiang, Ji define the real part of $\omega_x^{adp,2nd,c}$ as the phase transformation for the 2nd-order adaptive FSST (see [15]). The phase transformation $\omega_x^{adp,2nd,c}$ is defined by

$$\omega_x^{adp,2nd,c}(t,\eta) = \begin{cases} Re\{\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\} - Re\{\frac{\widetilde{V}_x^{g_1}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}P_0(t,\eta)\} + \frac{\sigma'(t)}{\sigma(t)}Re\{\frac{\widetilde{V}_x^{g_3}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\}, \\ \qquad if \quad \frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta)}{\widetilde{V}_x(t,\eta)}) \neq 0 \quad and \quad \widetilde{V}_x(t,\eta) \neq 0, \\ Re\{\frac{\frac{\partial}{\partial t}\widetilde{V}_x(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\} + \frac{\sigma'(t)}{\sigma(t)}Re\{\frac{\widetilde{V}_x^{g_3}(t,\eta)}{i2\pi\widetilde{V}_x(t,\eta)}\}, if \quad \frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta)}{\widetilde{V}_x(t,\eta)}) = 0 \quad and\, \widetilde{V}_x(t,\eta) \neq \end{cases}$$

The adaptive FSST with $\widetilde{V}_x(t,\eta) \neq 0$ and $\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}}{\widetilde{V}_x(t,\eta)}) \neq 0$ described by thresholds $\gamma_1 > 0$, $\gamma_2 > 0$. More precisely:

$$\omega_{x\gamma_1,\gamma_2}^{adp,2nd,c}(t,\eta) = \begin{cases} quantity \quad in(2.6.8), \quad if \left|\widetilde{V}_x(t,\eta)\right| > \gamma_1 \quad and \quad \left|\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta)}{\widetilde{V}_x(t,\eta)})\right| > \gamma_2 \\ quantity \quad in(2.6.4), \quad if \left|\widetilde{V}_x(t,\eta)\right| > \gamma_1 \quad and \quad \left|\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta)}{\widetilde{V}_x(t,\eta)})\right| \leq \gamma_2 \end{cases} \tag{2.6.11}$$

Again, let $h(t)$ be a compactly supported function with certain smoothness and $\int_{\mathbb{R}} h(t)dt = 1$.
The 2nd-order adaptive FSST, $R_{x\gamma_1,\gamma_2}^{adp,2nd,\lambda}$, is defined as

$$R_{x\gamma_1,\gamma_2}^{adp,2nd,\lambda}(t,\xi) = \int_{\{\eta:\left|\widetilde{V}_x(t,\eta)\right|>\gamma_1,\left|\partial\eta(\widetilde{V}_x^{g_1}(t,\eta)/\widetilde{V}_x(t,\eta)\right|>\gamma_2\}} \widetilde{V}_x(t,\eta)\frac{1}{\lambda}h(\frac{\xi - \omega_{x\gamma_1,\gamma_2}^{adp,2nd}(t,\eta)}{\lambda})d\eta \tag{2.6.12}$$

2.6.2 Analysis of adaptive FSST

The analysis of adaptive FSST was studied in [19]. A sinusoidal signal locally approximates each component of $x_k(t) = A_k(t)e^{i2\pi\phi_k(t)}$. We assume $A_k'(t)$ and $\phi_k'(t)$ are small.

$$|A_k'(t)| \leq \varepsilon_1, \quad |\phi_k''(t)| \leq \varepsilon_2, \quad t \in \mathbb{R}, \quad 1 \leq k \leq K \tag{2.6.1}$$

$$\widetilde{V}_x(t,\eta) = \int_{\mathbb{R}} x(\tau)g(\tau - t)e^{-i2\pi\eta(\tau-t)}d\tau \tag{2.6.2}$$

for some positive number $\varepsilon_1, \varepsilon_2$. Let $D_{\varepsilon_1,\varepsilon_2}$ denote the set of the set of multicomponent signals.

Let $x(t) \in D_{\varepsilon_1,\varepsilon_2}$. We write $x_k(t+\tau)$ as

$$\begin{aligned} x_k(t+\tau) \;&=\; x_k(t)e^{i2\pi\phi_k'(t)\tau} + (A_k(t+\tau) - A_k(t))e^{i2\pi\phi_k(t+\tau)} \\ &\quad + x_k(t)e^{i2\pi\phi_k'(t)\tau}(e^{i2\pi(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau)} - 1). \end{aligned}$$

If we simplify the expansion, we obtain

$$x_k(t+\tau) = (A_k(t+\tau) - A_k(t))e^{i2\pi\phi_k(t+\tau)} + x_k(t)e^{i2\pi(\phi_k(t+\tau)-\phi_k(t))}.$$

Then, we have

$$\begin{aligned} \widetilde{V}_x(t,\eta) \;&=\; \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t+\tau)\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \\ &=\; \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t)e^{i2\pi\phi_k'(t)\tau}\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau + rem_0 \end{aligned}$$

$$\widetilde{V}_x(t,\eta) \;=\; \sum_{k=1}^{K} x_k(t)\int_{\mathbb{R}} e^{i2\pi\phi_k'(t)\tau}\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau + rem_0 \tag{2.6.3}$$

$$\;=\; \sum_{k=1}^{K} x_k(t)\hat{g}(\sigma(t)(\eta - \phi_k'(t))) + rem_0 \tag{2.6.4}$$

where rem_0 is the remainder for the expansion of $V_x(t,\eta)$ in (2.6.4) given by

$$\begin{aligned} rem_0 &= \sum_{k=1}^{K}\int_{\mathbb{R}}(A_k(t+\tau)-A_k(t))e^{i2\pi\phi_k(t+\tau)}.\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \\ &+\sum_{k=1}^{K}\int_{\mathbb{R}}x_k(t)e^{i2\pi\phi_k'(t)\tau}(e^{i2\pi(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau)}-1)\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \end{aligned}$$

with $|A_k(t+\tau)-A_k(t)|\le\varepsilon_1|\tau|$ and
$\left|e^{i2\pi(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau)}-1\right|\le 2\pi\left|(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau\right|\le\pi\varepsilon_2|\tau|^2$,
we have

$$\begin{aligned} |rem_0| &\le \sum_{k=1}^{K}\int_{\mathbb{R}}\varepsilon_1|\tau|\frac{1}{\sigma(t)}\left|g(\frac{\tau}{\sigma(t)})\right|d\tau+\sum_{k=1}^{K}A_k(t)\int_{\mathbb{R}}\pi\varepsilon_2|\tau|^2\frac{1}{\sigma(t)}\left|g(\frac{\tau}{\sigma(t)})\right|d\tau \\ &= K\varepsilon_1 I_1\sigma(t)+\pi\varepsilon_2 I_2\sigma^2(t)\sum_{k=1}^{K}A_k(t), \end{aligned}$$

where I_n is defined in (2.5.11). Hence we have

$$|rem_0|\le\sigma(t)\Lambda_0(t) \tag{2.6.5}$$

where

$$\Lambda_0(t)=K\varepsilon_1 I_1+\pi\varepsilon_2 I_2\sigma(t)\sum_{k=1}^{K}A_k(t). \tag{2.6.6}$$

We can extend $\widetilde{V}_x^{g'}(t,\eta)$ as (2.6.4) with remainder rem_0'. In (2.6.5), rem_0' is defined as rem_0 in (2.6.5) with $g(\tau)$ replaced by $g'(\tau)$. Then we have the estimate for the remainder similar to (2.6.5). More precisely, we have

$$|rem_0'|\le\sigma(t)\widetilde{\Lambda}_0(t), \tag{2.6.7}$$

where

$$\widetilde{\Lambda}_0(t) = K\varepsilon_1\widetilde{I}_1 + \pi\varepsilon_2\widetilde{I}_2\sigma(t)\sum_{k=1}^{K}A_k(t) \tag{2.6.8}$$

with $\widetilde{I}_n$ defined in (2.5.11).

If the remainder rem_0 in (2.6.4) is small, then the term $x_k(t)\hat{g}(\sigma(t)(\eta - \phi_k'(t))$ in (2.6.4) gives the time-frequency zone of the STFT $\widetilde{V}_{x_k}$ of the kth component $x_k(t)$ of $x(t)$. Additionally, if g is band-limited, that is $\hat{g}$ is compactly supported. Therefore, if $(\hat{g}) \subset [-\triangle, \triangle]$, then $x_k(t)\hat{g}(\sigma(t)(\eta - \phi_k'(t))$ lies within the zone:

$$\{(t,\eta) : |\eta - \phi_k'(t)| < \frac{\triangle}{\sigma(t)}, t \in \mathbb{R}\}.$$

The multicomponent signal $x(t)$ is well-separated (that is $Z_k \cap Z_l = \varnothing, k \neq l$), provided that $\sigma(t)$ satisfies

$$\sigma(t) \geq \frac{2\triangle}{\phi_k'(t) - \phi_{k-1}'(t)}, t \in \mathbb{R}, k = 2, ..., K. \tag{2.6.9}$$

If $\hat{g}$ is not compactly supported, we must define the"support" of $\hat{g}$ outside of $\hat{g}$ $(\xi) \approx 0$. Specifically, for a given threshold $0 < \tau_0 < 1$, if $|\hat{g}(\xi)| \leq \tau_0$ for $|X| \geq \alpha$, then we say $\hat{g}(\xi)$ is "supported" in $[-\alpha, \alpha]$. When $|\hat{g}(\xi)|$ is even and decreasing for $\xi \geq 0$, then α can be obtained by solving

$$|\hat{g}(\alpha)| = \tau_0. \tag{2.6.10}$$

For instance, when g is the Gaussian function with $\hat{g}(\xi) = e^{-2\pi^2\xi^2}$, the corresponding α is given by

$$\alpha = \frac{1}{2\pi}\sqrt{2ln(\frac{1}{\tau_0})}. \tag{2.6.11}$$

Thus, g with $\hat{g}(\xi)$ is "supported" in $[-\alpha, \alpha]$, and we define the time-frequency zone Z_k of the kth-component $x_k(t)$ of $x(t)$ by

$$Z_k = \{(t,\eta) : |\hat{g}(\sigma(t)(\eta - \phi_k'(t))| > \tau_0, t \in \mathbb{R}\} = \{(t,\eta) : |\eta - \phi_k'(t)| < \frac{\alpha}{\sigma(t)}, t \in \mathbb{R}\} \quad (2.6.12)$$

If $\sigma(t)$ satisfies

$$\sigma(t) \geq \frac{2\triangle}{\phi_k'(t) - \phi_{k-1}'(t)}, t \in \mathbb{R}, k = 2, ..., K, \qquad (2.6.13)$$

the multicomponent signal $x(t)$ is well-separated.
In this case $Z_k \cap Z_l = \varnothing, k \neq l$. When $\sigma(t)$ satisfies (2.6.13), since $\phi_k'(t)$ is bounded, we can say

$$\left|\left|\frac{1}{\sigma(t)}\right|\right|_\infty .$$

In this case, we may also say

$$\sigma(t)\left|\phi_k'(t) - \phi_l' t)\right| \geq 2\sigma\left|k - l\right|. \qquad (2.6.14)$$

Theorem 2.6.1. *[19] Let $x(t) \in D_{\varepsilon_1,\varepsilon_2}$ for small $\varepsilon_1, \varepsilon_2 > 0$. Then we have the following*

(a) Suppose $\widetilde{\varepsilon_1}$ satisfies $\widetilde{\varepsilon_1} \geq \sigma(t)\Lambda_0(t) + \tau_0 \sum_{k=1}^K A_k(t)$. Then for (t,η) with $\left|\widetilde{V}_x(t,\eta)\right| > \widetilde{\varepsilon_1}$, there exists $k \in \{1, 2, 3, ..K\}$ such that $(t,\eta) \in Z_k$

(b) For (t,η) with $\left|\widetilde{V}_x(t,\eta)\right| \neq 0$, we have

$$\omega_x^{apd,c}(t,\eta) - \phi_k'(t) = \frac{Rem_1}{i2\pi\widetilde{V}_x(t,\eta)}$$

where

$$Rem_1 = i2\pi(\eta - \phi_k'(t))rem_0 - \frac{rem_0'}{\sigma(t)} + i2\pi\sum_{k\neq l} x_l(t)(\phi_l'(t) - \phi_k'(t))\hat{g}(\sigma(t)(\eta - \phi_k'(t)))$$

Hence, for (t,η) *satisfying* $\left|\widetilde{V}_x(t,\eta)\right| > \widetilde{\varepsilon_1}$ *and* $(t,\eta) \in Z_k$*, we have*

$$\left|\omega_x^{apd,c}(t,\eta) - \phi_k'(t)\right| < bd_1$$

$$\begin{aligned} bd_1 &= \frac{1}{\widetilde{\varepsilon_1}}(\alpha\Lambda_0(t) + \frac{1}{2\pi}\widetilde{\Lambda}_0(t)) \\ &+ \frac{1}{\widetilde{\varepsilon_1}} max_{k\in\{1,\ldots,K\}}\Big\{\sum_{k\neq l} A_l(t)\Big|\phi_l'(t) - \phi_k'(t)\Big| sup_{u:|u|<\alpha}\Big|\hat{g}(u + \sigma(t)(\phi_k'(t) - \phi_l'(t)))\Big|\Big\} \end{aligned}$$

(c) *Suppose that* $\widetilde{\varepsilon_1}$ *satisfies the condition in part (a) and that* bd_1 *in part (b) satisfies* $bd_1 \leq \dfrac{\alpha}{\sigma(t)}$*. Then for* $\widetilde{\varepsilon_3}$ *satisfying* $bd_1 \leq \widetilde{\varepsilon_3} \leq \dfrac{\alpha}{\sigma(t)}$ *we have*

$$\Big|\lim_{\lambda\to 0}\frac{\sigma(t)}{g(0)}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon_3}} R_{x,\widetilde{\varepsilon_1}}^{adp,\lambda}(t,\xi)d\xi - x_k(t)\Big| \leq bd_2, \tag{2.6.15}$$

where

$$bd_2 = \frac{1}{|g(0)|}\Big\{2\alpha(\sigma(t))\Lambda_0(t) + \widetilde{\varepsilon_1}) + A_k(t)\Big|\int_{|u|\geq\alpha}\hat{g}(u)du\Big| + \sum_{l\neq k} A_l(t)m_{l,k}(t)\Big\} \tag{2.6.16}$$

with

$$m_{l,k}(t) = \Big|\int_{|u|\geq\alpha}\hat{g}(u + \sigma(t)(\phi_k'(t) - \phi_l'(t)))du\Big|$$

