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The Effect of Using Virtual Manipulatives on Students' Ability to Mentally Compare Proper Fractions

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Abstract

This study proposes a method to analyze the effects of the use of virtual fraction models (circle/bar/number line) on students' ability to mentally compare proper fractions. Since developing a sense of magnitude with both whole numbers and rational numbers is highly correlated with improved performance on standardized assessments and improved performance in later algebra classes, special attention is directed to the bar and number line as they are linear representations.

The study used an experimental pretest/posttest group design by randomly assigning subjects within class sections to a control group (physical fraction circles) and treatment groups with seven different methods of comparing fractions (virtual fraction circle, virtual bar model, virtual number line, and all combinations). The pretest and posttest instruments identifying student reasoning in fraction comparison used in the study were developed by the Education Development Center's Eliciting Mathematics Misconceptions Project. The instruments were designed to gauge students' dependence on whole number reasoning, the unit fraction, and gap reasoning (the difference between the numerator and denominator)

The use of the virtual fraction circle should determine whether a technology bias is inherent in the study, while the bar model and the number line model show a more linear view of the fractions. A t-test showed significant improvement in the overall sample, but analysis of variance by manipulative on the differences between pretest and posttest scores and the differences between a pre/post classification on a scale of student comparison method showed no significant differences between the manipulatives.

Dedication

I would like to dedicate this to my family. My parents, Bill and Peggy McNeary, have supported me in so many ways throughout my life and educational career. Thank you for your enduring love and affection. Likewise, my wife, Charlotte, has been my rock throughout our marriage. She complains that I have been in school the whole time we have been married, but she has been there every step of the way. To my three children, Wilson, Mari and Maggie, each of you serves as an inspiration to me every day of my life. Your spirit, work, dedication, and expectations challenge me to be better.

Acknowledgements

A great many people have supported me in this endeavor. I would like to recognize those who were the most instrumental. To Dr. Tamela Randolph, my mentor and my friend, this would never have happened without you. Dr. Keith Miller has been there literally from day one of this doctoral journey. He was part of my interview committee for the program, taught my first class, and served as my advisor. I took a class from Dr. Natalie Bolton in my second semester and her encouragement, support and friendship have been vital to getting me through this journey. She sponsored all of my internships and allowed me to participate in her joint projects with Maryville University and the elementary mathematics teachers in the St. Louis school district. While I never took a class from Dr. Amber Candela, she offered me a unique opportunity to observe her class on a regular basis during the semester when she was recognized as an outstanding faculty member as UMSL. Finally, Dr. Helene Sherman was the last to join my committee, but her experience, encouragement and thoroughness helped make me better and helped make this thesis a much better document. Thank you all from the bottom of my heart.

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Definition of Terms

Area Model – A fraction model that uses the two dimensional area of a geometric shape to designate the unit which is then subdivided to indicate fractional parts

Benchmarking – A method of fraction comparison where the two fractions of interest are compared to a third fraction of known size such as ½ . In attempting to compare 2/5 and ¾, understanding that 2/5 is less than ½ and ¾ is greater than ½ allows you to state that ¾ must be greater than 2/5.

Biologically primary/secondary – In the Privileged Domain Theory of numbers, the central principles that serve as the basis for understanding numbers are counting and one to one correspondence. The fact that infants recognize the relative size of sets of objects makes the counting numbers primary and relegates other numbers that are derived operationally from counting numbers such as integers and rationals to secondary status.

Discrete Model – A model that uses a quantity of separate items such as counters to define the unit. If four counters make up the unit, then two counters would represent ½.

Gap Reasoning – The idea that the difference between the numerators and denominators of two fractions defines their relative size. When using gap reasoning, ¾ and 2/3 would be equivalent since the difference between each numerator and denominator is 1.

Linear Model – A category of models that uses the length of a segment to define the unit. This is generally different from a number line in that the linear model has a finite length associated with the unit and is not mapped to the set of real numbers.

Part/Whole Model – A model that represents a unit divided into equal parts.

Partitioning – The act of dividing a whole into parts; equi-partitioning would result in equal parts.

Residual thinking – A method of fraction comparison that involves understanding the relative distance from one. This is more complete than gap reasoning because it involves understanding that the gap represents a fractional piece. In comparing 2/3 and ¾, residual thinking deduces that they are both one "away" from the unit, but the 2/3 is 1/3 of a unit away while the ¾ is ¼ of a unit away. Since ¼ is smaller than 1/3 then ¾ must be closer to one so it is the larger fraction.

The Effect of Using Virtual Manipulatives on Students' Ability to Mentally Compare Proper Fractions

Chapter 1

Introduction

"Why should we pay the same amount for a third of a pound of meat as we do for a quarter pound of meat at McDonald's? You're overcharging us." (Taubman, 2009, p. 62) So said the focus groups organized by A&W after their campaign to sell a larger, better-tasting burger at the same price as McDonald's quarter-pounder failed to gain traction during the early 1980s. The potential customers assumed that the fraction of meat with the larger denominator was the larger portion demonstrating one of the more significant misconceptions regarding the comparison of fractions.

"The teaching and learning of fractions is not only very hard, it is, in the broader scheme of things, a dismal failure" (Davis et al., 1993, p. 1) Fractions are often introduced using an area model (Simon et al., 2018) with a pizza or pie or cookie for context. This can limit students to always seeing fractions as less than one, and the lack of a connection between the model and the number line deemphasizes the fact that the fraction represents a real number with a location (and magnitude). For students trained to operate with numerals, the numerator and the denominator appear to be separate numbers which must be analyzed accordingly. This separation of the numbers in a fraction is further accentuated by a focus on parts and wholes when constructing fractions. This leads students to add and subtract numerators and denominators rather than finding common denominators when performing fraction arithmetic. (Siegler, et al., 2010)

While ancient Egyptians and Babylonians left evidence of the use of fractions, Flemish mathematician Simon Stevin was among the first to propose the existence of a continuous magnitude of number in his work, *Arithme'tique*, in 1585 (Malet, 2006). This means that the conception of rational numbers as "numbers" is only a little older than Calculus, developed in the mid-1600s by Liebnitz and Newton.

Using a measurement model in mathematics also dates back to the ancient Egyptians, but "the earliest recorded instance in a US textbook of the words 'number line' paired with an infinite line marked with both integers and rational number representations occurs in Merrill' Modern Algebra" (McNeary, 2012, p. 4) published in 1962. Standards documents such as the Common Core State Standards now include a standard for locating a fraction on a number line at around the fifth-grade level, but most schools still introduce fractions at earlier grade levels fairly exclusively using part/whole fraction models (circles and bars).

The sequence of the introduction of number systems in school mathematics has followed the historical "discoveries" of these systems. Counting numbers (1,2,3,…) come first, followed by whole numbers (0 and the counting numbers). Next are the positive rationals (fractions) followed by integers (whole numbers and their opposites). Negative rationals and irrationals (numbers that cannot be represented as fractions) complete the real number system. When analyzing number systems based on the concept of closure (arithmetic operations in a set result in a number in the set), integers result from the lack of closure in the whole numbers on the operation of subtraction while rationals arise due to the lack of closure on the operation of division.

Math Education Since 1950

According to Woodward (2004), mathematics education in the US over the last half of the twentieth century was divided into three time periods that all carried implications specifically for the teaching of rational numbers.

1) The 1950s and 1960s – The New Math
2) The 1970s and 1980s – Back to Basics
3) The 1990s – Excellence in Education

The New Math phase resulted from developments in the Cold War and is often tied directly to the launch of Sputnik and the Space Race of the 1960s. The federal government diverted extensive funding for research and training in mathematics and the development of new curricula. This resulted in more focus on discovery and understanding and a move away from the three decades of focus on connectionist theory advocated by Thorndike (Woodward, 2004) and the more recent advent of Skinner's operant conditioning (Woodward, 2004). "Behaviorism placed a premium on the efficient development of bonds through rote practice and memorization" (Woodward, 2004, p. 6). The material stressed topics "such as set theory, operations, and place value through different base systems … and alternative algorithms for division and operations on fractions" (Woodward, 2004, p. 5). This period also saw a rise in the influence of Piaget's theories of child development and Bruner's work in educational psychology (Woodward, 2004).

The New Math era failed to deliver on expectations and the resulting backlash led to the Back to Basics movement of the 1970s (Woodward, 2004). Part of the problem with New Math was that some teachers were not prepared for discovery learning or higher order mathematical concepts traditionally taught in secondary curriculum or higher education and had a tendency to guide students through the learning process in a very structured way. (Woodward, 2004). As well, scores on national standardized tests geared to more procedural types of questions rather than developmental, did not improve. (Kena, 2016) The Back to Basics movement returned to an emphasis on rote memorization of math facts and procedural competence as opposed to understanding.

When a researcher in the 1970s advocating the removal of fractions from the curriculum, "[s]ince both the metric system and the hand-held calculator use decimals, in twenty-five years common fractions will be as obsolete as Roman numerals are today" (Usiskin, 1979, p. 1), Usiskin (1979) argued strenuously against this idea by pointing out that every use of division results in the use of a fraction, and the use of fractions is pervasive in algebraic expressions and equations where the calculator has no particular advantage. He categorized the uses of fractions beyond measurement as Splitting up (dividing a portion equally), Rate (any comparison of units begins as a fraction), Proportion (an equality of two fractions), Formulas (many important formulas incorporate fractions, such as the area of a triangle, A=1/2 base*height), and Sentence-solving (use of division to solve equations such as 7x=1).

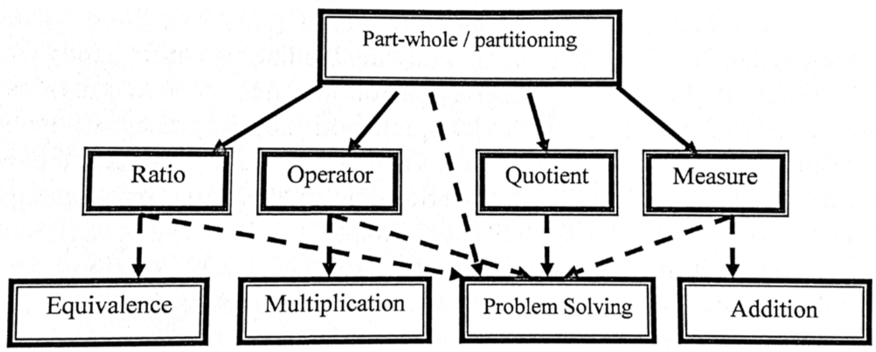
According to Woodward (2004), Project Follow Through, one of the largest federally funded quantitative studies of early education conducted between 1967 and 1977, was used as justification for the efficacy of the formulaic direct/active instruction model which breaks curricular units into lessons that start with a brief review followed by the "development portion of the lesson (20 minutes), independent seatwork (15 minutes), and a homework assignment." (Woodward, 2004) The tide began to turn again in the late 1970's and early 1980's as cognitive science gained influence as a new framework for educational research. “By the 1980’s, problem solving had become a central theme in mathematics education, … and [b]y the mid-1980’s, cognitive research was the dominant framework in mathematics education. Cognitive scientists attempted to articulate the fundamental role of visual imagery as a representational form of memory.” (Woodward, 2004) By the end of the decade, cognitive researchers, influenced by information processing theory, were including constructivist theory in their work. Broader educational policy initiatives from the 1980’s reignited many of the reform ideas of the “New Math” era as part of the Excellence in Education movement of the 1990’s. According to Woodward (2004), one of the primary drawbacks in this era was that researchers focused on basic skills continued to hold sway in the areas of special education and LD (learning disabilities).

A Theoretical Model of Fractions

Throughout the various periods of education, the primary methods for introducing fractions have not changed significantly. (Simon et al., 2018) They include the use of set (discrete) models, area models, linear models, and number line models and the teaching of procedural competence. (Kieren, 1976) Circle models and part/whole modeling have long dominated rational number instruction even after Kieren (1976) introduced the idea of interrelated subconstructs for rational numbers beyond the idea of part/whole - ratio, operator, quotient and measure. In his conceptualization, the subconstructs worked together to demonstrate the part/whole construct which he expressly avoided identifying as a fifth subconstruct. Later work by Behr et al. (1983) extended the subconstructs to include part/whole as one of the five areas of fraction conceptualization- part/whole, ratio, operator, quotient and measure. Behr et al. (1983) developed a theoretical model tying the five constructs to the basic operations of fractions, fraction equivalence and problem-solving (see Figure 1). Fundamental to the idea was that "[e]quivalence and partitioning are constructive mechanisms operating across the … subconstructs to extend images and build mathematical ideas." (Behr M. et al., 1983, p. 3) Later research by Hannula (2003) added decimal as a possible sixth construct.

Figure 1

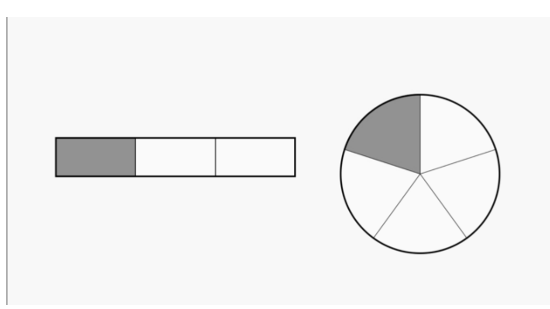
The theoretical model linking the five subconstructs of fractions to the different operations of fractions and to problem solving (Behr M. et al., 1983)



Charalambus & Pitta-Pantazi (2007) describe the five subconstructs in detail. In the part/whole subconstruct, the fraction represents a comparison between the number of parts selected and the whole unit where the unit is partitioned into equal parts (Figure 2).

Figure 2

Partitioning in fraction models



To master the part/whole subconstruct, students must grasp the partitioning of the whole into equal parts. This can be demonstrated through the partitioning of a discrete set into equal size groups or partitioning a continuous length or area into equal parts. Several ancillary ideas contribute to complete understanding such as- all of the parts taken together exhaust the whole; the more parts, the smaller the part; the relationship between the parts and the whole is conserved over size, shape, and arrangement of equivalent parts. A full understanding of the part/whole subconstruct depends on the student's ability to unitize and reunitize. Charalambos (2007) describes this as a reconstruction of the whole based on its parts and repartitioning already equipartitioned wholes (construct 3/8 from a whole partitioned into fourths). Area models are the most commonly used methods for teaching about part/whole relationships. (Tunc-Pekkan, 2015)

The ratio subconstruct relates the notion of a comparison of two numbers by the operation of division, but this relationship extends beyond just part/whole. In that regard, it is considered a "comparative index rather than a number" (Carraher, 1996, p. 245). Ratios can compare separate quantities or measures of different units (which are more specifically identified as rates). Students demonstrate a full understanding of ratios when they construct the idea of relative amounts and grasp the notion of the covariance between the quantities as well as the invariance of the relationship (multiplication of the ratio by a whole number retains the value of the ratio). Since covariance/invariance is a property of ratios, it becomes a distinguishing factor between the understanding of ratio and part/whole interpretations.

The operator subconstruct regards rational numbers as a scale factor or as pair of functions to be applied to some other number, object, or set. Mastering the operator subconstruct requires students to interpret the fractional multiplier in several ways. Three fourths can be seen either 3 x [one fourth of a unit] (dubbed stretcher/shrinker context by Behr et al. (1993)) or one fourth x [3 units] (duplicator/partition-reducer). Charalambos (2007) also describes how students should be able to name a single fraction to perform a composite operation and relate outputs to inputs. This subconstruct requires students to move beyond understanding multiplication as repeated addition and see it as a scaling operation. Multiplication can result in larger products when the factors are whole numbers OR smaller products when one of the factors is a proper fraction.

The quotient subconstruct requires students to see the fraction as the result of a division. The fraction represents the numerical value that is obtained by the division. The quotient subconstruct (and division in general) is often introduced under the idea of "fair share"- I have three pizzas to share among four friends, how much pizza does each friend get? Like the ratio subconstruct, the quotient subconstruct potentially deals with different units within the subconstruct (pizzas vs friends) as opposed to equal parts of a whole. In mastering the concept, students need to understand the two types of division- partitive (dividing a quantity into shares resulting in the size of each share) and quotitive (dividing a quantity by the size of each share to determine the number of shares)- and "the role of the dividend and the divisor" (Charalambos, 2007, p. 106). The dividend refers to the number of parts in each share, and the divisor names the fraction of each share.

The pizza example cited above demonstrates partitive division. The three pizzas are divided into fourths and each person gets three shares. In partitive division, the result is the amount each person receives. Quotitive division results in the number of equal shares- three pizzas are to be shared among some friends, if each friend gets three fourths of a pizza, how many friends are there? The pizza model context does lend weight to the use of a circle model, but division can also be demonstrated with rectangular area models and linear models.

Fractional Scheme Theory

Tunc-Pekkan (2015) combined the work of Charalambos & Pitta-Pantazi (2007), (Kieren, 1976), and Steffe (2001) to propose Fractional Scheme Theory where schemes are defined as "goal-directed activities that consist of three parts: an assimilated situation, an activity, and a result." (Tunc-Pekkan, 2015, p. 422) Fraction Scheme Theory consists of the following-

1. Part/Whole subconstruct
    a. Parts within wholes fraction scheme- Only partitioning is observed. Students partition wholes, but not necessarily equally.
    b. Part-whole fraction scheme- Partitioning and disembedding (seeing a fraction of the whole as related to the whole) are observed. Students partition wholes equally and recognize fractional parts in the context of the whole.
2. Beyond part/whole subconstuct (and leading to the measurement subconstruct)
    a. Partitive unit fraction scheme- Partitioning, disembedding, and iterating (replicating the unit fraction to the whole to ensure the unit is correct) are observed. Students can also take a unit fraction and iterate to find the whole.

b. Partitive fractional scheme- Partitioning, disembedding, and iterating are observed. Given a whole, students can find a proper fraction by partitioning to the unit and iterating to the desired fraction.

c. Iterative fractional scheme- Splitting (a combination of partitioning and iterating) and disembedding are observed. Students can find an improper fraction based on the whole by partitioning and iterating or find the whole by splitting an improper fraction into the appropriate unit fraction (based on the numerator rather than the denominator) and iterating to the whole.

This study focuses on the measure subconstruct which is addressed to some degree by the last three schemes in Fractional Scheme Theory. As with whole numbers, each fraction has a place on the number line that represents its magnitude, but it also represents the length or space over which a unit fraction defined by the denominator can be iterated to its numerator. It is a subtle difference, but the place is absolute and tied to a distance from zero, while the space is relative and can start from anywhere. The fraction ¾ corresponds to a distance of 3 (one fourth-units) which can be measured from 0, stopping at the place, three fourths or it can be measured from any other marker, like 1 and stop at the place three fourths of a unit past the starting point (in this case 1 ¾). The number line is the primary tool used for learning about the magnitude of fractions, but students struggle with partitioning and the fact that fractions do not follow the counting sequence. Overcoming this struggle provides the opportunity to consider the density of rational numbers which implies that between any two fractions lies an infinite number of fractions. (Charalambos, 2007) Students also demonstrate difficulty with the number line through the counting of marks as opposed to partitions. That particular issue is not confined to fractions as they often include 0 in the natural number counting sequence. Understanding magnitude can be closely tied to understanding order and equivalence in rational numbers.

In considering the different contexts of fractions, selecting one model as better than any other is shortsighted. The strengths of the various models apply to the different interpretations of fractions in different ways and so where one may be better in terms of a particular construct, the "best" method is the use of multiple models to work with students to understand multiple representations of rational numbers. If the models do not provide a particular differentiation for the overall learning of fractions, then one might consider how the models are used to try and find a better way of teaching fractions.

Curricular Issues

Gearhar, et al. (1999) studied the difference in the use of a problem-solving curriculum and a skills-based curriculum while providing professional development support in both scenarios. They found that professional development was especially critical to the implementation of the problem-solving curriculum. This finding supports the ideas that led to the implementation of New Math as a response to the "Back to Basics" by trying to develop a deeper understanding before attempting to apply procedural routines to operations. The finding also points to some of the reasons for a lack of success due to insufficient professional development. (Gearhart, et al., 1999)

Cramer et al. (2002) also studied the use of contrasting curricula by using reform material from the Rational Number Project (RNP) and comparing it to commercially available curricula. Interestingly, the RNP curriculum was built in a way that minimized the need for professional development during its implementation. The researchers saw significant gains for the students in classrooms using the RNP material which is primarily built around unit circles and sets. The commercially available curricula provided little modeling for students in the control group and were almost entirely focused on procedural fluency. The key element in the RNP is not the specific fraction model, but the use of multiple representations in the introduction of fractions and the transfer between the representations to address a variety of subconstructs of fractions. (Cramer et al., 2002)

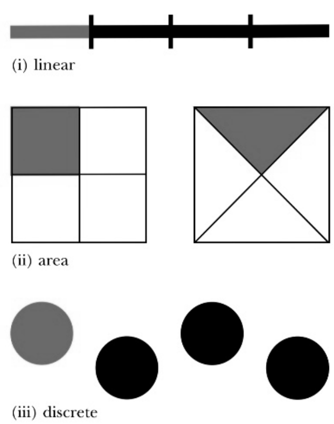
Bailey et al. (2015) argue that procedural fluency aids the development of fraction concepts which then in turn aids in the development of procedural fluency. In the researchers' attempt to resolve the dilemma as to which comes first, they studied the development of fraction concepts in US and Chinese children. Bailey et al. (2015) determined that the development of conceptual knowledge of fraction magnitude contributed to procedural fluency in fraction addition which then resulted in a better conceptual understanding of fraction addition.

For particular models, Jigyel & Afamasaga-Fuata'I (2007) studied the performance of year 4, 5 and 6 students in Australia on tests of fractions and equivalence. The predominant model used in these classrooms was the unit circle. Unlike the results of the RNP, they found disappointing results on the equivalence tests for these students. Their struggles with fraction equivalence suggest that it may not be just the model that determines success.

On the other hand, Gould (2013) suggests moving away from the area and discrete models in Australia and toward a linear model, see Figure 3. He contends that the students do not have a well-defined understanding of area, and the fractions that they create come from a counting perspective rather than a true understanding of the fractional area that is displayed. Often they lose sight of the fact that the pieces of an area must be partitioned equally to properly model a fraction. The use of a linear model (not a number line) can focus them on the need to partition equally based on units of length rather than units of area.

Figure 3

Linear, Area, and Discrete Models (Gould, 2013)



Mills (2011) devised a different approach by using "body fractions" to introduce fractional concepts. Despite the loss of precision in comparing body parts, the idea of introducing kinesthetic activity to learn is a good one. In essence, each student can represent a unit from fingertip to fingertip. That means that one arm represents one half and the length from fingertip to elbow is one fourth. Students can then stand together to represent the same fractions in different ways or display different fractions.