2.6.3 Analysis of 2nd-order adaptive FSST

For a given t, $G_k(\xi)$ denotes the Fourier transform of $e^{i\pi\sigma(t)\phi_k''(t)\tau^2}g(\tau)$, namely (see [19]),

$$\begin{aligned} G_k(\xi) &= \mathcal{F}\Big(e^{i\pi\sigma(t)\phi_k''(t)\tau^2}g(\tau)\Big)(\xi) \\ &= \int_{\mathbb{R}} e^{i\pi\sigma(t)\phi_k''(t)\tau^2}g(\tau)e^{-i2\pi\xi\tau}d\tau \end{aligned}$$

$$\begin{aligned} x_k(t+\tau) &= x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)} + \big(A_k(t+\tau)-A_k(t)\big)e^{i2\pi\phi_k(t+\tau)} \\ &+ x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)}\Big(e^{i2\pi\left(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau-\frac{1}{2}\phi_k''(t)\tau^2\right)}-1\Big). \end{aligned}$$

Then, we have

$$\begin{aligned} \widetilde{V}_x(t,\eta) &= \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t+\tau)\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \\ &= \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)}\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau + res_0. \end{aligned}$$

where

$$res_0 = \sum_{k=1}^{K}\int_{\mathbb{R}}\{\big(A_k(t+\tau)-A_k(t)\big)e^{i2\pi\phi_k(t+\tau)} \tag{2.6.1}$$

$$+ \; x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)}\Big(e^{i2\pi\left(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau-\frac{1}{2}\phi_k''(t)\tau^2\right)}-1\Big)\}\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \tag{2.6.2}$$

$$\widetilde{V}_x(t,\eta) = \sum_{k=1}^{K}x_k(t)G_k\Big(\sigma(t)(\eta-\phi_k'(t))\Big) + res_0 \tag{2.6.3}$$

with

$$\Big|A_k(t+\tau)-A_k(t)\Big|\le \varepsilon_1|\tau|$$

and

$$\Big|e^{i2\pi\left(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau-\frac{1}{2}\phi_k''(t)\tau^2\right)}-1\Big|\le 2\pi\frac{1}{6}sup_{\eta\in\mathbb{R}}|\phi_k^{(3)}(\eta)\tau^3|\le\frac{\pi}{3}\varepsilon_2|\tau|^3.$$

we have

$$\begin{aligned}|res_0| &\le \sum_{k=1}^{K}\int_{\mathbb{R}}\varepsilon_1|\tau|\frac{1}{\sigma(t)}|g(\frac{\tau}{\sigma(t)})|d\tau+\sum_{k=1}^{K}A_k(t)\int_{\mathbb{R}}\frac{\pi}{3}\varepsilon_3|\tau|^3\frac{1}{\sigma(t)}|g(\frac{\tau}{\sigma(t)})|d\tau\\ &= K\varepsilon_1 I_1\sigma^3(t)\sum_{k=1}^{K}A_k(t)\end{aligned}$$

where I_n is defined in (2.5.11). Hence,

$$|res_0|\le\sigma(t)\prod_0(t), \tag{2.6.4}$$

where

$$\prod_0(t)=K\varepsilon_1 I_1+\frac{\pi}{3}\varepsilon_3 I_3\sigma^2(t)\sum_{k=1}^{K}A_k(t). \tag{2.6.5}$$

Therefore, if $\varepsilon_1,\varepsilon_2$ are small enough, then $|res_0|$ is small as well. $G_k\Big(\sigma(t)(\eta-\phi_k'(t))\Big)$ also provides the time-frequency zone for $\widetilde{V}_{x_k}(t,\eta)$. To describe those time-frequency zones mathematically, let $0<\tau_0<1$ be a given small number and the threshold:

$$O_k=\{(t,\eta):|G_k\Big(\sigma(t)(\eta-\phi_k'(t)\Big)|>\tau_0,t\in\mathbb{R}\}. \tag{2.6.6}$$

Assuming $|G_k(\xi)|$ is even and decreasing for $\xi \geq 0$, then we may write O_k as

$$O_k = \{(t, \eta) : |\eta - \phi'_k(t)| < \frac{\alpha_k}{\sigma(t)}, t \in \mathbb{R}\} \tag{2.6.7}$$

where $\alpha_k = \alpha_k(t)$ is obtained by solving $|G_k(\xi)| = \tau_0$.In this instance, we will say that multicomponent signal $x(t)$ is well-separated and there is $\sigma(t)$ such that

$$O_k \cap O_l = \emptyset, k \neq l.$$

We can use a Gaussian function defined in (2.5.24) as an example. For this g, we can obtain (see [15]),

$$G_k(u) = \frac{1}{\sqrt{1 - i2\pi\phi''_k(t)\sigma^2(t)}} e^{-\frac{2\pi^2 u^2}{1 + (2\pi\phi''_k(t)\sigma^2(t))^2}(1+i2\pi\phi''_k(t)\sigma^2(t)))}.$$

Thus,

$$|G_k(u)| = \frac{1}{(1 + (2\pi\phi''_k(t)\sigma^2(t))^2)^{\frac{1}{4}}} e^{-\frac{2\pi^2}{1 + (2\pi\phi''_k(t)\sigma^2(t))^2)} u^2}.$$

The solution of $|G_k(u) = \tau_0| \Leftrightarrow e^{-\frac{2\pi^2 u^2}{1 + (2\pi\phi''_k(t)\sigma^2(t))^2}} = \tau_0((1+(2\pi\phi''_k(t)\sigma^2(t))^2)^{\frac{1}{4}}$

Therefore, in this case, assume $\tau_0((1 + (2\pi\phi''_k(t)\sigma^2(t))^2)^{\frac{1}{4}} \leq 1$,

$$\alpha_k = \sqrt{1 + (2\pi\phi''_k(t)\sigma^2(t))^2} \frac{1}{2\pi} \sqrt{2ln(\frac{1}{\tau_0}) - \frac{1}{2}ln(1 + (2\pi\phi''_k(t)\sigma^2(t))^2)}.$$

The main theorem on the 2nd-order adptive FSST can be written in more notations observe:

$$\begin{aligned} G_{j,k}(t,\eta) &= \int_{\mathbb{R}} e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)} \frac{\tau^j}{\sigma(t)^{j+1}} g(\frac{\tau}{\sigma(t)}) e^{-i2\pi\eta\tau} d\tau. \quad (2.6.8)\\ &= \mathcal{F}\Big(e^{i\pi(\phi_k''(t)\tau^2)}\tau^j g(\tau)\Big)\big(\sigma(t)(\eta-\phi_k'(t))\big). \quad (2.6.9)\end{aligned}$$

Clearly,

$$G_{0,k}(t,\eta) = G_k\big(\sigma(\eta-\phi_k'(t))\big),$$

and, when $j \geq 1$, we can see that

$$G_{j,k} = \frac{1}{(-i2\pi)^j} G_k^{(j)}\big(\sigma(t)(\eta-\phi_k'(t))\big). \quad (2.6.10)$$

Let res_1, res_2, res_0', and res_1' be the residuals defined as res_0 in (2.6.2) with $g(\tau)$ replaced respectively by $g_1(\tau), g_2(\tau), g'(\tau)$, and $g_3(\tau) = \tau g'(\tau)$. We therefore have the estimates for these residuals that are similar to (2.6.4).

$$|res_1| \leq \sigma(t)\prod_1(t), \quad |res_2| \leq \sigma(t)\prod_2(t), \qquad |res_0'| \leq \sigma(t)\widetilde{\prod}_0(t),\ |res_1'| \leq \sigma(t)\widetilde{\prod}_1(t),$$

where

$$\begin{aligned} &\prod_1(t) = K\varepsilon_1 I_2 + \frac{\pi}{3}\varepsilon_3 I_4 \sigma^2(t)\sum_{k=1}^K A_k(t), \quad \prod_2(t) = K\varepsilon_1 I_3 + \frac{\pi}{3}\varepsilon_3 I_5 \sigma^2(t)\sum_{k=1}^K A_k(t),\\ &\widetilde{\prod}_0(t) = K\varepsilon_1 \widetilde{I}_1 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_3 \sigma^2(t)\sum_{k=1}^K A_k(t), \quad \widetilde{\prod}_1(t) = K\varepsilon_1 \widetilde{I}_2 + \frac{\pi}{3}\varepsilon_3 \widetilde{I}_4 \sigma^2(t)\sum_{k=1}^K A_k(t),\end{aligned}$$

with $I_n, \widetilde{I}_n$ defined in (2.5.11).

Denote:

$$B_k(t,\eta) = \sum_{l\neq k} x_l(t)(\phi_l'(t) - \phi_k'(t))G_{0,l}(t,\eta), \quad D_k(t,\eta) = \sum_{l\neq k} x_l(t)(\phi_l''(t) - \phi_k''(t))G_{1,l}(t,\eta),$$

$$E_k(t,\eta) = \sum_{l\neq k} x_l(t)(\phi_l'(t) - \phi_k'(t))G_{1,l}(t,\eta), \quad F_k(t,\eta) = \sum_{l\neq k} x_l(t)(\phi_l''(t) - \phi_k''(t))G_{2,l}(t,\eta)$$

and

$$Res_1 = i2\pi B_k(t,\eta) + i2\pi\sigma D_k(t,\eta) + i2\pi(\eta - \phi_k'(t))res_0 - \frac{res'0}{\sigma(t)} - i2\pi\phi_k''(t)\sigma(t)res_1 \quad (2.6.11)$$

$$\begin{aligned} Res_2 &= 4\pi^2\sigma(t)E_k(t,\eta) + 4\pi^2\sigma^2(t)F_k(t) & (2.6.12)\\ &+ i2\pi res_0 + 4\pi^2(\eta - \phi_k'(t))\sigma(t)res_1 + i2\pi res_1' - 4\pi^2\phi_k''(t)\sigma^2(t)res_2 & (2.6.13)\end{aligned}$$

Lemma 2.6.2. *[19] Let Res_1 be the quantity defined by (2.6.11). Then,*

$$\partial_t\widetilde{V}_x(t,\eta) = \Big(i2\pi\phi_k'(t) - \frac{\sigma'(t)}{\sigma(t)}\Big)\widetilde{V}_x(t,\eta) + i2\pi\phi_k''(t)\sigma(t)\widetilde{V}_x^{g_1}(t,\eta) - \frac{\sigma'(t)}{\sigma(t)})\widetilde{V}_x^{g_3}(t,\eta) + Res_1 \quad (2.6.14)$$

Lemma 2.6.3. *[19] For (t,η) satisfying $\widetilde{V}_x(t,\eta) \neq 0$ and $\frac{\partial}{\partial\eta}(\frac{\widetilde{V}_x^{g_1}(t,\eta}{\widetilde{V}_x(t,\eta)}) \neq 0$, we have*

$$P_0(t,\eta) = i2\pi\sigma\phi_k''(t) + Res_3 \quad (2.6.15)$$

where

$$Res_3 = \frac{\widetilde{V}_x(t,\eta)Res_2 - \partial_\eta\widetilde{V}_x(t,\eta)Res_1}{\widetilde{V}_x(t,\eta)\partial_\eta\widetilde{V}_x^{g_1}(t,\eta) - \widetilde{V}_x^{g_1}(t,\eta)\partial_\eta\widetilde{V}_x(t,\eta)}, \quad (2.6.16)$$

with Res_1 and Res_2 defined by (2.6.11)and (2.6.13) respectively.

Theorem 2.6.4. *[19] Suppose $x(t) \in D^{(2)}_{\varepsilon_1,\varepsilon_2}$ with a window function $g(t)$ for some small $\varepsilon_1, \varepsilon_2 > 0$. Then, we have the following:*

(a) Suppose $\widetilde{\varepsilon_1}$ satisfies $\widetilde{\varepsilon_1} \geq \varepsilon_0 \sum_{k=1}^{K} A_k(t) + \sigma(t)\Pi_0(t)$. Then for (t,η) with $\left|\widetilde{V}_x(t,\eta)\right| > \widetilde{\varepsilon_1}$, there exists $k \in \{1, 2, ..., K\}$ such that $(t,\eta) \in O_k$.

(b) Suppose (t,η) satisfies $|V_x(t,\eta)| > \widetilde{\varepsilon_1}$, $\left|\partial_\eta(\widetilde{V}^{g_1}_x(t,\eta)/\widetilde{V}_x(t,\eta))\right| > \widetilde{\varepsilon}_2$, and $(t,\eta) \in O_k$. Then

$$\omega^{apd,2nd,c}(t,\eta) - \phi x'_k(t) = Res_4, \tag{2.6.17}$$

where

$$Res_4 = \frac{Res_1}{i2\pi\widetilde{V}_x(t,\eta)} - \frac{\widetilde{V}^{g_1}_x(t,\eta)Res_3}{i2\pi\widetilde{V}_x(t,\eta)}. \tag{2.6.18}$$

Furthermore,

$$\left|\omega^{apd,2nd}(t,\eta) - \phi'_k(t)\right| < Bd_1 \tag{2.6.19}$$

where

$$Bd_1 = \max_{1\leq k\leq K} \sup_{\eta\in O_k} \left\{ \frac{|Res_1|}{2\pi\widetilde{\varepsilon}_1} + \frac{1}{2\pi\widetilde{\varepsilon}_1^3\widetilde{\varepsilon}_2} \left|\widetilde{V}^{g_1}_x(t,\eta)\right| \left(\left|\partial_\eta\widetilde{V}_x(t,\eta)\right| |Res_1| + \widetilde{\varepsilon}_1 |Res_2| \right) \right\}$$

(c) Suppose that ε_1 satisfies the condition in part (a) and $Bd_1 \leq \frac{1}{2}L_k(t)$, where

$$L_k(t) = \frac{1}{\sigma(t)} min\{\alpha_k + \alpha_{k-1}, \alpha_k + \alpha_{k+1}\}. \tag{2.6.20}$$

Then for any $\widetilde{\varepsilon}_3 = \widetilde{\varepsilon}_3(t) > 0$ satisfying $Bd_1 \leq \widetilde{\varepsilon}_3 \leq \frac{1}{2}L_k(t)$,

$$\Big|\lim_{\lambda\to 0}\frac{\sigma(t)}{g(0)}\int_{|\xi-\phi_k'(t)|<\widetilde{\varepsilon}_3} R^{adp,2nd,\lambda}_{x,\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2}(t,\xi)d\xi - x_k(t)\Big| \le Bd_2, \quad (2.6.21)$$

where $Bd_2 = Bd_2' + Bd_2''$ *with*

$$\begin{aligned} Bd_2' &= \frac{1}{|g(0)|}\Big\{2\alpha_k(\widetilde{\varepsilon}_1+\sigma(t)\Pi_0(t)) + A_k(t)\Big|\int_{|u|\ge\alpha_k} G_k(u)du\Big| + \sum_{l\neq k} A_l(t)M_{l,k}(t)\Big\}, \\ Bd_2'' &= \frac{1}{|g(0)|}\{2\Pi_0(t)) + A_k(t)\parallel g\parallel_1 |Z_t| + \sum_{l\neq k} A_l(t)M_{l,k}(t)\} \end{aligned}$$

and $|Z_t|$ *represents the Lebesgue measure of the set* Z_t *:*

$$Z_t = \Big\{\eta : (t,\eta)\in O_k, \Big|\widetilde{V}_x(t,\eta)\Big| > \widetilde{\varepsilon}_1, , \Big|\partial(V_x^{g_1}(t,\eta)/\widetilde{V}_x(t,\eta))\Big| \le \widetilde{\varepsilon}_2\Big\}.$$

Chapter 3

Higher-order SST

3.1 High-order Synchrosqueezing transform

3.1.1 The higher-order wavelet synchrosqueezing transform

Higher-order WSST

Consider the Nth-order polynomial-phase signal

$$x(t) = Ae^{i2\pi\phi(t)} \tag{3.1.1}$$

with $\phi(t) = r_1 t + \frac{1}{2} r_2 t^2 + \cdots + \frac{1}{N} r_N t^N$, and $\phi'(t) = r_1 + \cdots + r_N t^{N-1}$.
We have $x'(t) = i2\pi\phi'(t)x(t)$ and

$$\phi'(b + at) = \phi'(b) + \frac{\phi''(b)at}{1!} + \cdots + \frac{\phi^{(N)}(b)a^{N-1}t^{N-1}}{(N-1)!}. \tag{3.1.2}$$

We recall the CWT definition of the signal $x(t)$

$$W_x(a, b) \quad = \quad \int_{-\infty}^{\infty} x(b + at)\overline{\psi(t)}dt. \tag{3.1.3}$$

Thus, we have

$$
\begin{aligned}
\partial_b W_x(a,b) &= \int_{-\infty}^{\infty} x'(b+at)\overline{\psi(t)}dt = \int_{-\infty}^{\infty} i2\pi\phi'(b+a)x(b+at)\overline{\psi(t)}dt \\
&= \int_{-\infty}^{\infty} i2\pi(r_1 + r_2(b+at) + \cdots + r_N(b+at)^{N-1})x(b+at)\overline{\psi(t)}dt \\
&= i2\pi\int_{-\infty}^{\infty}\Big(\phi'(b) + \frac{\phi''(b)at}{1!} + \cdots + \frac{\phi^{(N)}(b)a^{N-1}t^{N-1}}{(N-1)!}\Big).x(b+at)\overline{\psi(t)}dt \\
&= i2\pi\Big(\phi'(b)W_x(b,a) + \frac{\phi^{(2)}(b)a}{1!}W_x^{\psi_1}(b,a) + \cdots + \tfrac{a^{N-1}\phi^{(N)}(b)}{(N-1)!}W_x^{\psi_{N-1}}(b,a)\Big)
\end{aligned} \tag{3.1.4}
$$

where

$$
W_x^{\psi_k}(b,a) = \int_{-\infty}^{\infty} x(b+at)t^k\overline{\psi(t)}dt. \tag{3.1.5}
$$

We can then write $\frac{\partial}{\partial b}W_x(b,a) = \displaystyle\sum_{k=1}^{N}\frac{i2\pi a^{k-1}}{(k-1)!}\phi^{(k)}(b)W_x^{\psi_{k-1}}(b,a)$.

The goal is to determine $\phi^{(1)}(b)$ according to $W_x(b,a), W_x^{\psi_1}(b,a), \ldots, W_x^{\psi_{N-1}}(b,a)$

$$
\frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)} = \phi^{(1)}(b) + \sum_{k=2}^{N}\frac{a^{k-1}}{(k-1)!}\frac{W_x^{\psi_{k-1}}(b,a)}{W_x(b,a)}\phi^{(k)}(b). \tag{3.1.6}
$$

We can write the equation in the form of a scalar product:

$$
w_x(a,b) = [x_1(b), \ldots, x_N(b)][1, V_{2,1}(b,a), \ldots, V_{N,1}(b,a)]^T,
$$

where $w_x(a,b) = \dfrac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}$ and $x_k(b) = \phi^{(k)}(b),$ for $k = 1, \ldots, N.$

To solve the problem, we pass through successive derivatives of equation

(3.1.6) according to the variable a.
To get sequence $\big(x_k(b)\big)_{1\le k\le N}$, we create up a system of N equations with variables $x_k(b)$. Let us denote

$$y_1 = V_N.X_N^T \tag{3.1.7}$$

with $V_N = [1, V_{2,1}(b,a), \dots, V_{N,1}(b,a)]$ and $X_N = [x_1(b), \dots, x_N(b)]$.

Computing the partial derivatives:

$$y_2(b,a) = \frac{\partial a y_1(b,a)}{\partial a V_{2,1}(b,a)} \text{ and } V_{k,2}(b,a) = \frac{\partial a V_{k,1}(b,a)}{\partial a V_{2,1}(b,a)}, \tag{3.1.8}$$

which implies the following expression:

$$y_2(b,a) = [0, 1, V_{3,2}(b,a), \dots, V_{N,2}(b,a)]X_N^T \tag{3.1.9}$$

$$V_{k,2}(b,a) = \frac{\partial a V_{k,1}(b,a)}{\partial a V_{2,1}(b,a)}, \quad \text{for } k = 3, \dots, N.$$

To get the j^{th} equation. We repeat the same process iteratively. We define the new parameter for the A_N matrix for $j = 2, \dots, N$ and $k = j, \dots, N$ by:

$$y_j(b,a) = \frac{\partial a y_{j-1}(b,a)}{\partial a V_{j,j-1}(b,a)}, \text{ and } V_{k,j}(b,a) = \frac{\partial a V_{k,j-1}(b,a)}{\partial a V_{j,j-1}(b,a)}. \tag{3.1.10}$$

Then,

$$y_j(b,a) = [0, 0, \dots, 1, V_{j+1,j}, \dots, V_{N,j}]X_N^T.$$

We group the N equations and get a good linear system

$$\begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix} = A_N \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} \tag{3.1.11}$$

where

$$A_N = \begin{pmatrix} 1 & V_{2,1} & \dots & \dots & V_{N,1} \\ 0 & 1 & V_{3,2} & \dots & V_{N,2} \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & V_{N,N-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{3.1.12}$$

Since the A_N is an upper triangular matrix with a nonzero diagonal, the solution of the linear system is given by

$$\begin{aligned} x_N(b) &= y_N(b,a) \\ x_j(b) &= y_j(b,a) - \sum_{k=j+1}^{N} V_{k,j}(b,a) x_k(b), \text{ for } j = N-1, \dots, 1. \end{aligned} \tag{3.1.13}$$

We can write this idea in the form of an algorithm.

Determination of the Nth-order local phase transformation

Step 1. We define the matrix A_N by (3.1.12) with $V_{k,j}$ obtained by the following formula:

- $V_{k,1}(b,a) = \frac{a^{k-1}}{(k-1)!} \frac{W_x^{\psi_{k-1}}(b,a)}{W_x(b,a)}$, for $k = 2, \dots, N$
- and $V_{k,j}(b,a) = \frac{\partial_a V_{k,j-1}(b,a)}{\partial_a V_{j,j-1}(b,a)}$, for $j = 2, \dots, N, \;\; k = j, \dots, N$
- The matrix A_N is defined by :

$$A_N = \begin{pmatrix} 1 & V_{2,1} & \dots & \dots & V_{N,1} \\ 0 & 1 & V_{3,2} & \dots & V_{N,2} \\ \vdots & & & & \\ 0 & 0 & \dots & 1 & V_{N,N-1} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Step 2. We compute $y_1, y_2, ..., y_N$ by

$$y_1(b,a) = w_x(a,b) = \frac{\partial_b W_x(b,a)}{i2\pi W_x(b,a)}$$
$$y_j(b,a) = \frac{\partial_a y_{j-1}(b,a)}{\partial_a V_{j,j-1}(b,a)}, \text{ for } j = 2, \dots, N$$

Step 3. We solve (3.1.11) and obtain

$$x_N(b) = y_N(b,a)$$
$$x_j(b) = y_j(b,a) - \sum_{k=j+1}^{N} V_{k,j}(b,a) x_k(b), \text{ for } j = N-1, \dots, 1$$

Step 4. The Nth-order phase transformation ω_x^N is defined by

$$\omega_x^N(a,b) = \begin{cases} w_x(b,a) - \sum_{k=2}^{N} V_{k,1}(b,a) x_k(b), & \text{if } W_x(b,a) \neq 0 \text{ and } \partial a V_{j,j-1}(b,a) \neq 0 \\ w_x(b,a), & \text{if } W_x(b,a) \neq 0 \end{cases}$$

Adaptive higher-order WSST

For a signal with Nth-order polynomial-phase, we define adaptive higher order synchrosqueezing transform $\widetilde{\omega_x}^N$. We start with the CWT with a time-varying parameter $\sigma(t)$.

$$\widetilde{W}_x(a,b) = \int_{-\infty}^{\infty} x(b+at)\overline{\psi_{\sigma(b)}(t)}dt = \int_{-\infty}^{\infty} x(b+at)\frac{1}{\sigma(b)}\overline{g\left(\frac{t}{\sigma(b)}\right)e^{-i2\pi\mu t}}dt \quad (3.1.14)$$

The partial derivative of $W(a,b)$ is given by

$$\begin{aligned}\frac{\partial}{\partial b}\widetilde{W}_x(a,b) &= \int_{-\infty}^{\infty} x'(b+at)\frac{1}{\sigma(b)}\overline{g\big(\frac{t}{\sigma(b)}\big)}e^{-i2\pi\mu t}dt \\ &+ \int_{-\infty}^{\infty} x(b+at)\frac{-\sigma'(b)}{\sigma(b)^2}\overline{g\big(\frac{t}{\sigma(b)}\big)}e^{-i2\pi\mu t}dt \\ &+ \int_{-\infty}^{\infty} x(b+at)\frac{-\sigma'(b)t}{\sigma(b)^3}\overline{g'\big(\frac{t}{\sigma(b)}\big)}e^{-i2\pi\mu t}dt.\end{aligned}$$

We can simplify the expression

$$\partial_b\widetilde{W}_x(a,b) = \int_{-\infty}^{\infty} x'(b+at)\frac{1}{\sigma(b)}\overline{g\big(\frac{t}{\sigma(b)}\big)}e^{-i2\pi\mu t}dt - \frac{\sigma'(b)}{\sigma(b)}\widetilde{W}_x(a,b) - \frac{\sigma'(b)}{\sigma(b)}\widetilde{W}_x^{\psi^2}(a,b)$$

where

$$\begin{aligned}\widetilde{W}_x(a,b) &= \int_{-\infty}^{\infty} x(b+at)\frac{1}{\sigma(b)}\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt \\ \widetilde{W}_x^{\psi^2}(a,b) &= \int_{-\infty}^{\infty} x(b+at)\frac{t}{\sigma(b)^2}\overline{g'(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt.\end{aligned}$$

Observe that

$$\partial_b\widetilde{W}_x(a,b) + \frac{\sigma'(b)}{\sigma(b)}(\widetilde{W}_x(a,b) + \widetilde{W}_x^{\psi^2}(a,b)) = \int_{-\infty}^{\infty} x'(b+at)\frac{1}{\sigma(b)}\overline{g\big(\frac{t}{\sigma(b)}\big)}e^{-i2\pi\mu t}dt.$$

we denote: $\widetilde{\beta}(a,b) = \partial_b\widetilde{W}_x(a,b) + \frac{\sigma'(b)}{\sigma(b)}(\widetilde{W}_x(a,b) + \widetilde{W}_x^{\psi^2}(a,b))$.

Therefore,

$$\widetilde{\beta}(a,b) = \int_{-\infty}^{\infty} x'(b+at)\frac{1}{\sigma(b)}\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\eta t}dt. \tag{3.1.15}$$

In addition for a signal x defined in (3.1.1), $\widetilde{\beta}(a,b)$ satisfies

$$\begin{aligned}\widetilde{\beta}(a,b) &= \int_{-\infty}^{\infty} i2\pi\Big(\phi^{'}(b)+\cdots+\frac{(at)^{N-1}}{(N-1)!}\phi^{(N)}\Big)x(b+at)\frac{1}{\sigma(b)}\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt \\ &= i2\pi\Big(\widetilde{W}_x(a,b)\phi'(b)+\cdots+\frac{a^{N-1}}{(N-1)!}\widetilde{W}_x^{\psi_{N-1}}(a,b)\phi^{(N)}(b)\Big)\end{aligned}$$

where for $k=1,\dots,N-1$

$$\widetilde{W}_x^{\psi_k}(a,b)=\int_{-\infty}^{\infty} x(b+at)\frac{t^k}{\sigma(b)}\overline{g(\frac{t}{\sigma(b)})}e^{-i2\pi\mu t}dt.$$

Thus, we have

$$\frac{\widetilde{\beta}(a,b)}{i2\pi\widetilde{W}_x(a,b)}=\phi'(b)+a\frac{\widetilde{W}_x^{\psi_1}(a,b)}{\widetilde{W}_x(a,b)}\phi^{(2)}(b)+\cdots+\frac{a^{N-1}}{(N-1)!}\frac{\widetilde{W}_x^{\psi_{N-1}}(a,b)}{\widetilde{W}_x(a,b)}\phi^{(N)}(b)$$

$$\phi'(b)=Re\Big\{\frac{\widetilde{\beta}(a,b)}{i2\pi\widetilde{W}_x(a,b)}-\sum_{k=2}^{N}\phi^{(k)}(b)T_{k,1}(a,b)\Big\},$$

with $T_{k,1}(a,b)=\frac{\widetilde{W}_x^{\psi_{k-1}}(a,b)}{\widetilde{W}_x(a,b)}\frac{a^{k-1}}{(k-1)!}$. It remains only to determine $\phi^{(k)}(b)$ for $k=1,\dots,N$

$$\frac{\widetilde{\beta}(a,b)}{i2\pi\widetilde{W}_x(a,b)}=\phi'(b)+\sum_{k=2}^{N}T_{k,1}(a,b)\phi^{(k)}(b).$$

We denote

$$\begin{aligned}y_1(a,b) &= \frac{\widetilde{\beta}(a,b)}{i2\pi\widetilde{W}_x(a,b)}=\frac{\sigma(b)\partial_b\widetilde{W}_x(a,b)+\sigma'(b)\Big(\widetilde{W}_x(a,b)+\widetilde{W}_x^{\psi^2}(a,b)\Big)}{i2\pi\sigma(b)\widetilde{W}_x(a,b)} \\ y_1(a,b) &= \chi_1(b)+\sum_{k=2}^{N}T_{k,1}(a,b)\chi_k(b),\ \text{ with }\ \chi_k(b)=\phi^{(k)}(b),\ k=1,\dots,N.\end{aligned}$$

We can also put y_1 in the form of a scalar product defined as follows

$$y_1(a,b) = [1, T_{2,1}(a,b), \ldots, T_{N,1}(a,b)] \begin{pmatrix} \chi_1(b) \\ \chi_2(b) \\ \vdots \\ \chi_N(b) \end{pmatrix} =< \begin{pmatrix} 1 \\ T_{2,1} \\ \vdots \\ T_{N,1} \end{pmatrix}, \begin{pmatrix} \chi_1(b) \\ \chi_2(b) \\ \vdots \\ \chi_N(b) \end{pmatrix} > .$$