Integrated Theory of Number Development

Students develop conclusions early in elementary school around whole numbers that often do not hold for the real number system. These include operational perceptions such as addition/multiplication make larger, subtraction/division make smaller; language-based ideas perpetuated by the teacher like “ you can’t take away a larger number from a smaller number”; and student-created understanding that teachers fail to correct.

These understandings, especially around subtraction and multiplication are influenced by teachers’ operational understandings. In an anecdotal survey (McNeary, 2012), three teachers- one primary, one middle school, and one high school- responded to the question, “What is subtraction?” with three different answers – take away, counting backward, and a difference. Each one demonstrated a larger and more inclusive understanding of the operation that often escapes students because they most often understand subtraction as take away.

Teachers limit students’ understanding of operations like subtraction by teaching it only as take away and multiplication by focusing on repeated addition. (Devlin, 2008) Teaching multiplication as repeated addition inhibits the understanding of proportionality and scaling that is essential to multiplicative reasoning. (Devlin, Devlin's Angle, 2011) The lack of understanding of the scaling nature of multiplication also impedes the understanding of fraction operations.

This can lead to the idea that rational numbers are completely different from whole numbers. As such, some researchers (and many students) treat the transition from whole numbers to rational numbers as less of a transition and more as the development of a completely different understanding of numbers (Gelman & Williams, 1998; Geary, 2006; Vosniadou et al., 2008).

In contrast, Siegler et al. (2011) proposed an integrated theory of numerical development in which they consider the transition from natural numbers to rational numbers by emphasizing the properties and concepts that carry forward.

> This theory proposes that numerical development is at its core a process of progressively broadening the class of numbers that are understood to possess magnitudes and of learning the functions that connect that increasingly broad and varied set of numbers to their magnitudes. In other words, numerical development involves coming to understand that all real numbers have magnitudes that can be ordered and assigned specific locations on number lines. … (T)he central conceptual structure for whole numbers, a mental number line, is eventually extended to other types of numbers, including rational numbers.
> (Siegler et al., 2011, p. 274)

The comparison (and addition) of numbers provides an example where common concepts between number systems could be emphasized. If one person has five apples and another three oranges and you want to compare (or add) the quantities, you have to understand that they are all pieces of fruit, a common unit. In that scenario, you see that the person with the apples has two more pieces of fruit than the person with the oranges or they have eight pieces of fruit together. Using a number line and the idea of a common unit when working with whole numbers can lay the groundwork for an easier transition to understanding rational numbers and the need to have a common denominator.

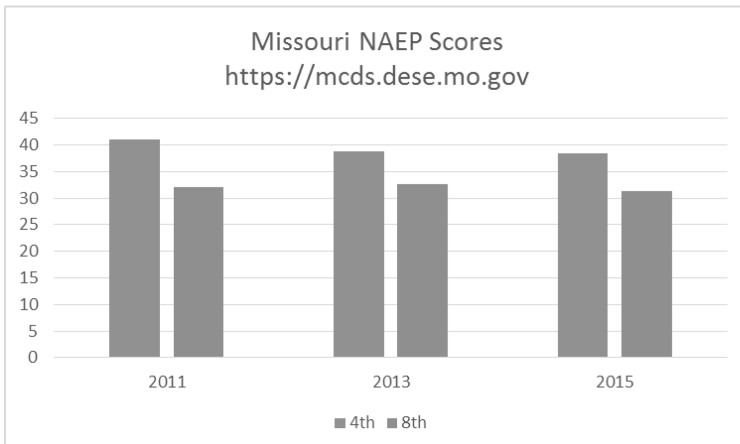
As an algebra teacher, the researcher observed that students' failure to understand the basic nature of numbers including properties of equality, operations, and identity; the need for common units in addition and subtraction; and the difference in context for numbers in multiplication and division inhibits their success. They see two numbers, an operation, and an equal sign as the signal to perform a rote calculation without understanding the context or relationship of the numbers to each other and to the answer of their calculation. For the four basic arithmetic operations, the relationships of the two numbers involved in the operation are critical to the performance of the operation. For addition and subtraction, the two numbers must have identical units; for multiplication, one of the numbers is a scale factor (the multiplier) and the other is a unit-based number (the multiplicand). Because division can be defined as the inverse of multiplication, the operation leads to two scenarios- dividing a unit based number by a scale factor to obtain a unit based answer or dividing a unit based number by a unit based number resulting in a scale factor. Consider a cookie sharing example. If a teacher has 12 cookies to share among 3 students, then each student gets 4 cookies, an example of a partitive or sharing division. In contrast, if a teacher has 12 cookies and wants to share 4 cookies each with a group of students, 3 students would receive cookies, which demonstrates quotitive or measurement division. The basic understanding of the need for common units to add, subtract and perform one form of division, while one of the numbers in a multiplication operation is a scalar can serve as a bridge to understanding the need for finding common denominators when adding and subtracting fractions, but not when multiplying them. It turns out that the second division scenario (unit by unit) also lends itself to using common denominators, but that is not taught as much as the "invert and multiply" method of dividing fractions. While the Common Core State Standards address the acquisition of whole number operations knowledge mostly by fifth grade, the acquisition of knowledge regarding fractions begins to ramp up in fourth grade with a heavier emphasis on operations in fifth grade.

Test Scores in the State of Missouri

As seen in Figures 4 and 5, student test scores show a significant decrease in percent Proficient/Advanced between the fourth and eighth grade in the state of Missouri on both nationally administered and state-administered standardized tests. (Missouri DESE - NAEP, 2019)

Figure 4

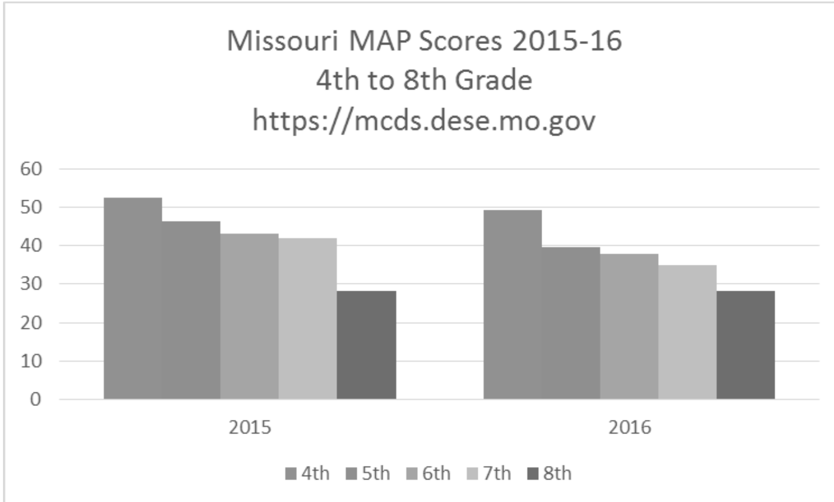
Missouri NAEP Scores, 2011/2013/2015



The decrease between fourth and eighth grade shows up consistently across most state and national tests. While part of the decrease is attributable to the introduction of Algebra in sixth and seventh grade, Figure 5 shows the first large decrease happens between fourth and fifth grade- the year that most students begin to work with rational numbers. (Missouri DESE - NAEP, 2019)

Figure 5

Missouri MAP Scores, 2015-2016



Purpose of the Study

The purpose of this study is to describe the effect of the use of various manipulatives on the changes in performance (and classification of same) on an identified test instrument for students at a midwestern regional university. The treatments specifically address the comparison of two fractions by displaying area models and/or relative positions on the number line. They are intended to help students overcome common perceptions in fraction comparison such as whole number reasoning, reliance on the unit fraction, and *gap reasoning*. In gap reasoning, students understand the gap between the numerator and the denominator of a fraction to be the determining factor in the relative size of the fractions. (Fagan et al., 2016) Many students think if the gap is the same, the fractions are equal, otherwise, the fraction with the greater gap is the smaller amount. For instance, a student operating under this idea would state that ½ is equal to 2/3 since the difference between the numerator and the denominator in each case is one.

Much of the previous research on rational number development, and gap reasoning in particular, tends to explain the difficulty in understanding fractions as whole number or natural number bias. This research posits that understanding rational numbers requires a different framework than the one used for understanding whole numbers (Gelman & Williams, 1998; Geary, 2006; Vosniadou et al., 2008). Other research looks at proposed solutions, several focused specifically on the use of technology (Bulut et al., 2014; Fazio et al., 2016; Neshar, 1987; Olive et al, 2010). However, none of the research emphasizes the conversion between early fraction models and number line placement. This study will address that specific deficiency concerning the particular classifications of understanding known as whole number reasoning, reliance on the unit fraction, and gap reasoning in fraction comparison by applying a specific treatment that converts fraction models to comparative locations on a number line and attempting to determine if the treatments improve student performance on proper fraction comparison activities.

Research Questions, Hypotheses, and Objectives

This study is framed with the research question "How does the use of multiple virtual fraction models affect students' mental comparison of the size of two proper fractions?" The hypotheses are the following:

H_0 1: There is no significant difference in the test score differences between groups of test subjects using assigned manipulatives to complete their exercises.

H_0 2: There is no significant difference in the classification differences on the scale of fraction understanding (described in Chapter 3) between groups of test subjects using assigned manipulatives.

The virtual manipulatives were created using the Scratch programming environment, the outcome of a project of the Lifelong Kindergarten Group at the MIT Media Lab (Massachusetts Institute of Technology, 2019). Scratch enables users to visually write scripts in a proprietary web-based scripting language to tell stories, perform animations, and play games and was originally conceived as a way to introduce 8 to 16-year-olds to computer programming.

Delimitations

This study proposes using a fraction modeling tool that emphasizes the connection between the early models and locating a fraction on a number line in introductory university mathematics classes. Students in most of these classes have generally displayed weak computational skills as evidenced by placement based on lower ACT scores (15-21). The study will be limited to university classrooms on a single campus in southeast Missouri. The student population spans a cross-section of socioeconomic status. In addition to lacking a significant ethnic diversity, the population is slightly more female since the initial mathematics content for teacher classes (which traditionally contain over 90% female students) comprise about one-fourth of the classes in the study. These classes also contain students with higher ACT scores because the classes have no upper limit on the ACT score (>15).

Students may complete the pretest using procedural comparison methods such as cross products, common denominators, and conversion to decimal as opposed to making a mental comparison. Administrators of the tests will read a script emphasizing the use of mental comparison and exhorting students to compare the fractions they see without changing them in any way. The specific methods are not mentioned in the script to not encourage their use if the students had not considered them.

The fact that students will not receive any additional instruction is of some concern. However, Boaler (2016) discusses a study that gave subjects a 10-minute exercise to work over 15 days, and participants experienced structural brain changes.

Significance

The unique elements of the treatment include the use of technology to create more accurate models than drawing by hand and the use of circular and bar area models as well as positions on a number line. Torbeyns et al. (2015) showed a correlation between the ability to locate fractions on a number line and improved general mathematical achievement. If successful, further study using this tool to introduce fractions at lower grade levels would be a natural extension.

Organization of the Study

The remainder of this study is structured in four additional chapters. The second chapter contains the literature review discussing mathematical misconceptions- the constructivist view of misconceptions, theories of number development, rational number learning, use of technology for identification and remediation- as well as research on fraction comparison. The third chapter specifies the research design and methodology, the fourth includes the data analysis and findings, and the fifth summarizes the study, discusses the conclusions, and makes recommendations regarding future research.