Similarly, applying the algorithm (3.1.1) in the case of σ is constant. The Nth-order phase transformation or the reference IF function estimate $\widetilde{\omega_x}^N$ is defined by

$$\widetilde{\omega_x}^N(a,b) = \begin{cases} \dfrac{\widetilde{\beta}(a,b)}{i2\pi \widetilde{W}_x(a,b)} - \displaystyle\sum_{k=2}^{N} T_{k,1}(b,a)\chi_k(b), & \text{if } \widetilde{W}_x(b,a) \neq 0 \text{ and } \partial_a T_{j,j-1}(b,a) \neq 0 \\ \frac{\widetilde{\beta}(a,b)}{i2\pi \widetilde{W}_x(a,b)}, & \text{if } \widetilde{W}_x(a,b) \neq 0 \end{cases}$$

3.1.2 The higher-order short time Fourier synchrosqueezing transform

Higher-order FSST

Definition 3.1.1. Given a signal $x(\tau) = A(\tau)e^{i2\pi\phi(\tau)}$ in $L^2(\mathbb{R})$ with $A(\tau)$ and $\phi(\tau)$ are equal to their Lth-order and N-order respectively, the Taylor expansion for τ close to t (see[17]):

$$\begin{aligned} A(\tau) &= e^{\log(A(\tau))} = \exp\Big(\sum_{k=0}^{L} \frac{(\log(A))^{(k)}(t)}{k!}(\tau - t)^k\Big) \\ \phi(\tau) &= \sum_{k=0}^{N} \frac{\phi^{(k)}(t)}{k!}(\tau - t)^k. \end{aligned} \tag{3.1.1}$$

The signal x is defined as above, with $L \leq N$, can be written as:

$$\begin{aligned} x(\tau) &= \exp\Big(\sum_{k=0}^{L} \frac{(\log(A))^{(k)}(t)}{k!}(\tau - t)^k\Big) \exp\Big(i2\pi \sum_{k=0}^{N} \frac{\phi^{(k)}(t)}{k!}(\tau - t)^k\Big) \\ &= \exp\Big(\sum_{k=0}^{N} \frac{1}{k!}([\log(A)]^{(k)}(t) + i2\pi\phi^{(k)}(t))(\tau - t)^k\Big). \end{aligned}$$

Since $(\log(t))^{(k)}(t) = 0$ if $L+1 \leq K \leq N$, we define the STFT for a signal x by:

$$V_x^g(t,\eta) = \int_{\mathbb{R}} x(\tau+t)g(\tau)e^{-i2\pi\eta\tau}d\tau. \tag{3.1.2}$$

Applying the derivative of STFT, we have:

$$\begin{aligned}
\partial t V_x^g(t,\eta) &= \int_{\mathbb{R}} \partial_t \Big[\exp\Big(\sum_{k=0}^{N} \frac{1}{k!}([\log(A)]^{(k)}(t) + i2\pi\phi^{(k)}(t))(\tau)^k \Big) g(\tau)e^{-i2\pi\eta\tau} \Big] d\tau \\
&= \sum_{k=0}^{N} \frac{1}{k!}\Big([\log(A)]^{(k+1)}(t) + i2\pi\phi^{(k+1)}(t) \Big) V_x^{t^{k-1}g}(t,\eta) \\
&= \sum_{k=1}^{N} \frac{1}{(k-1)!}\Big(\frac{1}{i2\pi}[\log(A)]^{(k)}(t) + \phi^{(k)}(t) \Big).i2\pi V_x^{t^{k-1}g}(t,\eta)
\end{aligned} \tag{3.1.3}$$

$$w_x(t,\eta) = \frac{\partial_t V_x^g(t,\eta)}{2i\pi V_x^g(t,\eta)} = \sum_{k=1}^{N} \frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^g(t,\eta)} r_k(t) \tag{3.1.4}$$

$$= \frac{(\log(A))'(t)}{i2\pi} + \phi'(t) + \sum_{k=2}^{N} \frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^g(t,\eta)} r_k(t) \tag{3.1.5}$$

where $r_k(t) = \frac{1}{(k-1)!}\Big(\frac{1}{i2\pi}[\log(A)]^{(k)}(t) + \phi^{(k)}(t)\Big)$, for $k = 1, \ldots, N$

$$w_x(t,\eta) - \sum_{k=2}^{N} \frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^g(t,\eta)} r_k(t) = \frac{(\log(A))'(t)}{i2\pi} + \phi'(t).$$

The Nth-order IF estimate follows:

$$Re\Big\{ w_x(t,\eta) + \sum_{k=2}^{N} r_k(t)\big(- P_{k,1}(t,\eta)\big) \Big\} = \phi'(t) \tag{3.1.6}$$

where $P_{k,1}(t,\eta) = \frac{V_x^{t^{k-1}g}(t,\tau)}{V_x^g(t,\tau)}$, for $k = 2, \ldots, N$.

Moreover, we can put equation (3.1.5) in the form

$$w_x(t,\eta) = [1, P_{2,1}(t,\eta), \ldots, P_{N,1}(t,\eta)] \begin{pmatrix} r_1(t) \\ r_2(t) \\ \vdots \\ r_N(t) \end{pmatrix} = P_N.R_N^T.$$

In the same way, we can present the previous cases to the algorithm 3.1.1 to provide the parameters $r_1, \ldots, r_N$.

Definition 3.1.2. If $s \in L^2(\mathbb{R})$, the Nth-order local complex IF estimate or phase transformation w_x^N is defined by (see [17]):

$$w_x^N(t,\eta) = \begin{cases} w_x(t,\eta) + \sum_{k=2}^{N} r_k(t)\big(-P_{k,1}(t,\eta)\big) & \text{if } V_x^g(t,\eta) \neq 0 \text{ and } \partial_\eta P_{j,j-1}(t,\eta) \neq 0 \\ w_x(t,\eta) & \text{if } V_x^g(t,\eta) \neq 0 \text{ and } \partial_\eta P_{j,j-1}(t,\eta) \neq 0. \end{cases} \tag{3.1.7}$$

Adaptive higher-order FSST

We recall the STFT of the signal $x(t)$ denoted by

$$\widetilde{V_x^g}(t,\eta) = \int_{\mathbb{R}} x(t+\tau)\frac{1}{\sigma(t)} g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau.$$

Taking the derivative by t

$$\begin{aligned} \frac{\partial}{\partial t}\widetilde{V_x^g}(t,\eta) &= \int_{\mathbb{R}} x'(t+\tau)\frac{1}{\sigma(t)} g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau \\ &+ \int_{\mathbb{R}} x(t+\tau)\Big(\frac{-\sigma'(t)}{\sigma^2(t)}\Big) g\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau \\ &+ \int_{\mathbb{R}} x(t+\tau)\Big(\frac{-\sigma'(t)}{\sigma^3(t)}\tau\Big) g'\big(\frac{\tau}{\sigma(t)}\big) e^{-i2\pi\eta\tau} d\tau. \end{aligned}$$

To simplify the calculation, we note the following expression

$$\begin{aligned}
\frac{\sigma'(t)}{\sigma(t)}\widetilde{V_x^g}(t,\eta) &= \int_{\mathbb{R}} x(t+\tau)\Big(\frac{-\sigma'(t)}{\sigma^2(t)}\Big)g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \\
\frac{-\sigma'(t)}{\sigma(t)}\widetilde{V_x}^{g^2}(t,\eta) &= \int_{\mathbb{R}} x(t+\tau)\Big(\frac{-\sigma'(t)}{\sigma^3(t)}\tau\Big)g'(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau
\end{aligned}$$

$$\frac{\partial_t \widetilde{V_x^g}(t,\eta)}{\widetilde{V_x^g}(t,\eta)} = I - \frac{\sigma'(t)}{\sigma(t)} - \frac{\sigma'(t)}{\sigma(t)}\frac{\widetilde{V_x^g}^{g^2}(t,\eta)}{\widetilde{V_x^g}(t,\eta)} \tag{3.1.8}$$

where

$$\begin{aligned}
I &= \frac{1}{\widetilde{V_x^g}(t,\eta)}\left(\int_{\mathbb{R}} x'(t+\tau)\frac{1}{\sigma(t)}g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau\right) \\
&= \frac{i2\pi}{\widetilde{V_x^g}(t,\eta)}.\sum_{k=1}^{N}\frac{1}{(k-1)!}\Big(\frac{1}{i2\pi}[\log(A)]^{(k)}(t)+\phi^{(k)}(t)\Big)\int_{\mathbb{R}} x(t+\tau)\tau^k g(\frac{\tau}{\sigma(t)})e^{-i2\pi\eta\tau}d\tau \\
&= \frac{i2\pi}{\widetilde{V_x^g}(t,\eta)}\sum_{k=1}^{N} r_k(t)\widetilde{V_x}^{t^k g}(\eta,t)
\end{aligned}$$

with $r_k(t) = \frac{1}{(k-1)!}\Big(\frac{1}{i2\pi}[\log(A)]^{(k)}(t)+\phi^{(k)}(t)\Big)$, for $k = 1,\dots,N$.

Thus, we have

$$\frac{1}{i2\pi}\Big(\frac{\partial_t \widetilde{V_x^g}(t,\eta)}{\widetilde{V_x^g}(t,\eta)} + \frac{\sigma'(t)}{\sigma(t)} + \frac{\sigma'(t)}{\sigma(t)}\frac{\widetilde{V_x}^{g^2}(t,\eta)}{\widetilde{V_x^g}(t,\eta)}\Big) = \frac{\log^{(1)}(A)(t)}{i2\pi} + \phi^{(1)}(t) + \sum_{k=2}^{N} r_k(t)\frac{\widetilde{V_x}^{t^k g}(\eta,t)}{\widetilde{V_x^g}(\eta,t)}.$$

we denote:

$$\widetilde{w}_x(t,\eta) = \frac{1}{i2\pi}\Big(\frac{\partial_t \widetilde{V_x^g}(t,\eta)}{\widetilde{V_x^g}(t,\eta)} + \frac{\sigma'(t)}{\sigma(t)} + \frac{\sigma'(t)}{\sigma(t)}\frac{\widetilde{V_x}^{g^2}(t,\eta)}{\widetilde{V_x^g}(t,\eta)}\Big).$$

Then,

$$\frac{\log(A)^{(1)}(t)}{i2\pi} + \phi^{(1)}(t) = \widetilde{w}_x(t,\eta) - \sum_{k=2}^{N} r_k(t)\frac{\widetilde{V_x}^{t^k g}(\eta,t)}{\widetilde{V_x^g}(\eta,t)}. \tag{3.1.9}$$

The IF function is obtained by

$$\phi^{(1)}(t) = Re\left\{\widetilde{w}_x(t,\eta) - \sum_{k=2}^{N} r_k(t)\frac{\widetilde{V_x}^{t^k g}(t,\eta)}{\widetilde{V_x^g}(t,\eta)}\right\}.$$

We can use the equation (3.3.2) to apply the unknowns $r_1(t),\ldots,r_N(t)$ in the form of a scalar product previously solved by the algorithm (3.1.1):

$$\widetilde{w}_x(t,\eta) = [1, S_{2,1}(t,\eta),\ldots,S_{N,1}(t,\eta)]\begin{pmatrix} r_1(t)\\ r_2(t)\\ \vdots \\ r_N(t)\end{pmatrix} = S_N.R_N^T$$

where $S_{k,1}(t,\eta) = \dfrac{\widetilde{V_x}^{t^k g}(t,\eta)}{\widetilde{V_x^g}(t,\eta)}$, for $k = 2,\ldots,N$.

Let us denote $y_1(t,\eta) = \widetilde{w}_x(t,\eta)$, $y_j(t,\eta) = \dfrac{\partial_\eta y_{j-1}(t,\eta)}{\partial\eta S_{j,j-1}(t,\eta)}$, and $S_{k,j}(t,\eta) = \dfrac{\partial\eta S_{k,j-1}(t,\eta)}{\partial\eta S_{j,j-1}(t,\eta)}$. In the same way, algorithm (3.1.1) solves the problem.

3.2 New FSST transform

A new phase transformation for the 2nd-order adaptive FSST was proposed in [26]. We consider the new second order FSST associated with STFT. For a signal $x(t)$ defined by

$$x(t) = Ae^{pt+\frac{q}{2}t^2}e^{i2\pi(ct+\frac{1}{2}rt^2)},$$

The STFT of $x(t) \in Ł_2(\mathbb{R})$ with a window function $g(t) \in L_2(\mathbb{R})$ is defined as

$$V_x(t,\eta) = \int_{\mathbb{R}} x(\tau)g(\tau - t)e^{-i2\pi\eta(\tau - t)}d\tau \tag{3.2.1}$$

where t is the time variable and η are is the frequency variable.
As in previous cases, we have

$$\partial_t V_x(t,\eta) = (p + qt + i2\pi(c + rt))V_x(t,\eta) + (q + i2\pi r)V_x^{g_1}(t,\eta).$$

Thus at (t,η) on which $V_x(t,\eta) \neq 0$

$$\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)} = \big(p + qt + i2\pi(c + rt)\big)\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)} + (q + i2\pi r).$$

Taking partial derivative ∂_η, then we have

$$\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) = \big(p + qt + i2\pi(c + rt)\big)P_0(t,\eta) \tag{3.2.2}$$

where we use $P_0(t,\eta)$ to denote

$$P_0(t,\eta) = \frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) \quad \Rightarrow \quad \big(p + qt + i2\pi(c + rt)\big) = \frac{1}{P_0(t,\eta)}\frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big).$$

Thus,

$$c + rt = -\frac{p + qt}{i2\pi} + \frac{1}{i2\pi P_0(t,\eta)}\frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big).$$

Thus for a general $x(t)$, we define a new phase transformation for the 2nd-order FSST, denoted by $\omega_x^{New,2nd}$, as

$$\omega_x^{New,2nd}(t,\eta) = \begin{cases} Re\Big\{\frac{1}{i2\pi\frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)}\frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)\Big\}. & \text{if } \frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) \neq 0,\ V_x(t,\eta) \neq 0 \\ Re\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\}, & \text{if } \frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) = 0,\ \ V_x(t,\eta) \neq 0 \end{cases}$$

3.2.1 New higher-order FSST

Definition 3.2.1. Given a signal $x(\tau) = A(\tau)e^{i2\pi\phi(\tau)}$ in $L^2(\mathbb{R})$ with $A(\tau)$ and $\phi(\tau)$ are equal to their Lth-order and N-order respectively, the Taylor expansion for τ close to t:

$$\begin{aligned} A(\tau) &= e^{\log(A(\tau))} = \exp\Big(\sum_{k=0}^{L}\frac{(\log(A))^{(k)}(t)}{k!}(\tau-t)^k\Big) \\ \phi(\tau) &= \sum_{k=0}^{N}\frac{\phi^{(k)}(t)}{k!}(\tau-t)^k, \end{aligned} \tag{3.2.1}$$

the signal x is defined as above, with $L \leq N$, can be written as:

$$\begin{aligned} x(\tau) &= \exp\Big(\sum_{k=0}^{L}\frac{(\log(A))^{(k)}(t)}{k!}(\tau-t)^k\Big)\exp\Big(i2\pi\sum_{k=0}^{N}\frac{\phi^{(k)}(t)}{k!}(\tau-t)^k\Big) \\ &= \exp\Big(\sum_{k=0}^{N}\frac{1}{k!}([\log(A)]^{(k)}(t) + i2\pi\phi^{(k)}(t))(\tau-t)^k\Big). \end{aligned}$$

Since $(\log(t))^{(k)}(t) = 0$ if $L+1 \leq K \leq N$. Next we define the STFT for a signal x by:

$$V_x^g(t,\eta) = \int_{\mathbb{R}} x(\tau+t)g(\tau)e^{-i2\pi\eta\tau}d\tau. \tag{3.2.2}$$

Applying the derivative of STFT, we have:

$$
\begin{aligned}
\partial_t V_x^g(t,\eta) &= \int_{\mathbb{R}} \partial_t \Big[\exp\left(\sum_{k=0}^{N} \Big(\frac{(\log(A))^{(k)}(t)}{k!} + i2\pi\phi^{(k)}(t) \Big) \tau^k \right) g(\tau) e^{-i2\pi\eta\tau} \Big] d\tau \\
&= \sum_{k=0}^{N} \Big(\frac{(\log(A))^{(k+1)}(t)}{k!} + i2\pi\phi^{(k+1)}(t) \Big) V_x^{t^{k-1}g}(t,\eta) \qquad (3.2.3) \\
&= \sum_{k=1}^{N} \Big(\frac{(\log(A))^{(k)}(t)}{i2\pi(k-1)!} + \phi^{(k)}(t) \Big) i2\pi V_x^{t^{k-1}g}(t,\eta)
\end{aligned}
$$

$$
\begin{aligned}
w_x(t,\eta) &= \frac{\partial_t V_x^g(t,\eta)}{2i\pi V_x^{tg}(t,\eta)} = \sum_{k=1}^{N} \frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^g(t,\eta)} r_k(t) \\
&= \Big(\frac{(\log(A))'(t)}{i2\pi} + \phi'(t) \Big) \frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)} + \frac{(\log(A))''(t)}{i2\pi} + \phi''(t) + \sum_{k=3}^{N} \frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^{tg}(t,\eta)} r_k(t)
\end{aligned}
$$

where $r_k(t) = \frac{(\log(A))^{(k)}(t)}{i2\pi(k-1)!} + \phi^{(k)}(t)$, for $k = 1, \ldots, N$.

Taking the derivative by η for the equation (3.2.4), we have:

$$
\partial_\eta w_x(t,\eta) = \Big(\frac{(\log(A))'(t)}{i2\pi} + \phi'(t) \Big) \partial_\eta \Big(\frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)} \Big) + \sum_{k=3}^{N} \partial_\eta \Big(\frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^{tg}(t,\eta)} \Big) r_k(t).
$$

Therefore, if in addition, $\partial_\eta \Big(\frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)} \Big) \neq 0$, then

$$
w_x^{New}(t,\eta) = \Big(\frac{(\log(A))'(t)}{i2\pi} + \phi'(t) \Big) + \sum_{k=3}^{N} \frac{1}{\partial_\eta \Big(\frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)} \Big)} \partial_\eta \Big(\frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^{tg}(t,\eta)} \Big) r_k(t). \qquad (3.2.4)
$$

where the function w_x^{New} is defined by

$$
w_x^{New}(t,\eta) = \frac{1}{\partial_\eta \Big(\frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)} \Big)} \partial_\eta \Big(\frac{\partial_t V_x^g(t,\eta)}{2i\pi V_x^{tg}(t,\eta)} \Big). \qquad (3.2.5)
$$

In addition, we can write equation (3.2.4) in the form

$$w_x^{New}(t,\eta) - \sum_{k=3}^{N} W_{k,1}(t,\eta) r_k(t) = \frac{(\log(A))'(t)}{i2\pi} + \phi'(t)$$

where $W_{k,1}(t,\eta) = \frac{1}{\partial_\eta\Big(\frac{V_x^g(t,\eta)}{V_x^{tg}(t,\eta)}\Big)} \partial_\eta\Big(\frac{V_x^{t^{k-1}g}(t,\eta)}{V_x^{tg}(t,\eta)}\Big)$ for $k = 3, \ldots, N$.

The new version for the Nth-order IF estimate is defined by the STFT:

$$\Re\Big\{ w_x^{New}(t,\eta) + \sum_{k=3}^{N} r_k(t)\big(- W_{k,1}(t,\eta)\big)\Big\} = \phi'(t). \tag{3.2.6}$$

Moreover, we can put the equation (3.2.4) in the form:

$$w_x^{New}(t,\eta) = [1, W_{3,1}(t,\eta), W_{4,1}(t,\eta), \ldots, W_{N,1}(t,\eta)] \begin{pmatrix} r_1(t) \\ r_3(t) \\ r_4(t) \\ \vdots \\ r_N(t) \end{pmatrix} = W_{N-1}.R_{N-1}^T.$$

In the same way the previous cases used the algorithm (3.1.1) to provide the parameters $r_1, r_3, r_4, \ldots, r_N$, we can denote $k = 3, \ldots, N$ by computing the partial derivatives:

$$y_2(t,\eta) = \frac{\partial_\eta \omega_x^{New}(t,\eta)}{\partial_\eta W_{3,1}(t,\eta)} \text{ and } W_{k,2}(t,\eta) = \frac{\partial_\eta W_{k,1}(t,\eta)}{\partial_\eta W_{3,1}(t,\eta)} \tag{3.2.7}$$

which implies the following expression

$$y_2(t,\eta) = [0, 1, W_{4,2}(t,\eta), \ldots, W_{N,2}(t,\eta)] R_{N-1}^T. \tag{3.2.8}$$

To find the j^{th} equation, we do the same process iteratively. We define the new parameter for the A_{N-1} matrix for $j = 2, \ldots, N-1$ and $k = j+1, \ldots N-1$ by:

$$y_j(t,\eta) = \frac{\partial_\eta y_{j-1}(t,\eta)}{\partial_\eta W_{j+1,j-1}(t,\eta)}, \text{ and } W_{k,j}(t,\eta) = \frac{\partial_\eta W_{k,j-1}(t,\eta)}{\partial_\eta W_{j+1,j-1}(t,\eta)}. \tag{3.2.9}$$

Then,

$$y_j(t,\eta) = [0, 0, \ldots, 1, W_{j+2,j}, \ldots, W_{N,j}] R_{N-1}^T.$$

We group the $N-1$ equations and get a good linear system:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & W_{3,1} & \ldots & \ldots & W_{N,1} \\ 0 & 1 & W_{4,2} & \ldots & W_{N,2} \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & W_{N,N-2} \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_3 \\ r_4 \\ \vdots \\ r_N \end{pmatrix} \tag{3.2.10}$$

$$Y_{N-1} = A_{N-1}.[R_{N-1}]^T \tag{3.2.11}$$

Since the A_{N-1} is an upper triangular matrix with a nonzero diagonal, the solution of the linear system is given by

$$\begin{aligned} & r_N(t) = y_{N-1}(t,\eta) \\ & r_j(t) = y_{j-1}(t,\eta) - \sum_{k=j+1}^{N} W_{k,j-2}(t,\eta) r_k(t), \text{ for } j = N-1, \ldots, 3. \end{aligned} \tag{3.2.12}$$

Definition 3.2.2. Let $x \in L^2(\mathbb{R})$. The New version for Nth-order local complex IF estimate or phase transformation $\omega_x^{N,New}$ is defined by

$$\omega_x^{N,New}(t,\eta) = \begin{cases} w_x^{New}(t,\eta) + \sum_{k=3}^{N} r_k(t)\big(-W_{k,1}(t,\eta)\big) & \text{if } \partial_\eta W_{j,j-1}(t,\eta) \neq 0 \\ w_x^{New}(t,\eta) & \text{if } \partial_\eta W_{j,j-1}(t,\eta) = 0. \end{cases}$$

Chapter 4

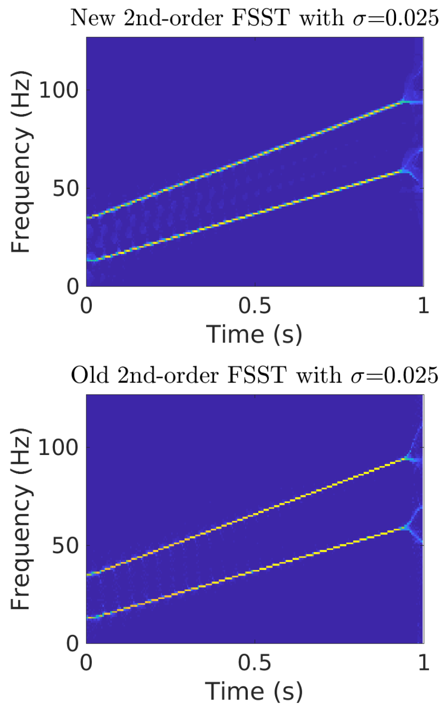
Numerical simulation

4.1 Numerical Simulation

In this section, we present some experimental results for the new second order of the phase transformation ω_x^{2nd}. Let $x(t)$ be a signal with two linear chirps:

$$x(t) = x_1(t) + x_2(t) = \cos\big(2\pi(c_1 + \frac{1}{2}b_1 t)t\big) + \cos\big(2\pi(c_2 + \frac{1}{2}b_2 t)t\big), t \in [0, 1] \tag{4.1.1}$$

where the reference frequencies are $c_1 = 12$, $c_2 = 34$, and the chip rates are $b_1 = 50$, $b_2 = 64$. Here, $x(t)$ is sampled uniformly with $N = 256$ sample points.



We proceed with two representative signal types.
Example 1: Our first signal is a signal with three components, given by $s(t) = s_1(t) + s_2(t) + s_3(t)$,
where

$$\begin{aligned}
s_1(t) &= \cos(118\pi(t-\tfrac{1}{2}) + 100\pi(t-\tfrac{1}{2})^2)1_{[\frac{1}{2},1]}. \\
s_2(t) &= \cos(94\pi t + 110\pi t^2 + 13\cos(4\pi t - \tfrac{pi}{2})). \\
s_3(t) &= \cos(194\pi t + 112\pi t.^2)1_{[0,\frac{3}{4}]}.
\end{aligned} \tag{4.1.2}$$

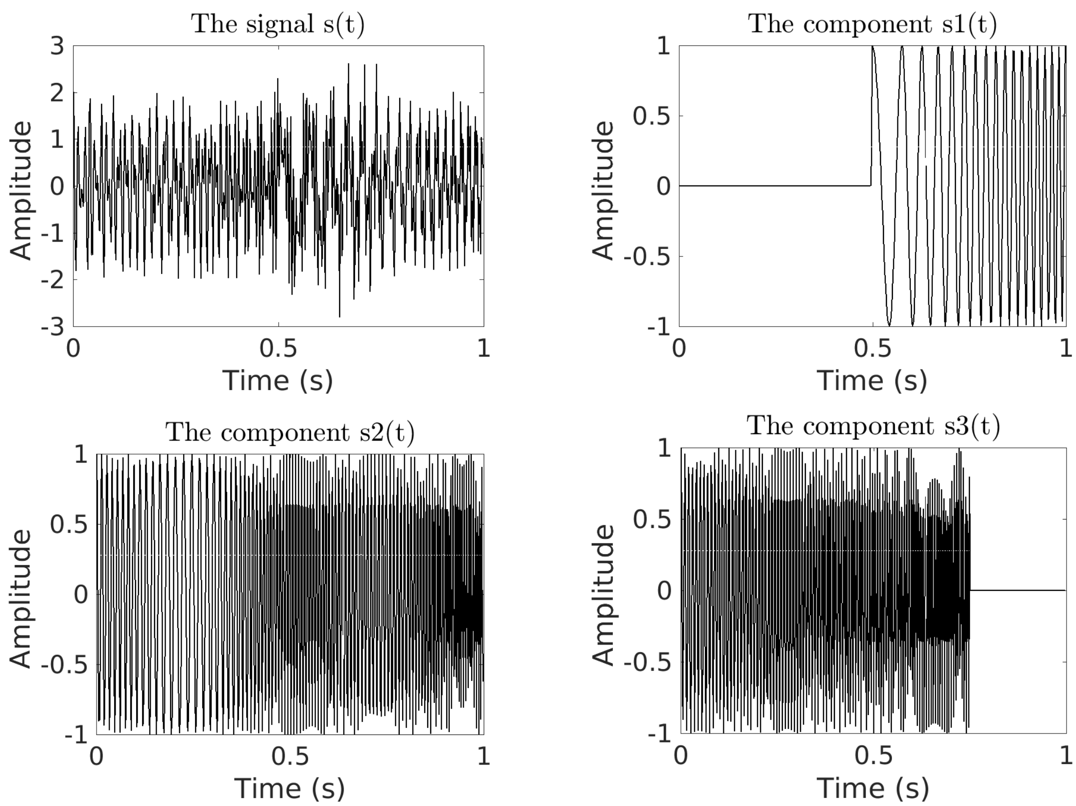


Figure 4.1: The signal $s(t)$ and its components $s_1(t)$, $s_2(t)$ and $s_3(t)$

We use the relative "root mean square error" (RMSE) to evaluate the separation performance, which is defined by

$$RMSE = \frac{1}{K}\sum_{k=1}^{K}\frac{||z_k - \hat{z}_k||_2}{||z_k||},$$

where $\hat{z}_k$ is the reconstruction result of z_k, K is the number of components.

Here, we test some parameters of σ from 0.001 to 0.1. The best value for the reconstrucation signal and its components from FSST2 can be obtained by minimizing the RMSE for both approaches of FSST2. Then, $\sigma_{Old} = \sigma_{New} = 0.023$. We colculated the following results.

σ	0.015	0.020	0.023	0.025	0.030
RMSE for Old	0.1375	0.0860	0.0798	0.0843	0.0922
RMSE for New	0.1091	0.0720	0.0717	0.0782	0.0888

Table 4.1: Some differents values of σ and their RMSE.

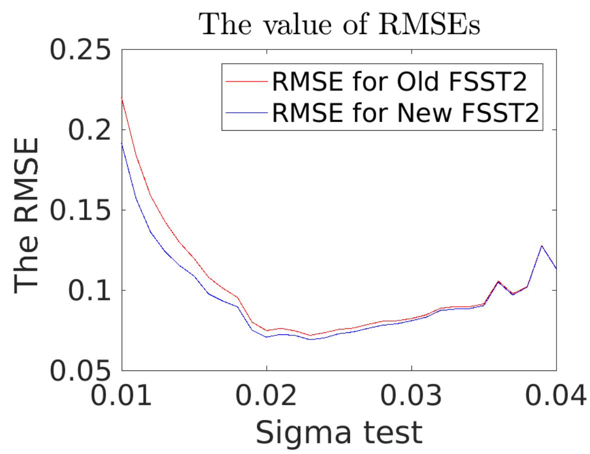


Figure 4.2: RMSE for FSST2 Old and New with $\sigma \in [0.001, 0.04]$

As we can see, the best value for minimizing the value of RMSE is $\sigma \approx 0.023$. This value shows the difference between the original IFs and the reconstrucation using the FSST2 old and FSST2 new.