Chapter 2

Literature Review

Misconceptions and Constructivism

Smith et al. (1993) attempt to reorient the traditional discussion of student misconceptions to a more constructivist framework. In the first part of the article the authors discuss misconceptions research and some of the central tenets that fly in the face of constructivism. Much of the research on misconceptions discussed in their article implies that misconceptions interfere with learning. As such, misconceptions should be identified specifically; confronted explicitly; and replaced with expert knowledge. Less emphasis in this previous research is noted on "modeling the learning of successful students in those domains, … characterizing how misconceptions (and the cognitive structures that embed them) evolve, or to describing the nature of instruction that successfully promotes such learning." (Smith et al., 1993, p. 123)

In one scenario from the text, students from novice to master were presented with a series of fraction tasks including comparison. (Smith et al., 1993) They make the point that students classified as masters use some of the same knowledge and structures in their reasoning that novices do, but masters have built and expanded upon that knowledge and structure. In the researchers' discussion of strategies of fraction comparison, they note that while novices focus on models, masters have used the divided whole concepts from the models to develop reasoning about the quantities themselves.

Even though textbooks and curriculum focus on two primary strategies for fraction comparison (and operations), conversion to common denominator and conversion to decimal, mastery depends on a wider variety of strategies, many only useful within a specific context. (Smith et al., 1993) Some of those strategies included benchmarking (comparing to a common reference point such as ½ or 1) and easy relationships between numerator and denominator (12/24 and 8/16 are one half). Both masters and novices tended to develop common strategies that they would use within the context of a problem type, falling back on the taught strategies when they could find no easy relationships. One of the key findings was that these student-developed strategies are rarely taught explicitly. (Smith et al., 1993)

Neshar (1987) proposes an instructional theory based on using student misconceptions to guide instruction. The author makes the point that cognitive dissonance is necessary for learning and that often specific student errors arise from more general misconceptions that will yield both correct and incorrect answers, depending on the question. The paper cites two approaches involving decimal comparison- either longer is larger (whole number thinking) or shorter is larger (tenths are bigger than hundredths). In both cases, students using these approaches will (potentially) correctly answer any question where the number of decimal places in the two numbers is the same. Also, the approaches can yield the correct answer to questions that are set up in a particular way. When comparing .4 vs .125 students using the shorter/larger approach will choose the correct answer while students using the longer/larger will not. On the other hand, when comparing .4 vs .675, the longer/larger approach will lead to a correct answer while the shorter/larger will not). Neshar (1987) discusses the implementation of the proposed instructional theory with a learning system/microworld containing an "articulation of the unit of knowledge" or "knowledge component" and an "exemplification component" which must be familiar but serve as a stepping stone to "new concepts and relationships". The fraction comparison application for this project provides just such a microworld with basic fraction knowledge serving as the knowledge component and the models as the exemplification component.

While the word misconception itself implies a deficit, it is the vocabulary that appears in much of the research that I used for my thesis. However, the use of the word "misconception" does not go as far as "mistake" or "error" in terms of labeling and creating an impression of "wrongness". Smith et al (1993) include an appendix with an extensive discussion of the language surrounding misconceptions. They point out that even "alternative conceptions" (which may, in some cases, still be correct) implies a difference from the "right" conception like "informal knowledge" is somehow not as good as "formal knowledge". They do not offer a solution to the discussion although the idea of a "preconception" does not carry a particular connotation. Still, in common usage, preconceptions tend to be somewhat negative- preconceived notions, etc. Regardless of the label, students will perform a fraction comparison in the way that they have developed to understand it, and my goal is to see if this tool can move them to a more complete understanding.

Theories of Number Development

The teaching of elementary mathematics from the perspective of privileged domain theory (Gelman & Williams, 1998), evolutionary theories of numerical development (Geary, 2006), and conceptual change theories (Vosniadou et al., 2008) treat the teaching of whole numbers and the teaching of rational numbers as completely different processes requiring a different framework for understanding the rational number system.

According to privileged domain theories (Gelman & Williams, 1998) and evolutionary number theories (Geary, 2006), whole number reasoning lies at the heart of many of the difficulties students have in learning about fractions and rational numbers. In some ways, the lack of understanding that fractions are rational (and real) numbers with associated locations on the real number line inhibits the transfer of operational knowledge from whole numbers to rational numbers. Students see the two numerals of a fraction, the numerator and denominator, as representing distinct values requiring separate analysis. This thinking is a logical extension of much of the fraction modeling that is used in elementary school, especially circular models where students learn to count the numerator and denominator separately.

Theories of numerical development that focus on the acquisition of whole number knowledge treat the development of knowledge of other types of numbers (integers and rationals) as distinct (Gelman & Williams, 1998) (separate number systems) and secondary (Geary, 2006) (whole numbers take precedence) and point to ways in which the interpretation of whole numbers inhibits the understanding of the other types of numbers. The theories emphasize the discontinuity between the number systems- whole numbers are different from integers and both are different from rational numbers. “Privileged domain theories argue that specialized learning mechanisms make it easier to learn about whole numbers than about fractions or other types of numbers.” (Gelman & Williams, 1998, p. 11)

Geary (2006) proposed an evolutionary theory that whole numbers are “biologically primary” and that other types of numbers are “biologically secondary”. The fact that infants recognize different sizes of sets establishes the primacy of counting as a way of understanding numbers. In that sense, whole numbers (or even more specifically, counting numbers) are tied to the innate understanding of numerosity. According to this theory (and privileged domain theories) the counting elements of whole numbers, like the one to one correspondence of sets to the counting numbers and the fact that the cardinal number is the last number counted in a set, make it harder to understand fractions because they have no analog in the rational number system.

According to Vosniadou et al. (2008) conceptual change theories place a greater emphasis on fraction knowledge development but still focus on the differences between learning whole numbers and fractions. Vamakoussi & Vosniadou (2010) speculate that children develop a framework for understanding numbers as counting numbers that "constitutes an initial, domain-specific theory of number". Using a framework theory approach to conceptual change, the misconceptions due to natural number reasoning are an indication that students use their understanding of counting numbers to try and make sense of rational numbers. This leads to ideas like larger numbers make larger fractions.

Siegler et al. (2011) propose an integrated theory of number development that contrasts with much of the research in support of privileged domain theories and evolutionary development theories of number development. This integrated theory recognizes the differences between whole numbers and fractions, but it posits that they share the important commonality of the centrality of numerical magnitudes in the overall understanding of numbers. The researchers found that:

> accuracy of fraction magnitude representations is closely related to both fraction arithmetic proficiency and overall mathematics achievement test scores, that fraction magnitude representations account for substantial variances in

> mathematics achievement test scores beyond that explained by fraction arithmetic proficiency and that developing effective strategies plays a key role in improved knowledge of fractions. (Siegler et al., 2011, p. 22)

Moving from misconception/alternative conception/preconception to a more complete understanding can be approached from a constructivist point of view. The integrated number theory work of Siegler et al. (2011) proposed that rational number learning should be treated as an extension of whole number learning instead of as something different. In that sense, the constructivist ideas for building on existing knowledge as opposed to trying to create a different understanding just for rational numbers come into play. The activities in this project attempt to transition the students from the part-whole circle model understanding to the idea of magnitude on a number line. The part-whole ideas are not necessarily replaced by the magnitude representation, but they are supplemented or expanded as the understanding of fractions and rational numbers are multilayered.

In the vein of supplementing as opposed to replacing, the transition between number systems should focus on the operational properties of numbers that do not change and the positioning of numerical values on a number line. Also, the further emphasis during whole number learning on the things that numbers represent in whole number operations can lay a better foundation for rational number learning. Emphasis needs to be placed on the fact that in addition and subtraction all of the numbers represent common units regardless of the number system, while in one step multiplication and division (involving three numbers), two of the numbers represent unit-based quantities while the third is a unitless scale factor.

Rational Number Learning

Steffe's schemes as noted in McCloskey & Norton (2009), Norton & McCloskey (2008), and Norton et al. (2018) describe fraction learning as a progression through seven schemes as noted in Figure 6.

Figure 6
Steffe's Schemes (McCloskey & Norton, 2009, p. 47)

Scheme	Operations	Sample Task
Simultaneous partitioning scheme	Unitizing the whole, partitioning the continuous whole using a composite unit as a template	Share this candy bar equally among you and two friends.
Part-whole scheme	Unitizing, partitioning, disembedding a part from the partitioned whole	Show me two-thirds of the candy bar.
Equi-partitioning scheme	Unitizing, partitioning, iterating any part to determine its identity with the other parts	If you share this candy bar equally among you and two friends, show me what your piece would look like.
Partitive unit fractional scheme	Iterating a given unit fraction to produce a continuous partitioned unitized whole	If I give you this much [show a one-third piece and an unpartitioned whole], what fraction of the candy bar would you have?
Partitive fractional scheme	Unitizing, disembedding a proper fraction from the whole, hypothetically partitioning the proper fraction to produce a unit fraction, iterating the unit fraction to produce the proper fraction and the whole, coordinating unit fractions within a composite fraction (units coordinating at two levels)	If I give you this much [show an unpartitioned two-thirds piece and unpartitioned whole], what fraction of the candy bar would you have?
Reversible partitive fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a larger whole to re-create the whole	If the bar is four-fifths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.
Iterative fractional scheme	Splitting (that is, partitioning and then iterating) an unpartitioned piece of a smaller whole to re-create the whole	If the bar is five-fourths as long as your candy bar [show an unpartitioned piece], draw what your candy bar would look like.

The test items described in the Methodology section of this document which were designed to elicit students' conceptions in a variety of situations roughly correlate to elements of Steffe's schemes. The reasoning behind each answer can be categorized using three main ideas regardless of the "correctness" of the answer:

Idea 1 – Bigger is greater

Idea 2 – Larger denominator is less

Idea 3 – Size of the gap indicates relative size – larger gap means a smaller fraction, vice versa; the same gap means fractions are equivalent

In looking at these three ideas, one can begin to see a progression through Steffe's schemes. The idea that larger numbers yield larger fractions is a counting based idea that prefaces the earliest Steffe scheme of Part/Whole as the student does not grasp that the whole is partitioned into equal pieces. The idea that a larger denominator indicates a smaller fraction indicates an understanding of a unit fraction under the Part/Whole scheme as the student understands the partitioning of the whole, but not the iteration of the parts. Gap reasoning falls somewhere in the equi-partitioning scheme as the student has an understanding of the division of the whole and counting of the parts to create the fractions, but they have not put it all together for comparison of two fractions. Leveraging the fact that an equal gap comparison transitions to a "common numerator" comparison, if one considers which fraction is closer to/further from 1 (benchmarking), means that it might be possible to modify a student's use of gap reasoning to better fit the circumstance rather than replacing it entirely. Finally, Steffe's more advanced schemes revolve around iteration which is easily seen in the context of the number line.

Norton et al. (2018) investigated whether Steffe's schemes were particular to US schools by studying Chinese students. The researchers found similar schemes in both countries even though students in the US are primarily introduced to fractions through part/whole concepts while Chinese students learn more from the measurement model.

> "Collectively, our findings suggest a common cognitive core in students' development of fractions knowledge, which is described in terms of the progression of fraction schemes shared in Table 1. Educators could foster student growth by building from primitive part-whole schemes toward measurement schemes (e.g., PUFS). Previous research has indicated that

engaging in tasks involving iterating unit fractions can support that growth (Tzur, 1999). Such tasks already play a prominent role in the elementary school curriculum in China (Li et al., 2009)." (Norton et al., 2018, p. 225)

Technology and Rational Number Learning

While Neshar (1987) describes the construction of microworlds using technology to create learning systems, Olive, et al (2010) provide a discussion of technology specifically related to mathematics education in their chapter of the 17th ICMI Study, Mathematics Education and Technology, Rethinking the Terrain. The chapter is divided into three main subjects- "1) mathematical knowledge and learning that results from the use of technology, 2) mathematical knowledge on which the technologies are based, and 3) mathematical practices that are made possible through the use of technology."