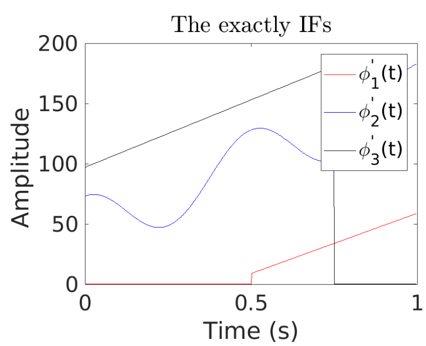


Figure 4.3: The original IF of the signal s(t)

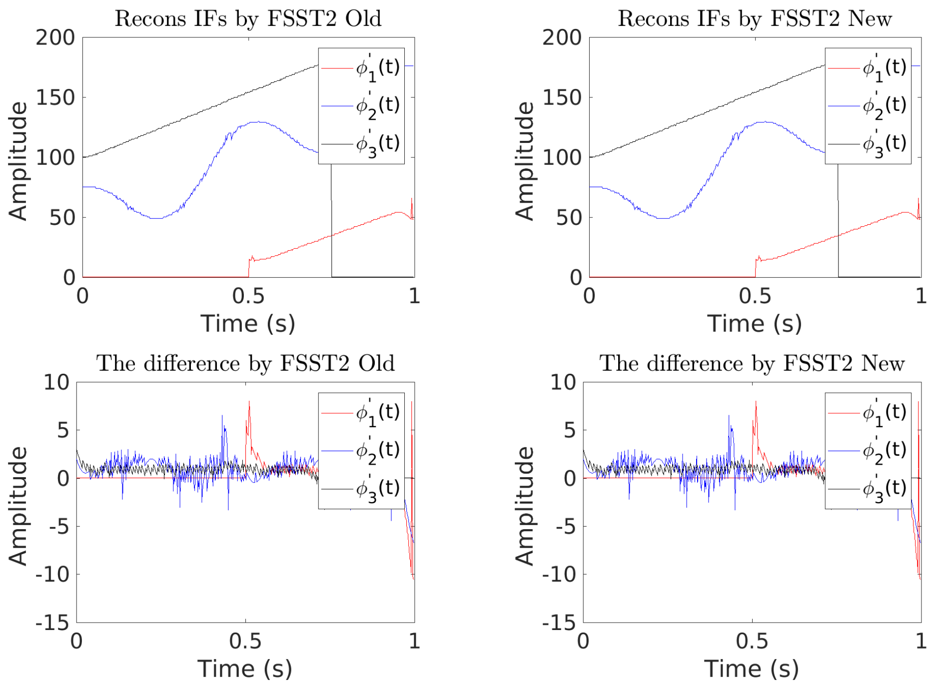


Figure 4.4: Difference of reconstructed IFs with original IFs by old 2nd-order and new 2nd-order FSST.

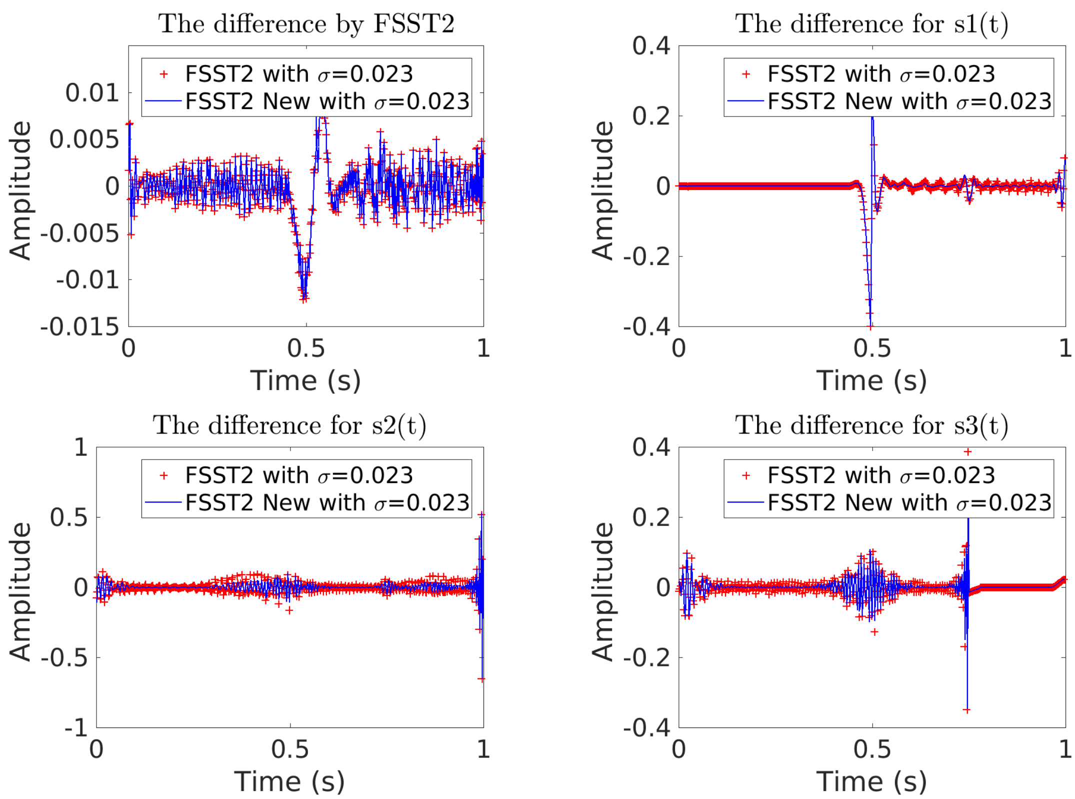


Figure 4.5: Difference for the reconstructed s_1, s_2, s_3 with original component $s_1(t), s_2(t), s_3(t)$ by old 2nd-order and new 2nd-order FSST.

Example 2: The second signal is a signal with two components, given by the signal $s(t) = s_1(t) + s_2(t)$, defined by

$$s_1(t) = \cos(2\pi(12t + 25t^2)) \quad \text{and} \quad s_2(t) = \cos(2\pi(34t + 32t^2)). \qquad (4.1.3)$$

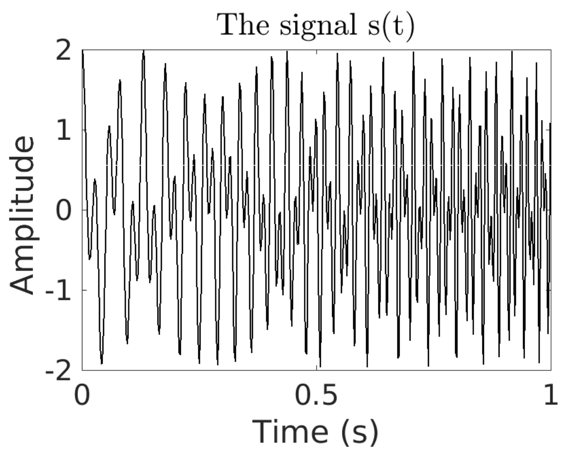


Figure 4.6: The signal $s(t)$

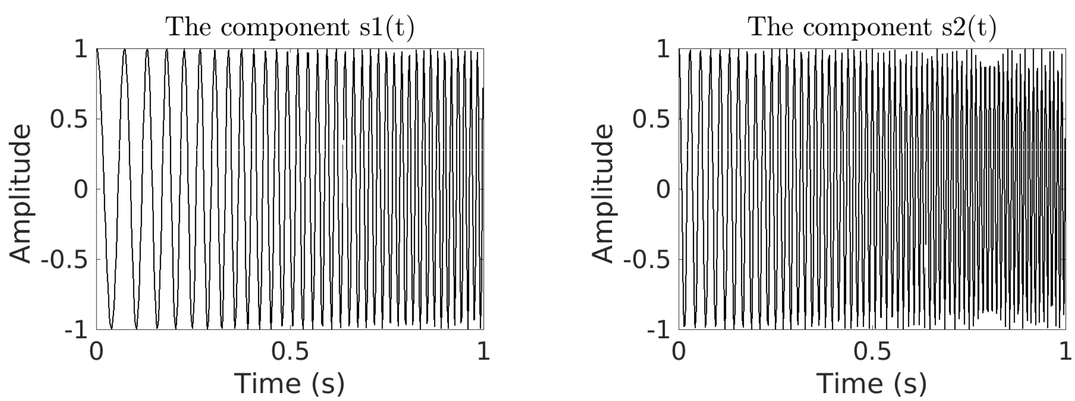


Figure 4.7: The components of the signal $s(t)$ one by one $s_1(t)$(left), and $s_2(t)$(right)

Here, we test some parameters of σ from 0.001 to 0.1. The best value for the reconstrucation signal and its components from FSST2 can be obtained by minimizing the RMSE for both approaches of FSST2. Then, $\sigma_{Old} = \sigma_{New} \approx 0.05$. We colculated the following results.

σ	0.04	0.042	0.045	0.047	0.050
RMSE for Old	0.0893	0.0897	0.0802	0.0827	0.0824
RMSE for New	0.0893	0.0893	0.0800	0.0823	0.0822

Table 4.2: Some Examples for differents parameter of σ.

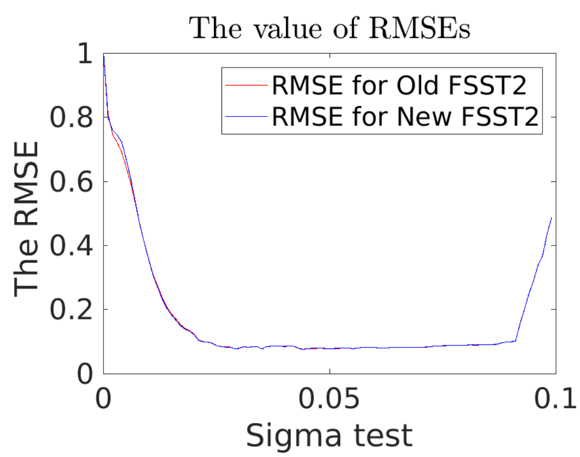


Figure 4.8: RMSE for Old and New FSST2 with $\sigma \in [0.001, 0.1]$

As we can see, the best value for minimizing the value of RMSE is $\sigma \approx 0.045$. This value demonstrates the difference between the original IFs and the reconstrucations using the FSST2 old and FSST2 new.

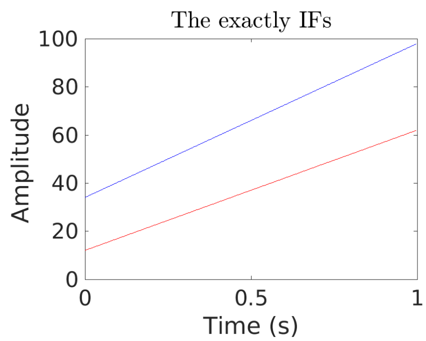


Figure 4.9: The original IF of the signal s(t)

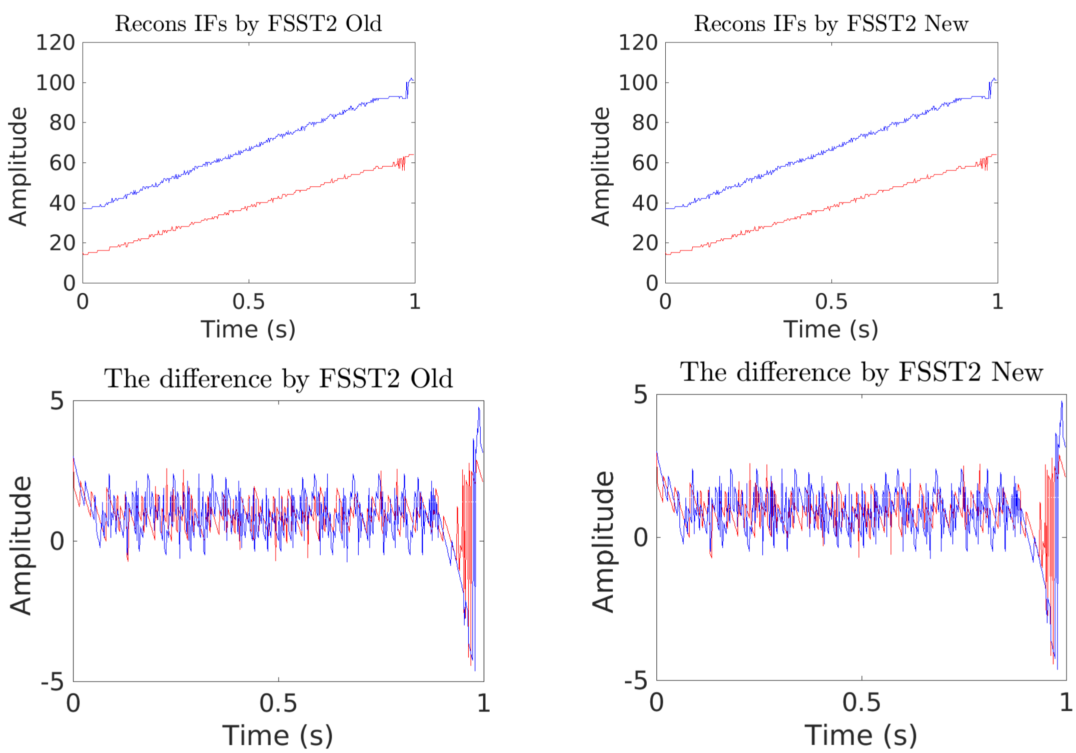


Figure 4.10: Difference of reconstructed IFs with original IFs by old 2nd-order and new 2nd-order FSST.

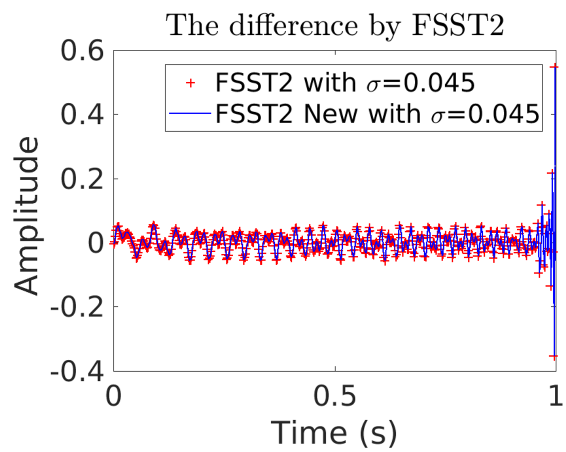


Figure 4.11: The difference for the signal $s(t)$

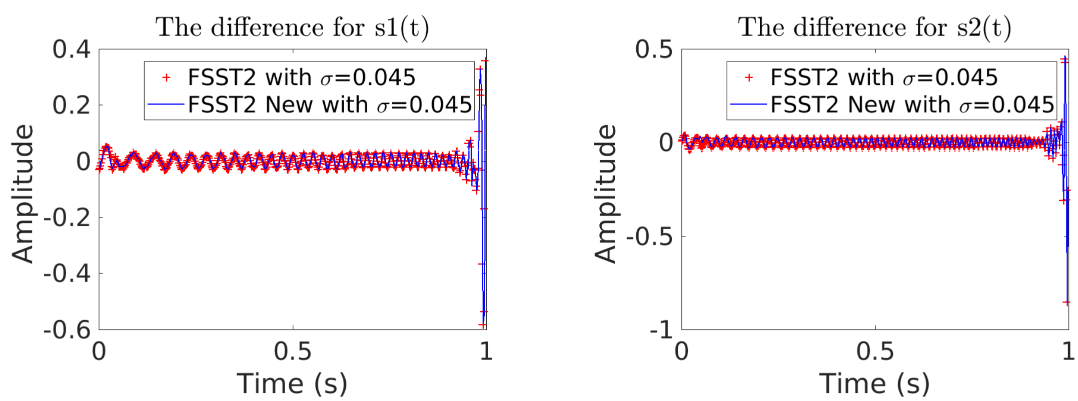


Figure 4.12: Difference for the reconstructed $s_1(t), s_2(t)$ with original component $s_1(t), s_2(t)$ by old 2nd-order and new 2nd-order FSST.