In the first section, the authors make the point that a significant application of technology is geared toward more efficiency in the same classroom environment, but that the opportunity exists for much more. They contrast the use of technology for efficiency with the TIMA software application that Olive and Steffe developed at the University of Georgia also documented in Steffe & Olive (2002). TIMA is a multi-faceted computer environment that allows students to access fractions as elements of sets, measurements, and area. It addresses a wide range of fraction learning allowing students to "enact their mathematical operations of unitizing, uniting, fragmenting, segmenting, partitioning, disembedding, iterating and measuring." (Steffe & Olive, 2002, p. 55)

The primary point they are trying to make is that through interaction within the context of a microworld such as TIMA, students are more able to construct mathematics and develop a deeper understanding. An interesting element of this idea is the context.

Much of the development of regular curriculum focuses on bringing “real world” aspects to the learning environment, but they contend that this strategy still misses the mark. The parameters have to be controlled to the point where the scenarios are not real outside the manufactured context developed by the curriculum. In a micro world, students are solving the problems in the context of that world and do not necessarily need the element of reality to develop an understanding of the mathematics. (Steffe & Olive, 2002)

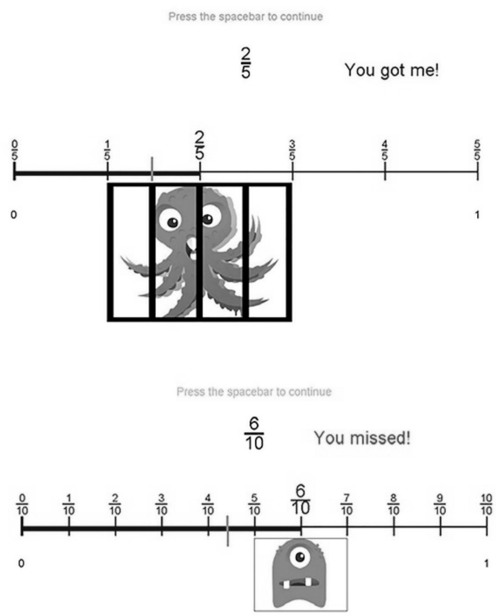
Bulut et al. (2014) developed a Dynamic Geometry Environment microworld for third graders in Turkey using Geogebra. Bulut et al (2014) presented a technology enhancement of the current classroom process by creating dynamic software models of the physical representations normally used. ”In the experimental group dynamic oriented activities were used by using [a] constructive approach.” (Bulut et al., 2014) Students were able to see a wider range of fractions and use the software to change the models in ways not possible with physical manipulatives.

Fazio, Kennedy & Siegler (2016) modified a program developed for decimal magnitude to create a microworld called Catch the Monster with Fractions and deployed it as an instructional supplement. Students received the same instruction, but the control group performed their activities using worksheets while the experimental group played the “Catch the Monster” game. The game itself was designed to emphasize fraction magnitude, understanding the measurement context of a fraction especially regarding the position on a number line (see Figure 7). “The [Common Core] standards’ focus on understanding fractions as numbers with magnitude dovetails with recent emphasis within cognitive psychological theories on the centrality of magnitude understanding to mathematical knowledge.” (Fazio et al, 2016)

Catch the Monster with Fractions was used for two studies. The first involved 26 fourth and fifth graders near Pittsburgh, PA, and the second expanded to 51 fifth graders in the same area. Even though the size and education level of the two studies were different, both indicated significant improvement in the experimental groups as compared to the control groups.

Figure 7

A sample correct trial (top) and a sample incorrect trial (bottom) from Catch the Monster (Fazio et al., 2016)



Olive, et al (2010) point out that a significant amount of potentially complicated mathematics can serve as the underpinnings for any microworld. A teacher or student does not need to understand all of the underlying mathematics to use the microworld, but anyone who is building a microworld needs to understand the consequences of changes to any particular aspect of the program.

The final element of technology use in mathematics education is the development of mathematical practices. As students are allowed to explore and discover they use more of the practices that form the fundamental element of what distinguishes the Common Core Standards. By interacting and receiving feedback, students can implement the mathematical practices listed in the Common Core standards.

- Make sense of problems and persevere in solving them.
- Construct viable arguments and critique the reasoning of others.
- Reason abstractly and quantitatively.
- Model with mathematics.
- Attend to precision.
- Use appropriate tools strategically.
- Look for and make use of structure.

Identification of Understanding

Mazzocco et al. (2013) demonstrated how a qualitative error analysis of early symbolic number knowledge reveals potential sources of differences that may affect mathematics outcomes. The article discusses some specific errors, but the objective was to show how qualitative analysis can augment test scores. They found that gaps in the number knowledge of second and third graders appeared to predict specific types of error on eighth-grade math assessments. They showed "that early whole number misconceptions predict slower and less accurate performance, and atypical computational errors, on Grade 8 arithmetic tests … (and) that basic number misconceptions can be detected by idiosyncratic responses to number knowledge items" (Mazzocco et al., 2013, p. 33).

Steinle & Stacey (2003) showed variations in the patterns of understandings and developed estimates of the lifetime prevalence of these misconceptions. While they focused specifically on two understandings related to decimal comparison- longer is larger, classified as (L) and shorter is larger, classified as (S), the larger point is the different ways in which related misconceptions manifest themselves. They found that second through fifth-grade students were more likely to exhibit (L), but that its appearance decreased over time. Of more interest was the fact that the (S) understanding increased over time and persisted in high school. In a different study using the same test data, Steinle & Stacey (2004) classified the (S) understandings in more specific ways based on students' comparisons of decimals with the same number of digits. They devised (S1) for *denominator based thinking* (since 1/100 is less than 1/10, anything with hundredths must be less than something with tenths) and *place value number line thinking* (since three-digit numbers follow two-digit numbers on the number line, it follows that the order is reversed on the other side of the decimal so all three-digit decimals must be less than all two-digit decimals); (S3) for *reciprocal thinking* (1/73<1/6, so .73 must be less than .6) or negative thinking (-73<-6, so .73 must be less than .6); and (A2) for *money thinking* (everything is truncated to two decimal places and the resulting two-digit decimal is compared). The authors found that younger students demonstrating (S1) and (A2) were more likely to move to expertise on their following tests where students falling into (S3) were more likely to stay there. However, older students for all three categories were less likely to ever move to expertise, possibly because they have demonstrated some type of learning disability. (Steinle & Stacey, 2004)

Finally, Kerr (2014) hypothesized that educational video games can reveal understandings in ways unavailable in traditional environments. One of the major issues proved to be separating mistakes in the video games from mathematical errors which became the basis of the research question – "Can mathematical misconceptions be identified solely from actions students take in an educational video game?" (Kerr, 2014, p. 8) Through the use of cluster analysis on two separate video games and surveys, the researcher showed that certain understandings could be isolated from difficulties with the structure of the games. The first game, *Save Patch*, was designed to test students' ability to understand the meaning of the unit, the meaning of addition as applied to fractions, and the meaning of the numerator and the denominator. The most common misconception was a misunderstanding of how to partition fractions. Students viewed a rectangular grid divided into equal sections by posts and counted the posts to construct their denominators rather than the spaces between the posts. This misunderstanding is reinforced by the use of circular models because the amount of cuts required to divide a circle is equal to the resulting denominator. The next most common misconception revolved around an inability to properly establish the unit upon which the fraction was based. About two-thirds of the problems in *Save Patch* were designed to use fractions greater than one, but students with the misconception consistently set their unit as one by including the entire grid.

As a check regarding the understandings, the researcher presented the students with a series of number line problems outside the game and found that the same students made the same errors. Finally, the researcher used a second game, *Wiki Jones*, which had a remediation element to it and found similar results thus concluding that common understanding identified in the video game matched real mathematical understandings encountered by students.

Remediation of Student Understanding

The development of the fraction comparison tool is supported by research on the remediation of student understanding. Riccomini (2005) studied teachers as a source of remediation for students and found that when they can identify errors, they often do not shift their instruction based on the errors they see. The researcher presented teachers with two systematic error patterns in subtraction and asked them to identify and propose remediation. About 60% of the teachers correctly identified both errors, but they did not base their instructional focus on the pattern of errors. It is not enough to identify errors; teachers must tie remediation to the pattern of error. If this project is successful, the design of the comparison tool can allow teachers to address misconceptions in fraction comparison.

According to Durkin & Rittle-Johnson (2012), misconceptions can be useful teaching tools. Researchers examined students' performance on a decimal understanding task based on learning with correct examples versus intervention with incorrect examples. They hypothesized that the students using the intervention with incorrect examples would do outperform the students learning with correct examples. The researchers designed two sets of tasks to follow similar instruction. In the set of tasks for the first group, students were presented with one correct and one incorrect placement of a decimal on a number line with explanations of the reasoning, and in the tasks for the second group, the students were presented with two correct placements and the associated explanations. The researchers found that the use of the incorrect examples supported greater learning of correct procedures and retention of correct concepts. The comparison tool for my research project was designed specifically to address misconceptions in fraction comparison. Students have the opportunity to consider their approach to fraction comparison when their predictions do not match the models.

Finally, according to Hewitt (2012) technology can support remediation. This researcher examined the use of software to introduce algebraic notation to 9-10-year-olds. The software, *Grid Algebra*, was designed to introduce formal notation and help students solve linear equations. The software is built around a multiplication grid and the idea that moving spaces to the right represents addition, moving spaces to the left represents subtraction, spaces down represents multiplication and moving spaces up represents division. By moving around the grid, students build a series of arithmetic operations. This alternative approach to linear equations avoided some common difficulties that often arise especially in understanding the building of equivalent expressions as opposed to creating calculations. This research, as well as the work of Fazio, Kennedy, & Siegler (2016) supports the idea of using technology as a remediation tool.

Fraction Comparison

According to Siegler et al. (2011), many theories of numerical cognition accept that whole number knowledge is organized around a mental number line. They also state that research has shown that number line estimation is an underutilized task that can be useful for studying the development of whole number magnitude representations. The advantage of number line estimation is that it is not limited to whole numbers, but it can also be used with any type of real number, large or small.

Siegler et al. (2011) propose five commonalities between magnitude representations of whole numbers and fractions in their proposal of an integrated theory of number development-

1) Alternative measures of fraction magnitude knowledge are highly correlated.
2) Numerical magnitude comparisons with fractions yield distance effects.
3) Knowledge of different ranges of fractions develops at different times (earlier for fractions from 0 to 1 than from 0 to 5).
4) Knowledge of fraction magnitudes varies greatly among individuals and correlates with both arithmetic proficiency and mathematics achievement test scores.
5) Relations between fraction magnitude representations and mathematics achievement test scores extend beyond their common relation to arithmetic knowledge.

These commonalities support the value of the development of a single integrated theory for the development of whole numbers and fractions as proposed by Siegler et al. (2011).

Clark & Roche (2009) studied students' mental fraction comparison strategies on a set of eight different pairs of fractions and broke the strategies down into four broad categories:

1) *Residual thinking* – how much left to get to the unit (2/3 is 1/3 away from 1, 2/5 is 3/5 away from 1)
2) *Benchmarking* (or transitive) – how close to benchmark fractions (1/4, 1/3, ½, 2/3, 3/4)
3) *Common denominators*- transform the fractions to equivalent fractions with equal denominators. Because fractions with common denominators are obtained through a series of three multiplications, it is a more procedural strategy that requires less conceptual knowledge of the relative size of the fractions.

4) *Gap thinking* – a comparison of the difference between the numerator and the denominator. Gap thinking is tied to the idea that the numerator and denominator exist as separate entities. When using gap thinking a student would look at ¾ and 3/5 and make their comparison based on the fact that the difference between the numerator and denominator in the first fraction is 1 while the difference between the numerator and denominator in the second is 2. Since ¾ has a smaller gap, then it must be a larger fraction. Four of the pairs yielded the correct answer through incorrect gap thinking, so the explanation from the students was key to determining how they arrived at their answers.

Students reported benchmarking and/or residual thinking the most on six out of the eight comparisons. However, in situations where those two strategies were most appropriate, the most widely used strategy was common denominators that favor procedural over conceptual knowledge.

Clark & Roche (2009) found that students with the greatest success tended to use residual thinking or benchmarking. Students with a better conceptual understanding leaned on benchmarking and residual thinking as well, but teachers did not use or teach these strategies. Many teachers were unable to offer a strategy other than common denominators leading the authors to speculate that teachers were generally unaware of these strategies.

If students do not recognize the relative size of fractions, they will struggle to conceptualize any associated operations on fractions. Post et al. (1986) note that "children's understandings about ordering whole numbers often adversely affect their early understandings about ordering fractions." (pg 33) For some children, these misunderstandings persist even after relatively intense instruction based on the use of manipulative aids such as diagrams and fraction circles, but Kilpatrick et al. (2001) argue that "of all the ways which rational numbers can be interpreted and used, the most basic is the simplest- rational numbers are numbers. That fact is so fundamental that it is easily overlooked." (pg. 235)

The ability to perceive the ordered pair in a fraction symbol as a conceptual unit rather than as two individual numbers was found to be an indicator for successful performance by Clark & Roche (2009). Also, the researchers noted that using models such as the circular type of fraction models often introduced with fractions to make comparison decisions caused problems because children are often "model poor". This idea is supported by Post et al. (1986) who found that "a crucial point in acquisition of the order and equivalence concept is reached when children's understanding of fractions becomes detached from concrete embodiments and children are able to deal with fractions as numbers." Moss & Case (1999) laid out an instructional program to address this that included a greater emphasis on the meaning of rational numbers as opposed to the procedures for manipulating them, greater emphasis on the proportionality of rational numbers with an attendant focus on the differences with whole numbers, and the use of an alternative visual representation between proportional quantities and their numeric representation (something other than pie charts).

Summary

The research of Smith, diSessa, & Roschelle supports the idea of constructivist approaches like the use of models in addressing students' perceptions of fractional comparison. Furthermore, Neshar's work supports the idea of a microworld such as the limited one built for this exercise.

The Integrated Theory of Number Development proposed by Siegler, Thompson, & Schneider lays the groundwork for understanding that fraction magnitude is similar to whole number magnitude and that modeling of fractions as positions on a number line provides the context for better understanding of fraction operations. This supports Steffe's schemes that indicate that fraction understanding is not complete until students can demonstrate measurement aspects of fractions.

Technology is ubiquitous today, and educators need to find ways to leverage it in their classrooms. While certain aspects of physical manipulatives are difficult to replicate, the ability to share technology across a wider field of students bends the arc toward its use, especially if student performance is at least the same when using technology versus not. In addition, the identification and remediaton of understanding in general, and fraction comparison in this study, can be enhanced by technology.

Students use a variety of strategies for fraction comparison that are often not taught. Tapping into these strategies directly and supporting them with models can help students not only with fraction comparison, but with understanding fraction magnitude which is highly correlated with improved performance on mathematics aptitude tests on a variety of subjects.

Chapter 3

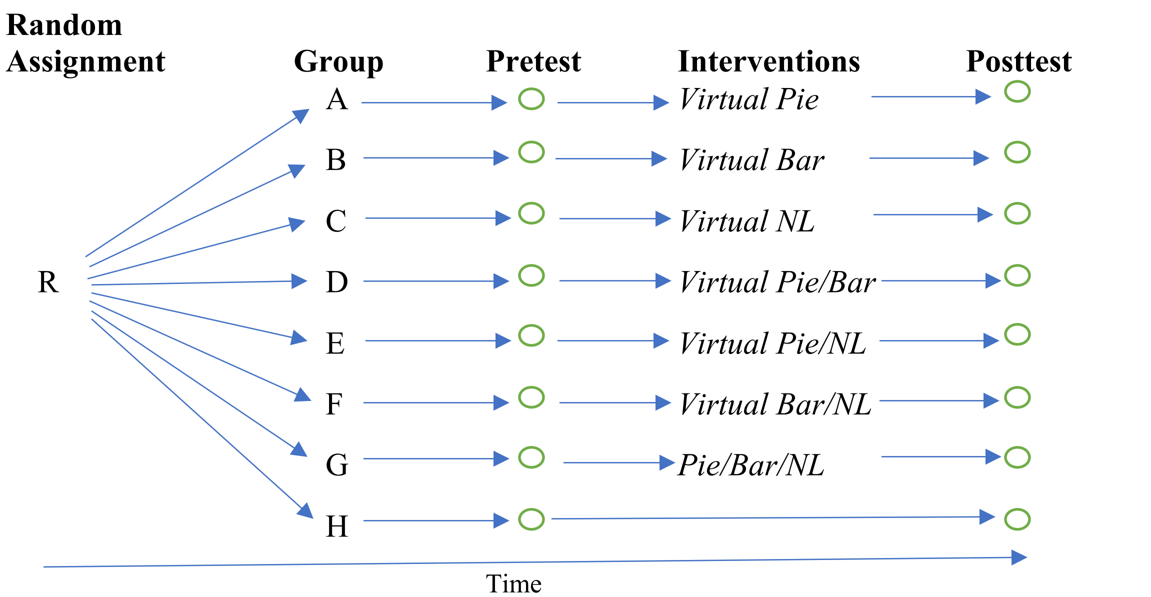
Methodology

This study is focused on answering the two research questions, “How does the use of virtual fraction models affect students’ mental comparison of the size of two proper fractions as reflected in performance on a test designed to identify fraction comparison strategies?” and “How does the use of virtual fraction models affect students’ mental comparison of the size of two proper fractions as reflected on a scale of fraction understanding?” by determining whether the associated hypotheses can be supported. Chapter three describes the research design, sampling, variables, data analysis, and ethics for the study.

Research Design

To investigate the effects of the treatment in addressing a specific mathematical skill, the study used a pretest/posttest control/comparison group design with random assignment of control and treatment groups within different class sections. Seven different treatment groups were created using all combinations of the manipulatives. The experiment involved eleven lower-division university level math classes, and the manipulatives were distributed randomly within these classes so that approximately the same number of students used each manipulative. Figure 8 shows a visual model of the design.

Figure 8
Randomized Pretest-Posttest Control/Comparison Group Design (per class section)



Scale of Fraction Understanding

According to the senior researcher on the EM^2 project, P. Clements (personal communication, April 24, 2020), the tests were not designed to measure a single construct like fraction comparison so they have relatively low internal reliability. They were designed to indicate the probability of whether a student has any of the identified misconceptions. The patterns of correct and incorrect answers indicate whether the student understands fraction comparison in a particular way so performing an analysis and classification of the data is recommended.

EM^2 provided a comprehensive analysis of the test instruments that can be used to identify whether a student exhibits one of the three comparison methods targeted by the Comparison of Fractions Assessment (included in Appendix A). Students took a pretest designed to identify particular approaches in the area of fraction comparison, and student scores on the pretest were classified based on the students' methods of comparing fractions – cross products, decimal conversion, common denominators, whole number reasoning, dependence on the unit fraction, and gap reasoning. Pretest answers were analyzed to classify each students' reasoning on a scale of fraction understanding as detailed below-

- 0 procedural application of comparison (cross products, decimal conversion, common denominator)
- 1 whole number reasoning (larger numbers make larger fractions)
- 2 unit fraction reasoning (larger denominators make smaller fractions)
- 3 gap reasoning (a larger gap between numerator and denominator makes smaller fraction)
- 4 mastery

The control group received practice work and a physical manipulative (chosen to determine whether technological bias might be an influence) while the treatment groups received the same practice work to be completed using the treatment designed for their assigned group. No additional instruction was administered for any of the groups. The practice work included sets of the following types of fraction comparisons (an example is included in Appendix A)-

- Proper fractions with a common difference between the numerator and denominator
- Proper fractions with numerator and denominator of one fraction greater than numerator and denominator of the other fraction
- Equivalent proper fractions with denominators less than or equal to twelve
- Equivalent proper fractions, one with a denominator less than twelve and the other with a denominator greater than twelve

The practice work was designed as seven separate assignments, each with four fraction comparison problems to be completed over a one to two-week period. The students used their assigned tool (physical manipulative vs virtual manipulative) to complete each worksheet designed to address the types of fraction comparisons mentioned in the preceding paragraph. Upon completion of the series of worksheets, the students completed the post-test.

After administering the posttest, an independent sample t-test was used to establish whether the pretest scores or pretest classifications varied significantly between the groups. Additionally, a paired samples t-test was performed to determine whether the students in the entire sample exhibited significant improvement in test scores and/or classification. ANOVA was used on the gain scores (differences of the pre-test and post-test mean scores) of the groups to show any between-group differences. Additional categorization of the pretest and post-test results assisted in identifying the presence of various types of comparison (whole number thinking, gap reasoning, and denominator focus) allowing for a separate analysis of the treatments and types of comparison, and ANOVA was used to compare the differences of the classifications from before and after administration of the treatments.

Threats to Internal Validity

Possible threats to internal validity include regression, selection, compensatory/resentful demoralization, compensatory rivalry, and instrumentation. Regression and selection were minimized by random assignment. Since this is a blind study and students will receive completion grades, I do not anticipate either of the compensatory items being a significant issue, but the nature of the treatment will be such that any student can take advantage of it after the study is complete. Concerning instrumentation, the exact questions were not the same on the pretest and post-test, but the corresponding questions test the same ideas using the same restrictions on the fractions in the question.

Threats to External Validity

Setting and the selection of participants are the most common threats to external validity as they often limit the generalizability of a study. In this case, the subjects are students in lower-division mathematics courses at a small regional university in the Midwest. As such, results should not extend beyond the population in the study.

Research Questions

1) "How does the use of virtual fraction models affect students' mental comparison of the size of two proper fractions as reflected in performance on a test designed to identify fraction comparison strategies?"
2) "How does the use of virtual fraction models affect students' mental comparison of the size of two proper fractions as reflected on a scale of fraction understanding?"

Hypotheses

H_0 1: There is no significant difference in the test score differences between groups of test subjects using assigned manipulatives to complete their exercises.

H_0 2: There is no significant difference in the classification differences on the scale of fraction understanding between groups of test subjects using assigned manipulatives.

Population and Sample

The study involved students from twelve mathematics classes at a regional university located in a Midwestern state. The total enrollment of classes in the study was approximately 350 students of which 211 ended up in the study. See Table 1 for the manipulative assignments. An approximately equal number of each manipulative type was assigned to each class section and manipulatives were distributed at random to all students in the section. Pretest and posttest scores were collected and classified according to the types of students' approaches to comparing fractions as described earlier in this section.