Chapter 5

Analysis of Adaptive Shor-time Fourier Transform-based Synchrosqueezing Transform

5.1 Analysis for new approach of FSST2

The Fourier transform of $e^{i\pi\phi_k''(t)\tau^2}g(\tau)$, which we denote by $G_k(\xi)$, is defined by (refer to [19]):

$$\begin{aligned} G_k(\xi) &= \mathcal{F}\Big(e^{i\pi)\phi_k''(t)\tau^2}g(\tau)\Big)(\xi) \\ &= \int_{\mathbb{R}} e^{i\pi\phi_k''(t)\tau^2}g(\tau)e^{-i2\pi\xi\tau}d\tau \end{aligned}$$

$$\begin{aligned} x_k(t+\tau) &= x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)} + \big(A_k(t+\tau)-A_k(t)\big)e^{i2\pi\phi_k(t+\tau)} \\ &+ x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)}\Big(e^{i2\pi\Big(\phi_k(t+\tau)-\phi_k(t)-\phi_k'(t)\tau-\frac{1}{2}\phi_k''(t)\tau^2\Big)}-1\Big). \end{aligned}$$

Then, we have

$$\begin{aligned} V_x(t,\eta) &= \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t+\tau)g(\tau)e^{-i2\pi\eta\tau}d\tau \\ &= \sum_{k=1}^{K}\int_{\mathbb{R}} x_k(t)e^{i2\pi(\phi_k'(t)\tau+\frac{1}{2}\phi_k''(t)\tau^2)}g(\tau)e^{-i2\pi\eta\tau}d\tau + res_0. \end{aligned}$$

where

$$
\begin{aligned}
res_0 &= \textstyle\sum_{k=1}^{K} \int_{\mathbb{R}} \{ \big(A_k(t+\tau) - A_k(t)\big) e^{i2\pi\phi_k(t+\tau)} \\
&+ x_k(t) e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)} \big(e^{i2\pi\left(\phi_k(t+\tau) - \phi_k(t) - \phi_k'(t)\tau - \frac{1}{2}\phi_k''(t)\tau^2\right)} - 1\big)\} g(\tau) e^{-i2\pi\eta\tau} d\tau.
\end{aligned} \tag{5.1.1}
$$

$$
|res_0| \leq \prod_0(t), \tag{5.1.2}
$$

where

$$
\prod_0(t) = K\varepsilon_1 I_1 + \frac{\pi}{3}\varepsilon_3 I_3 \sum_{k=1}^{K} A_k(t).
$$

We introduce more notations defined as follows

$$
\begin{aligned}
G_{j,k}(t,\eta) &= \int_{\mathbb{R}} e^{i2\pi(\phi_k'(t)\tau + \frac{1}{2}\phi_k''(t)\tau^2)} \tau^j g(\tau) e^{i2\pi\eta\tau} d\tau. \\
&= \mathcal{F}\Big(e^{i\pi(\phi_k''(t)\tau^2)} \tau^j g(\tau)\Big)\big(\eta - \phi_k'(t)\big).
\end{aligned}
$$

We also note

$$
\begin{gathered}
G_k\big(\eta - \phi_k'(t)\big) = G_{0,k}(t,\eta) \\
G_{j,k} = \frac{1}{(-i2\pi)^j} G_k^{(j)}\big(\eta - \phi_k'(t)\big).
\end{gathered}
$$

Lemma 5.1.1. *[19] Let $x(t) = A(t)e^{i2\pi\phi(t)} \in L^2(\mathbb{R})$, we have*

$$
\partial_t V_x(t,\eta) = i2\pi\phi_k'(t) V_x(t,\eta) + i2\pi\phi_k''(t) V_x^{g1}(t,\eta) + Res_1 \tag{5.1.3}
$$

where

$$
Res_1 = i2\pi B_k(t,\eta) + i2\pi D_k(t,\eta) + i2\pi(\eta - \phi_k'(t)) res_0 - res_0' - i2\pi\phi_k'(t) res_1.
$$

Proof. $V_x^{g'}$ is defined by (see [19]):

$$V_x^{g'} = i2\pi \sum_{\ell=1}^{K} x_\ell(t)\big(\eta - \phi'_\ell(t)\big)G_{0,\ell}(t,\eta) - i2\pi \sum_{\ell=1}^{K} x_\ell(t)\phi''_\ell(t)G_{1,\ell}(t,\eta) + res'_0.$$

Taking the derivatve by the time variable t, we obtain

$$\partial_t V_x(t,\eta) = i2\pi\eta V_x(t,\eta) - V_x^{g'}(t,\eta). \tag{5.1.4}$$

Using the derivative of the time variable (5.1.4), we have

$$\begin{aligned}
& \partial_t V_x(t,\eta) - i2\pi\phi'_k(t)V_x(t,\eta) - i2\pi\phi''_k(t)V_x^{g1}(t,\eta) \\
= \;& i2\pi\big(\eta - \phi'_k(t)\big)V_x(t,\eta) - V_x^{g'}(t,\eta) - i2\pi\phi''_k(t)V_x^{g1}(t,\eta) \\
= \;& i2\pi\big(\eta - \phi'_k(t)\big)V_x(t,\eta) - i2\pi \sum_{\ell=1}^{K} x_\ell(t)\big(\eta - \phi'_\ell(t)\big)G_{0,\ell}(t,\eta) - res'_0 \\
+ \;& i2\pi \sum_{\ell=1}^{K} x_\ell(t)\phi''_\ell(t)G_{1,\ell}(t,\eta) - i2\pi\phi''_k(t)V_x^{g1}(t,\eta).
\end{aligned}$$

In addition, we define the expression of $V_x(t,\eta)$ and $V_x^{g1}(t,\eta)$ in the form

$$\begin{aligned}
V_x(t,\eta) &= \sum_{\ell=1}^{K} x_\ell(t)G_{0,\ell}(t,\eta) + res_0 \\
V_x^{g1}(t,\eta) &= \sum_{\ell=1}^{K} x_\ell(t)G_{1,\ell}(t,\eta) + res_1.
\end{aligned}$$

Then, we have

$$
\begin{aligned}
& \partial_t V_x(t,\eta) - i2\pi\phi_k'(t)V_x(t,\eta) - i2\pi\phi_k''(t)V_x^{g1}(t,\eta) \\
= \;& i2\pi\big(\eta-\phi_k'(t)\big)\Bigg(\sum_{\ell=1}^{K} x_\ell(t)G_{0,\ell}(t,\eta)+res_0\Bigg) - i2\pi\sum_{\ell=1}^{K} x_\ell(t)\big(\eta-\phi_\ell'(t)\big)G_{0,\ell}(t,\eta) - res_0' \\
+ \;& i2\pi\sum_{\ell=1}^{K} x_\ell(t)\phi_\ell''(t)G_{1,\ell}(t,\eta) - i2\pi\phi_k''(t)\Bigg(\sum_{\ell=1}^{K} x_\ell(t)G_{1,\ell}(t,\eta)+res_1\Bigg) \\
= \;& i2\pi\sum_{\ell\neq k} x_\ell(t)\big(\phi_\ell'(t)-\phi_k'(t)\big)G_{0,\ell}(t,\eta) + i2\pi\sum_{\ell\neq k} x_\ell(t)\big(\phi_\ell''(t)-\phi_k''(t)\big)G_{1,\ell}(t,\eta) \\
+ \;& i2\pi\big(\eta-\phi_k'(t)\big)res_0 - res_0' - i2\pi\phi_k''(t)res_1 \\
= \;& i2\pi B_k(t,\eta) + i2\pi D_k(t,\eta) + i2\pi\big(\eta-\phi_k'(t)\big)res_0 - res_0' - i2\pi\phi_k''(t)res_1 \\
= \;& Res_1
\end{aligned}
$$

where

$$
\begin{aligned}
B_k(t,\eta) &= \sum_{\ell\neq k} x_\ell(t)\big(\phi_\ell'(t)-\phi_k'(t)\big)G_{0,\ell}(t,\eta) \\
D_k(t,\eta) &= \sum_{\ell\neq k} x_\ell(t)\big(\phi_\ell''(t)-\phi_k''(t)\big)G_{1,\ell}(t,\eta).
\end{aligned}
$$

$\square$

We recall the definition of the new phase transformation for the 2nd-order STFT, which $V_x(t,\eta)\neq 0$.:

$$
\omega_x^{new,2nd}(t,\eta) = \begin{cases} Re\Bigg\{\dfrac{1}{i2\pi\frac{\partial}{\partial\eta}\Big(\frac{V_x(t,\eta)}{V_x^{g1}(t,\eta)}\Big)}\frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g1}(t,\eta)}\Big)\Bigg\} & \text{if } \frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g1}(t,\eta)}\Big)\neq 0,\ V_x^{g_1}(t,\eta)\neq 0 \\[2ex] Re\Big\{\frac{\partial_t V_x(t,\eta)}{i2\pi V_x(t,\eta)}\Big\}, & \text{if } \frac{\partial}{\partial\eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g1}(t,\eta)}\Big) = 0,\ \ V_x^{g_1}(t,\eta)\neq 0. \end{cases}
$$

Then, the new second order complex IF is defined as follows

$$
\omega_x^{New,2nd,c}(t,\eta) = \frac{P_1(t,\eta)}{i2\pi Q_1(t,\eta)}
$$

where P_1 and Q_1 are defined by

$$P_1(t,\eta) = \frac{\partial}{\partial \eta}\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) \text{ and } Q_1(t,\eta) = \frac{\partial}{\partial \eta}\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big). \tag{5.1.5}$$

Lemma 5.1.2. *For (t,η) such that $V_x^{g_1}(t,\eta) \neq 0$ and $P_1(t,\eta) \neq 0$, as defined in the equation (5.1.5), we have*

$$P_1(t,\eta) = i2\pi\phi_k'(t)Q_1(t,\eta) + Res_3 \tag{5.1.6}$$

and we note that $Res_2 = \partial_\eta Res_1 (see[19])$, where

$$Res_3 = \frac{Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta) Res_1}{\big(V_x^{g_1}(t,\eta)\big)^2}.$$

Proof. Using the result of lemma (5.1.1), we have

$$\partial_t V_x(t,\eta) = i2\pi\phi_k'(t)V_x(t,\eta) + i2\pi\phi_k''(t)V_x^{g1}(t,\eta) + Res_1. \tag{5.1.7}$$

Therefore, in the equation (5.1.7), therefore, thus

$$\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)} = i2\pi\phi_k'(t)\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)} + i2\pi\phi_k''(t) + \frac{Res_1}{V_x^{g_1}(t,\eta)}. \tag{5.1.8}$$

Taking the derivative by η, then yields

$$\partial_\eta\Big(\frac{\partial_t V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) = i2\pi\phi_k'(t)\partial_\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big) + \partial_\eta\Big(\frac{Res_1}{V_x^{g_1}(t,\eta)}\Big). \tag{5.1.9}$$

We replace in the equation (5.1.9) with the expression of the P_1 and Q_1:

$$P_1(t,\eta) = i2\pi\phi_k'(t)Q_1(t,\eta) + Res_3. \tag{5.1.10}$$

This completes the proof of lemma (5.1.2).

$\square$

Theorem 5.1.3. *Let $x(t) \in D_{\varepsilon_1,\varepsilon_2}$ for small $\varepsilon_1, \varepsilon_2 > 0$. We have the following results:*

(a) Suppose ε_1 satisfies $\varepsilon_1 \geq \Pi_0(t) + \tau_0 \sum_{k=1}^{K} c_k A_k(t)$, and for (t,η) with $|V_x^{g_1}(t,\eta)| > \varepsilon_1$. Then, there exists $k \in \{1,\ldots,K\}$ such that $(t,\eta) \in O_k$

(b) If (t,η) such that $|V_x(t,\eta)| > \varepsilon_1$, $\left|\partial_\eta\left(\frac{V_x(t,\eta)}{V_x^{g1}(t,\eta)}\right)\right| > \varepsilon_2$ and $(t,\eta) \in O_k$, then, we have

$$\omega_x^{New,2nd,c}(t,\eta) - \phi_k'(t) = Res_4$$

where

$$Res_4 \quad = \quad \frac{Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta)Res_1}{i2\pi\Big(\partial_\eta V_x(t,\eta)V_x^{g1}(t,\eta) - \partial_\eta V_x^{g1}(t,\eta)V_x(t,\eta)\Big)}.$$

In addition, we have

$$|\omega^{New,2nd}(t,\eta) - \phi_k'(t)| < Bd_1$$

where

$$Bd_1 = \max_{1\leq k\leq K}\sup_{\eta\in O_k}\left\{\frac{1}{2\pi\varepsilon_1^2\varepsilon_2}\Big(|Res_2||V_x^{g_1}(t,\eta)| + |Res_1||\partial_\eta V_x^{g_1}(t,\eta)|\Big)\right\}$$

(c) If ε_1 satisfies the condition in part (a) and $Bd_1 \le \frac{1}{2}L_k(t)$, then

$$L_k(t) = \min\Big\{\alpha_k + \alpha_{k-1}, \alpha_k + \alpha_{k+1}\Big\}.$$

Then for any $\varepsilon_3 = \varepsilon_3(t) > 0$ satisfying $Bd_1 \le \varepsilon_3 \le \dfrac{1}{2}L_k(t)$,

$$\Big|\lim_{\lambda\to 0}\frac{1}{g(0)}\int_{|\xi-\phi_k'(t)|<\varepsilon_3} R^{2nd,\lambda}_{x,\varepsilon_1,\varepsilon_2}(t,\xi)d\xi - x_k(t)\Big| \le Bd_2,$$

where $Bd_2 = Bd_2' + Bd_2''$ with

$$\begin{aligned} Bd_2' &= \frac{1}{|g(0)|}\Big\{2\alpha_k(\prod_0(t)+\varepsilon_1) + A_k(t)|\int_{|u|\ge\alpha_k} G_k(u)du| + \sum_{l\ne k} A_l(t)M_{l,k}(t)\Big\}, \\ Bd_2'' &= \frac{1}{|g(0)|}\Big\{2\prod_0(t) + A_k(t)\parallel g\parallel_1 |Z_t| + \sum_{l\ne k} A_l(t)M_{l,k}(t)\Big\} \end{aligned}$$

and with $|Z_t|$ represents the Lebesgue measure of the set Z_t :

$$Z_t := \Big\{\eta : (t,\eta)\in O_k, |V_x^{g_1}(t,\eta)| > \varepsilon_1, \Big|\partial_\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)\Big| \le \varepsilon_2\Big\}.$$

Proof Part (a). Assume $(t,\eta) \notin \cup_{k=1}^K O_k$. Then for any k, by the definition of O_k in (2.6.6) with $\sigma = 1$, we have $|G_k(\eta - \phi_k'(t))| \le \tau_0$. Hence, by (5.1.1) and (5.1.2), we have

$$|G_{1,k}(\eta - \phi_k'(t))| \le c_k|G_k(\eta - \phi_k'(t))| \tag{5.1.11}$$

where c_k is a polynomial of $(|\phi_{k-1}(t) - \phi_k(t)| + \alpha_k)$, therefore, thus

$$\begin{aligned} |V_x^{g_1}(t,\eta)| &\le \sum_{k=1}^K \big|x_k(t)G_{1,k}(\eta - \phi_k'(t))\big| + |res_0| \\ &\le \tau_0\sum_{k=1}^K A_k(t)c_k + \Pi_0(t) \le \varepsilon_1, \end{aligned}$$

which contradiction the assumption $|V_x^{g_1}(t,\eta)| > \varepsilon_1$. Therefore, (a) holds.

Proof Part (b). Using the result of Lemma (5.1.2),

$$P_1(t,\eta) = i2\pi\phi_k'(t)Q_1(t,\eta) + Res_3. \tag{5.1.12}$$

Then we have

$$\omega_x^{New,2nd,c}(t,\eta) - \phi_k'(t) = Res_4$$

where

$$\begin{aligned} Res_4 &= \frac{Res_3}{i2\pi Q_1(t,\eta)}. \\ &= \frac{\left(V_x^{g1}(t,\eta)\right)^2 Res_3}{i2\pi\Big(\partial_\eta V_x(t,\eta)V_x^{g1}(t,\eta) - \partial_\eta V_x^{g1}(t,\eta)V_x(t,\eta)\Big)}. \end{aligned}$$

However, the formula of Res_3 is defined in the lemma (5.1.2), and we obtain

$$\left(V_x^{g_1}(t,\eta)\right)^2 Res_3 = Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta)Res_1.$$

Then,

$$Res_4 = \frac{Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta)Res_1}{i2\pi\Big(\partial_\eta V_x(t,\eta)V_x^{g1}(t,\eta) - \partial_\eta V_x^{g1}(t,\eta)V_x(t,\eta)\Big)}.$$

Next, we have

$$
\begin{aligned}
|Res_4| &= \Big|\frac{Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta) Res_1}{V_x^{g_1}(t,\eta)^2}\Big|\Big|\frac{V_x^{g_1}(t,\eta)^2}{i2\pi\Big(\partial_\eta V_x(t,\eta)V_x^{g1}(t,\eta) - \partial_\eta V_x^{g1}(t,\eta)V_x(t,\eta)\Big)}\Big| \\
&\leq \frac{1}{2\pi\varepsilon_2}\Big|\frac{Res_2 V_x^{g_1}(t,\eta) - \partial_\eta V_x^{g_1}(t,\eta) Res_1}{V_x^{g_1}(t,\eta)^2}\Big| \\
&\leq \frac{1}{2\pi\varepsilon_2}\Big(\frac{|Res_2|}{|V_x^{g_1}(t,\eta)|} + \Big|\frac{\partial_\eta V_x^{g_1}(t,\eta)}{V_x^{g_1}(t,\eta)^2}\Big||Res_1|\Big) \\
&\leq \frac{1}{2\pi\varepsilon_1^2\varepsilon_2}\Big(|Res_2||V_x^{g_1}(t,\eta)| + |Res_1||\partial_\eta V_x^{g_1}(t,\eta)|\Big)
\end{aligned}
$$

where

$$
Bd_1 = \max_{1\leq k\leq K}\sup_{\eta\in O_k}\Big\{\frac{1}{2\pi\varepsilon_1^2\varepsilon_2}\big(|Res_2||V_x^{g_1}(t,\eta)| + |Res_1||\partial_\eta V_x^{g_1}(t,\eta)|\big)\Big\}.
$$

Proof Part (c). First, we have the following result from [7] on p.254

$$
\lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\varepsilon_3} R^{2nd,\lambda}_{x,\varepsilon_1,\varepsilon_2}(t,\xi)d\xi = \int_{X_t} V_x(t,\eta)d\eta \qquad (5.1.13)
$$

where

$$
X_t := \Big\{\eta : \Big|V_x^{g_1}(t,\eta)\Big| > \varepsilon_1, \Big|\partial_\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)\Big| > \varepsilon_2 \text{ and } |\omega^{2nd}(t,\eta) - \phi_k'(t)| < \varepsilon_3\Big\}.
$$

Denote

$$
Y_t := \Big\{\eta :\ |V_x^{g_1}(t,\eta)| > \varepsilon_1, \Big|\partial_\eta\Big(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\Big)\Big| > \varepsilon_2 \text{ and } (t,\eta)\in\ O_k\Big\}.
$$

According to theorem (2.6.4) part(b), if $\eta\in Y_t$, then

$$
|\omega^{2nd}(t,\eta) - \phi_k'(t)| < Bd_1 \leq \varepsilon_3.
$$

Thus, $\eta \in X_t$. This which implies the first inclusion $Y_t \subset X_t$.

Using theorem (2.6.4) part(a), if $\eta \in X_t$,

then $|V_x^{g_1}(t,\eta)| > \varepsilon_1$ and there exists $l \in \{1, 2, \ldots, K\}$ such that $(t, \eta) \in O_l$. If $l \neq k$, then

$$|\omega_{x,\varepsilon_1,\varepsilon_2}^{New,2nd}(t,\eta) - \phi_k'(t)| \geq |\phi_k'(t) - \phi_l'(t)| - |\phi_l'(t) - \omega_{x,\varepsilon_1,\varepsilon_2}^{New,2nd}(t,\eta)|.$$

We use the following inequalities

$$|\phi_k'(t) - \phi_l'(t)| > L_k \ \text{ and } \ |\phi_l'(t) - \omega_{x,\varepsilon_1,\varepsilon_2}^{New,2nd}(t,\eta)| < Bd_1 \leq \varepsilon_3.$$

Therefore,

$$\left|\omega_{x,\varepsilon_1,\varepsilon_2}^{New,2nd}(t,\eta) - \phi_k'(t)\right| > L_k(t) - \varepsilon_3 \geq \varepsilon_3,$$

and contradicts the assumption $\eta \in X_t$.

Therefore, $l = k$ and $\eta \in Y_t$, implying the second inclusion $X_t = Y_t$. We recall

$$Z_t := \left\{\eta : (t,\eta) \in O_k, \left|V_x^{g_1}(t,\eta)\right| > \varepsilon_1, \left|\partial_\eta\left(\frac{V_x(t,\eta)}{V_x^{g_1}(t,\eta)}\right)\right| \leq \varepsilon_2\right\},$$

The fact that $X_t = Y_t$ and $Y_t \cap Z_t = \phi$, with the equation (5.1.13), imply that

$$\begin{aligned}
\int_{Y_t} V_x(t,\eta)d\eta &= \lim_{\lambda\to 0}\int_{|\xi-\phi_k'(t)|<\varepsilon_3} R^{2nd}_{x,\varepsilon_1,\varepsilon_2}(t,\eta)d\xi. \\
&= \int_{Y_t\cup Z_t} V_x(t,\eta)d\eta - \int_{Z_t} V_x(t,\eta)d\eta \qquad (5.1.14) \\
&= \int_{\{|V_x^{g_1}(t,\eta)|>\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - \int_{Z_t} V_x(t,\eta)d\eta
\end{aligned}$$

Using this equalition

$$\begin{aligned}
\int_{\{|V_x^{g_1}(t,\eta)|>\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta &= \int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta \\
&- \int_{\{|V_x^{g_1}(t,\eta)|\le\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta
\end{aligned}$$

we have

$$\begin{aligned}
\Big|\int_{\{|V_x^{g_1}(t,\eta)|>\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta) \ d\eta - g(0)x_k(t)\Big| \\
= \Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g(0)x_k(t) \\
- \int_{\{|V_x^{g_1}(t,\eta)|\le\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta\Big| \\
\le \underbrace{\Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g(0)x_k(t)\Big|}_{Term_1} \\
+ \underbrace{\Big|\int_{\{|V_x^{g_1}(t,\eta)|\le\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta\Big|}_{Term_2}
\end{aligned}$$

We recall that $O_k = \Big\{(t,\eta): \ \Big|\eta-\phi_k'(t)\Big| < \alpha_k, \ \ t\in\mathbb{R}\Big\}$
For the first term, we get the following results:

$$\begin{aligned}
Term_1 &= \Big|\int_{\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - g(0)x_k(t)\Big| \\
&\leq \Big|\int_{\{\eta:(t,\eta)\in O_k\}} \Big(\sum_{l=1}^{k} x_l(t)G_{0,l}(t,\eta) + res_0\Big)d\eta - g(0)x_k(t)\Big| \\
&\leq \underbrace{\int_{\{\eta:(t,\eta)\in O_k\}} \Big|res_0\Big|d\eta}_{T_1} + \underbrace{\Big|x_k(t)\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,k}(t,\eta) - g(0)\Big|}_{T_2} \\
&+ \underbrace{\sum_{l\neq k} x_l(t)\Big|\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,l}(t,\eta)d\eta\Big|}_{T_3}
\end{aligned}$$

$$T_1 = \int_{\{\eta:(t,\eta)\in O_k\}} \Big|res_0\Big|d\eta = |res_0|\int_{\{\eta:(t,\eta)\in O_k\}} 1d\eta = 2\alpha_k|res_0|$$

$$\begin{aligned}
T_2 &= \Big|x_k(t)\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,k}(t,\eta) - g(0)\Big| \\
&= \Big|x_k(t)\int_{\{\eta:(t,\eta)\in O_k\}} G_k(\eta - \phi_k'(t)) - g(0)\Big|
\end{aligned}$$

Using change of variable $u = \eta - \phi_k'(t)$,

$$\begin{aligned}
T_2 &= |x_k(t)\int_{|u|<\alpha_k} G_k(u)) - g(0)x_k(t)| \\
&= |x_k(t)g(0) - g(0)x_k(t) - x_k(t)\int_{|u|\geq\alpha_k} G_k(u)du)\ | \\
&= |x_k(t)|\Big|\int_{|u|\geq\alpha_k} G_k(u)du\Big| = A_k(t)\Big|\int_{|u|\geq\alpha_k} G_k(u)du\Big|
\end{aligned}$$

$$\begin{aligned}
T_3 &= \sum_{l\neq k} |\ x_l(t)\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,l}(t,\eta)d\eta\ | \\
&= \sum_{l\neq k} |\ x_l(t)\ |\ \Big|\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,l}(t,\eta)d\eta\Big| = \sum_{l\neq k} A_l(t)M_{l,k}(t)
\end{aligned}$$

substituting the parameters estimated above we get

$$Term_1 \leq 2|res_0|\alpha_k + A_k(t)\Big|\int_{|u|\geq\alpha_k} G_k(u)du\Big| + \sum_{l\neq k} A_l(t)M_{l,k}(t).$$

$$Term_2 \;\; \leq \;\; \varepsilon_0 \int_{(t,\eta)\in O_k} 1 d\eta = \varepsilon_0 \int_{\phi_k'(t)-\alpha_k}^{\phi_k'(t)+\alpha_k} 1dt \;\leq\; 2\varepsilon_0\alpha_k.$$

We group all parameter estimates, we find the following results

$$\begin{aligned} | \int_{\{|V_x^{g_1}(t,\eta)|>\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} & V_x \quad (t,\eta)d\eta - g(0)x_k(t) \; | \\ & \leq \quad 2(|res_0| + \varepsilon_0)\alpha_k \; + A_k(t)\Big|\int_{|u|\geq\alpha_k} G_k(u)du\Big| + \sum_{l\neq k} A_l(t)M_{l,k}(t) \end{aligned}$$

$$| \frac{1}{g(0)} \int_{\{|V_x^{g_1}(t,\eta)|>\varepsilon_1\}\cap\{\eta:(t,\eta)\in O_k\}} V_x(t,\eta)d\eta - x_k(t) \; |\leq Bd_2'$$

$$\int_{\mathbb{R}} G_k(u)du = \int_{\mathbb{R}} \mathcal{F}(e^{i2\pi\phi_k''(t)\tau^2} g(\tau))(u)du = g(0).$$

Hence, we have

$$\begin{aligned} |\int_{Z_t} V_x(t,\eta)d\eta| \;\; &= \;\; \Big|\int_{Z_t} \Big(\sum_{\ell=1}^{K} x_\ell(t)G_{0,\ell}(t,\eta) + res_0\Big)d\eta\Big| \\ &\leq \;\; |res_0|2\alpha_k + A_k(t)\sup_{\eta\in Z_t}\big|G_k(\eta-\phi_k'(t))\big||Z_t| \\ &+ \;\; \sum_{\ell\neq k} A_\ell(t)\Big|\int_{\{\eta:(t,\eta)\in O_k\}} G_{0,\ell}(t,\eta)d\eta\Big| \\ &\leq \;\; 2|res_0|\alpha_k + A_k(t)||g||_1|Z_t| + \sum_{\ell\neq k} A_\ell(t)M_{\ell,k}(t) \leq Bd_2'' \end{aligned}$$

Then we have the result

$$\Big| \lim_{\lambda \to 0} \frac{1}{g(0)} \int_{|\xi - \phi_k'(t)| < \varepsilon_3} R_{x,\varepsilon_1,\varepsilon_2}^{adp,2nd,\lambda}(t,\xi) d\xi - x_k(t) \Big| \leq B d_2,$$

This completes proof of Theorem. $\square$

Chapter 6

Conclusion and future work

In this study, we introduced a generalization of the STFT-based SST (FSST) with time-varying, CWT-based SST (WSST), the WSST with time-varying, and the FSST with a new phase transformation by using higher order amplitude and phase approximations. This generalization allows us to better assess a wide variety of multicomponent signals containing very strongly modulated AM-FM modes.

We also studyed the theoretical analysis of the 2nd-order FSST with a new phase transformation. The new phase transformation is much simpler than the convectional one. The new FSST performance in IF estimation and component recovery is comparable with that of the conventional 2nd-order FSST. In some cses, the new FSST perfermed even better than its conventional counterpart.

Since the result showed a better concentration and reconstruction for a wider variety of AM-FM modes, we will continue working on the adaptive FSST with a new phase transformation by using higher order approximations both for the amplitude and phase.

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