Table 1
Manipulative assignment

Manipulative	Student Count
Physical Fraction Circle	28
Virtual Fraction Circle	29
Virtual Bar	28
Virtual Number Line	25
Virtual Circle/Bar	28
Virtual Circle/Number Line	26
Virtual Bar/Number Line	22
Virtual Circle/Bar/Number Line	25
Total	211

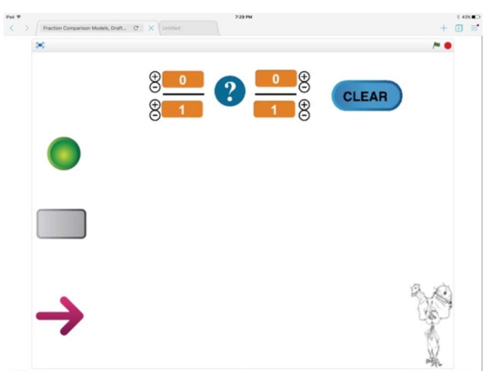
Instrumentation

Treatment

The researcher designed the specific treatment for this study using the Scratch scripting language developed at Massachusetts Institute of Technology (MIT). (Massachusetts Institute of Technology, 2019) Seven different levels of treatment were investigated. In the first level, students used an electronic version of the circle model. The intent was to account for any technology bias on the part of the students. The second level used a bar model for the fraction comparison. In this case, did the linearity of the bar model have the same effect as number line placement? The third group used a number line. The number line eliminates the use of an area construction in the fraction model. Also, groups were assigned combinations of the models to see if the interaction between the models made any difference. The fourth group used both a circle model and a bar model, the fifth group a circle model and number line, the sixth group a bar model and number line and the seventh used all three models.

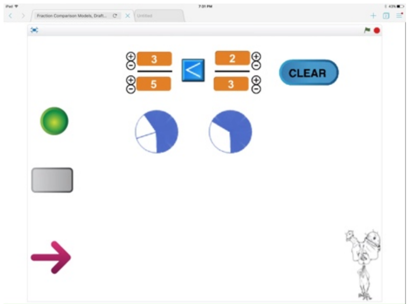
The treatment groups were intended to use their assigned manipulative in their practice. In opening the tool, the student was presented with a screen that allowed them to input two fractions for comparison (see Figure 9). Using the increment/decrement symbols next to each numerator and denominator, they can enter any proper fraction with a denominator up to twelve. At any point, the student can predict the relationship between the two fractions as <, >, or = by clicking the symbol between the two fractions.

Figure 9
Initial screen for fraction manipulative tool



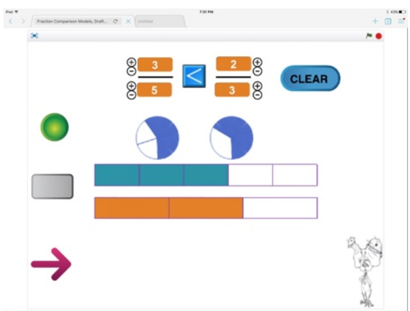
Clicking the circular symbol on the left creates circle models for the two fractions (Figure 10).

Figure 10
Creation of circle models in virtual manipulative



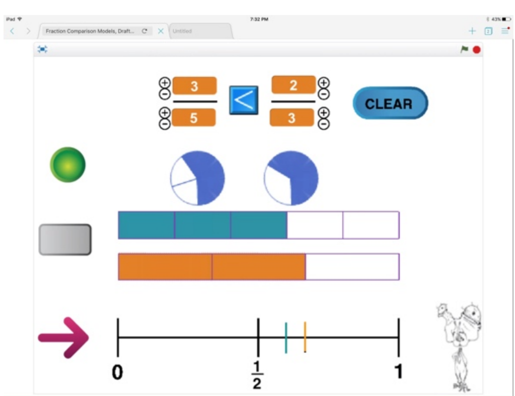
Clicking the rectangular symbol on the left creates bar models for the two fractions (Figure 11).

Figure 11
Creation of bar models in virtual manipulative



Finally, clicking on the arrow symbol draws two number lines, partitions them based on the denominator and iterates across the line to the location of the fraction (Figure 12).

Figure 12
Creation of a number line representation in virtual manipulative



The study did not require additional instruction since all of the participants had received previous instruction on fractions and operations. All students received a set of fraction comparison problems for practice. Those in the treatment groups used a version of the virtual manipulative to model each problem before completing their answer. Students in the control group worked the same set of problems, but they received a commercial set of fraction circle models divided into halves, thirds, fourths, fifths, sixths, eighths, ninths, tenths, and twelfths (see Figure 13).

Figure 13
Commercial circle model tool

Test Instrument

The Education Development Center initiated the Eliciting Mathematical Misconceptions project (Education Development Center, 2014) to develop open-source diagnostic assessments to specifically identify fraction related understanding. The project produced a series of assessments on Representing Fractions, Comparing Two Fractions, and Comparing Decimals. Each assessment is designed to identify specific understandings related to fractions and decimal understanding. They have published their testing material at em2.edc.org. (Education Development Center, 2014) For the fraction comparison assessments, each test consists of seven questions where the student is asked to select the appropriate comparison (<, >, or =) and explain their reasoning. Both the Pretest and Posttest are included in Appendix A.

According to EDC (2015), “The EM^2 diagnostic assessments help teachers identify which of their students are likely to have specific types of rational number misconceptions. Teachers can then use this information to inform their instruction.” (p. Research Foundations) They based the development on two areas of research- learning rational number concepts and formative assessment. The EM^2 project used “diagnostic cognitive modeling (DCM) methods described by Rupp, Templin, and Henson’s book on diagnostic measurement (2010). While the longer-term goal of the project is to use more sophisticated DCM analysis to empirically confirm the hypothesized structure of the assessments, analyses conducted to date have focused on qualitative scoring conducted by expert coders and item-level descriptive statistics (including the Kullback-Liebler Information index).” (Clements, Buffington, & Tobey, 2013)

The EM^2 research on rational number concepts closely tracks much of the work cited in this proposal’s literature review. The complexity of the rational number system can impede the mathematical development of students. The things that students learn, understand, and internalize regarding whole numbers can lead to misconceptions about rational numbers. “While whole number relationships are based on additive properties, rational numbers have relationships based on multiplicative relations. Moreover, rational numbers can be expressed in many different forms and can be designated by an infinite number of equivalent representations.” (Education Development Center, 2014, p. Research Foundations)

Further complicating matters is the use of two numerals in a fraction to represent a single number (1/2 to represent the value "one half"). Students also have difficulty distinguishing between the various meanings of a fraction, "referred to as 'sub-constructs' of rational numbers such as part-whole relation (4 of 5 equal shares), quotient interpretation (implied division, 2 sandwiches divided by 3 boys), measure (fixed quantity on a number line), ratio (5 girls to 6 boys), and multiplicative operator (scaling)." (EDC, 2015, p. Research Foundations)

EM^2 designed the Comparing Two Fractions assessment to diagnose three of the major misconceptions that students hold regarding fraction comparison. The first two misconceptions arise from a lack of understanding of the fraction symbol which leads students "to focus on either the numerators or denominators when ordering or comparing common fractions." (EDC, 2015, p. Research Foundations) When comparing two fractions such as 2/3 to 3/5, they may notice that either/both the 3 and the 5 are greater than the 2 and the 3 so, therefore, they would incorrectly conclude that 3/5 is greater than 2/3. In some cases, like comparing 2/3 to ½, they may obtain the correct answer using flawed reasoning.

Consistently focusing solely on the denominator is considered a separate misconception, misunderstanding the unit fraction. In this case, they rightly understand that a larger denominator makes a smaller unit fraction, fifths are less than fourths, but incorrectly extend that to an idea that 4/5 is less than ¾ because they are focused on the denominator.

"Students may also have difficulty with fact that the two numbers composing a common fraction--the numerator and denominator--are related through multiplication and division, not addition" (EDC, 2015, p. Research Foundations). Students exhibiting this last misconception, described earlier in this paper as gap reasoning, will focus on the difference between the numerator and the denominator of the fraction and inaccurately conclude that fractions such as ¾ and 5/6 are equivalent since the difference between each numerator and denominator is one.

"To develop diagnostic assessments that will support teachers' efforts to identify student misconceptions, the EM^2 Project used an iterative process that drew on the expertise of many individuals to develop each assessment." (EDC, 2015, p. Assessment Research). They assured validity (assessment accurately measures what it is supposed to measure) by employing a "principled and systematic approach" to each assessment design which allowed them to establish content validity and examine the convergent validity of each assessment. They used Susan Embretson's cognitive design framework, Embretson (1998), to develop each assessment. According to EDC (2015), components of the framework include:

- clearly articulating what we want to accomplish with each assessment,
- identifying relevant features in the "task domain" (i.e., what are we asking students to do,
- developing a cognitive model for the assessment (i.e., what are the different types of thinking in which we think students will engage to answer the items),
- generating items according to the cognitive model, and
- evaluating the cognitive model. (p. Assessment Research)

The Education Development Center (2015) describes the *Comparing Two Fractions* Assessment as follows-

> The *Comparing Two Fractions* assessment is designed to elicit information about several common misconceptions that students have when comparing two fractions:
>
> - *Misconception 1 (M1): Viewing a Fraction as Two Separate Numbers / Applying Whole-Number Thinking*
> - *Misconception 2 (M2): An Over-Reliance on Unit Fractions / A Focus on "Smaller Is Bigger"*
> - *Misconception 3 (M3): Numerator and Denominator Have an Additive Relationship / A Focus on the Difference from One Whole* (p. Assessment Research)

Generalizability

The sample selection of the original study combined with the demographics of the participants limits the generalizability of the initial study. Also, the generalizability of the study is limited due to the final sizes of the control and treatment groups. While the original sample was large enough to accommodate at least 30 subjects in one control and seven treatment groups, due to various factors, the actual group sizes ended up with 24 to 29 subjects.

Data Collection

The instructors administered the pretests with paper and pencil at the beginning of the study. Each test consists of seven questions where the student is asked to select the appropriate comparison (<, >, or =) and explain their reasoning. I collected the pretests and consent forms immediately. After receiving their packet of practice problems, the students were given a week to complete them. After collecting the practice problems, the class instructors administered the posttest and turned the problems and tests into me.

Data Analysis

Selection of the variables, construction of the models, and a decision on the appropriate statistical analysis preceded the sorting of the data. In addition, the project required Institutional Review Board (IRB) approval from two institutions prior to implementation.

Variables

The assignment to control and treatment groups is the independent variable. The dependent variable for the first research question is the test score difference between pretest and posttest. The difference in classification of comparison type as analyzed pretest and posttest serves as the dependent variable for the second research question. The main analysis used ANOVA on the test score and classification differences as the dependent variables. Table 1 describes the two models for the research questions.

Table 2
Models for the research questions

Model	Research Question	Independent Variable(s)	Dependent Variable	Comparison Group
1	How does the use of virtual fraction models affect students' ability to mentally compare fractions as reflected by performance on a test designed to identify fraction comparison strategies?	manipulative	Difference in test scores	control
2	How does the use of virtual fraction models affect students' ability to mentally compare fractions as reflected on a scale of fraction understanding?	manipulative	Difference in classification	control

The choice of ANOVA on gain scores over ANCOVA using the pretest score as a covariate for the posttest score is tied primarily to the difference in the research questions that they answer. According to Smolkowski (2020) the ANOVA answers the question of whether the group means change significantly over time (or test occurrence) while the ANCOVA answers the question of whether an individual in one group starting at the same level as an individual in another group can be expected to improve at the same rate. Smolkowski (2020) also identifies three additional factors that favor the choice of ANOVA over ANCOVA-

1. Covariate adjustment can bias results, especially in observational or quasi-experimental studies.
2. “[T]he difference score is an unbiased estimate of true change.” (Rogosa, 1988, p. 180)
3. ANCOVA assumes pretest measurements are made without error.

The raw scores were used to calculate descriptive statistics, specifically mean and standard deviation, for Pretest Scores, Posttest scores, test score differences, pretest classification, posttest classification, and classification differences. A paired-samples t-test showed overall improvement between test score means and classification means, but additional analysis of group differences in an ANOVA failed to reveal significant impacts on the students’ approaches to fraction comparison using the eight control/treatment groups.

Ethics and Human Relations

To maintain privacy and confidentiality, participants in the study were randomly assigned an identification number. Once the pretest and posttest scores were matched with an identification number, the data was entered into spreadsheets with no facility to match the summary data with individual students. All of the pretest and posttest items were completed on paper so there is no online record to access with student names or identifiers.

Chapter 4

Results

Two sets of results are described in this chapter. The first set came from a pilot study that only used a control and three types of virtual manipulatives due to the limited student population and the need to assign manipulatives randomly on a class-wide basis rather than randomly assigned to individuals. This was necessitated by the fact that students were doing the work in class, and it reduced the organizational load on the participating instructors. The second set of results came from the actual study that used a control group, three virtual manipulatives and the various combinations of those virtual manipulatives.

Pilot Study

A pilot study was used to test the process for administration of the pretests, exercises, and posttests as well as to preview the data analysis. In this case, three instructors with six lower-division university classes participated. All classes received the same instructions and tests and each class was randomly assigned a single manipulative which they were to use in class over two weeks to complete the exercises. The total number of students originally in the classes numbered around 250, but about two-thirds of them had to be excluded for various reasons (mostly missing the permission form, one of the tests or the set of exercises).

Sample Distributions

Graphs of the distributions of the pretest and posttest scores across the entire sample show a left skew to the data (Appendix B). This is partially due to the extensive use of algorithmic comparisons by a majority of the students which resulted in mostly perfect scores on the tests for those individuals. The classifications have more of a right skew. This is also likely due to the students using algorithmic calculations as they were classified with a 0 since their understanding of fraction magnitudes could not be determined. The differences between the test scores and the differences between the classifications look more normal, although the middle peak is particularly high indicating a significant number of students had similar differences in their scores/classifications regardless of their initial pretest score/classification.

Descriptive Statistics

Table 3 lists descriptive statistics for the variables in the study: the mean and standard deviation across the entire sample for Pretest Scores, Posttest Scores, Pre/Post-test score difference, Pretest Classification, Posttest Classification, and Pre/Post-test classification difference.

Table 3
Mean and Standard deviation for Pre/Post Test Scores, Pre/Post Classifications and Test/Classification Differences

Variable	N	Mean	Std Dev
Pretest Score	96	5.63	1.81
Posttest Score	96	6.15	1.31
PreClassification	96	1.27	1.48
PostClassification	96	1.39	1.64
Test Difference	96	.52	1.65
Classification Difference	96	.11	1.83

Paired Sample t-tests on Pretest/Posttest and Preclassification/Post classification

A t-test indicated on average, students scored higher on the posttest (M=6.15, SE=0.13) than they did on the pretest (M=5.62, SE=.19). This difference, -0.52, 95% CI [-0.86,-.0.19], was significant t(95)= -3.08, p=.003 and represented a relatively small-sized effect, d=.331. A t-test also indicated on average, students received a higher classification score after the posttest (M=1.39, SE=.17) than they did after the pretest

(M=1.27, SE=.15). This difference, however, -.11, 95% CI [-.49, .26], was NOT significant t(95)=-0.61, p=.542. Also, the Confidence Interval included 0. T-test results are in Table 4.

Table 4
t-test Results: Comparison of Pretest and Posttest Mean Scores of Test and Classification Differences for sample

	Pretest		Posttest			
Category	*M*	*SD*	*M*	*SD*	*t-test*	*df*
Test Score	5.63	1.81	6.15	1.31	-3.08**	95
Classification	1.27	1.48	1.39	1.64	.542	95

Note. M = mean. SD = standard deviation. *Test Score Range 0-7, Classification Range 0-4*.

**$p < .01$. ***$p < .001$.

ANOVA on Test Score Differences

The pretest/posttest score differences were approximately normally distributed across the sample, but after running an ANOVA, the Levene Statistic, p=.014 indicated variance between the groups which violates normality and calls into question using ANOVA for further analysis. In this case, mean pretest/posttest score differences showed no significant effect between manipulatives on Pre/Post-test score differences, F(3, 92)=.78, *p*=.507.

ANOVA on Pre/Post classification differences

An ANOVA on the classification differences yielded slightly different results. The Levene Statistic to test homogeneity of variances showed no significant variance between the groups as indicated by *p*=.811. Again, however, there was no significant effect between manipulatives on Pre/Post-test classification differences, F(3,92)=.45, *p* =.715. In this case, the result was expected since the t-test indicated no significant difference between classifications in the overall sample.

The Study

The study consisted of administering the same sequence of pretest/practice/posttest as the pilot to a different set of introductory university mathematics courses. In this case, the sample size was a bit larger than the pilot, and the out of class practice work allowed students in the same section to be assigned different manipulatives. As a result, various combinations of individual manipulatives were included in the study. The original sample consisted of about 350 students in seven Precalculus with Integrated Review courses (approximately 30 students per section) and five Math Content for Elementary Teachers courses (approximately 25 students per section). Instructors read from a script to describe the study and the sequence of actions, then administered the pretest. After completion of the pretest, students received a randomly assigned packet of practice materials with a manipulative to complete over a week. During that week, I sent messages encouraging the students to complete the packets and emphasizing the need to perform the comparisons non-algorithmically. At the end of the week, the instructor administered the posttest and turned the data over to the researcher. In this case, I still ended up discarding a large percentage of the student data due to missing paperwork or a lack of following directions (no explanation on the practice work, evidence of algorithmic comparison on the practice work, pictures on the practice not matching the assigned manipulative)

Sample Distributions

Graphs of the distributions for the pretest and posttest scores across the entire sample show a left skew to the data similar to the pilot, especially in the posttest scores. Unlike the pilot, the classifications have less skew but are still not very normally distributed. In looking at the differences between the test scores and the differences between the classifications the histograms appear much more normally distributed across the sample. However, Kolmogorov-Smirnov and Shapiro Wilk tests for normality indicate a lack thereof in the samples (See Appendix E). Transformations using logs, roots, reciprocals, and powers did not fix the normality issues. However, "Norton (1951, cit. Lindquist, 1953) analyzed the effect of distribution shape on robustness (considering either that the distributions had the same shape in all the groups or a different shape in each group) and found that, in general, F-test was quite robust, the effect being negligible." (Blanca, 2017) Histograms for the test differences and class differences appear in Appendix C.

Descriptive Statistics

Table 5 lists descriptive statistics for the variables in the study: mean and standard deviation across the entire sample for Pretest Scores, Posttest Scores, Pre/Post-test score difference, Pretest Classification, Posttest Classification, and Pre/Post-test classification difference.

Table 5
Mean and Standard deviation for Pre/Post Test Scores, Pre/Post Classifications and Test/Classification Differences

Variable	N	Mean	Std Dev
Pretest Score	211	4.32	1.97
Posttest Score	211	5.49	1.77
PreClassification	211	2.18	1.18
PostClassification	211	2.93	1.28
Test Difference	211	1.17	2.09
Classification Difference	211	.75	1.4

t-test for Equality of Means on Pretest scores and classifications

An independent sample *t*-test indicated no significance concerning the equality of means on the pretest scores (Table 6) or pretest classifications (Table 7) between the treatment and control groups. Equality of variance between the treatment and control groups showed no significance except for the comparison of treatment group 7 to the control group with regard to classification.

Table 6
Independent sample t-test: Pretest comparison for each group to the control.

	Levene's Test for Equality of Variance		t-test for Equality of Means						
								95% Confidence Interval of the Difference	
Group	F	Sig.	T	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
1	.307	.582	-.169	55	.866	-.0936	.5540	-1.204	1.0168
2	.016	.900	-.922	54	.361	-.5000	.5420	-1.587	.58748
3	.072	.790	-.372	51	.712	-.19429	.52259	-1.243	.85485
4	.023	.880	.419	54	.677	.21429	.51121	-.8106	1.2392
5	.007	.935	-1.28	52	.208	-.67582	.52967	-1.739	.38704
6	.702	.406	.708	48	.483	.37662	.53226	-.6936	1.4468
7	1.501	.226	1.454	51	.152	.72571	.49922	-.2765	1.72795

Table 7
Independent sample t-tetst: PreClassification comparisons for each group to the control

	Levene's Test for Equality of Variance		t-test for Equality of Means						
								95% Confidence Interval of the Difference	
Group	F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	Lower	Upper
1	.117	.734	-.310	55	.758	-.10653	.34409	-.7961	.58305
2	.413	.523	-1.31	54	.196	-.44643	.34109	-1.130	.23742
3	.023	.880	-.575	51	.568	-.20929	.36384	-.9397	.52115
4	2.249	.139	-1.06	54	.295	-.33929	.32075	-.9824	.30378
5	.021	.885	-.65	52	.448	-.28159	.36825	-1.021	.45735
6	3.127	.083	-.927	48	.358	-.31656	.34131	-1.003	.36970

7	4.565	.037	-1.58	51	.121	-.50929	.32321	-1.158	.13959

Paired Sample t-tests on Differences in Test Scores and Classification

A *t*-test indicated on average, students scored higher on the posttest (M=5.4882, SE=0.12) than they did on the pretest (M=4.3175, SE=.14). This difference, -1.17, 95% CI [-1.973,-.527], was significant t(210)= -8.125, *p*=.000 and represented a medium-sized effect, *d*=.63. A *t*-test also indicated on average, students received a higher classification score after the posttest (M=2.9336, SE=.088) than they did after the pretest (M=2.1825, SE=.081). This difference, -.75118, 95% CI [-.9412, -.5612], was significant *t*(210)=-7.793, *p*=.000 and represented a medium-sized effect *d*=0.61. The results for the *t*-tests are shown in Table 8.

Table 8
t-test Results: Comparison of Pretest and Posttest Mean Scores of Test Differences and Classification Differences for sample

	Pretest		Posttest			
Category	*M*	*SD*	*M*	*SD*	*t-test*	*df*
Test Score	4.3175	1.97089	5.4882	1.76571	-8.125***	210
Classification	2.1825	1.18166	2.9336	1.28001	-7.793***	210

Note. M = mean. SD = standard deviation. *Test Score Range 0-7, Classification Range 0-4*.

***p < .01. ***p < .001.*

ANOVA on Test Score Differences

While the pretest/posttest scores appeared somewhat normally distributed across the sample, the normality of the sample was only partially maintained at the level of each manipulative (see Appendix D). An ANOVA on the mean pretest/posttest score differences showed no significant variance between the groups as indicated by Levene Statistic with *p*=.813. However there was no significant effect between manipulatives on

Pre/Post-test score differences, F(7, 203)=1.236, *p*=.284. Therefore I cannot reject the null hypothesis for the first research question:

H_0 1: There is no significant difference in the test score differences between groups of test subjects using assigned manipulatives.

Table 9 displays the complete results of the ANOVA on test score gains.

Table 9
ANOVA results for test score gains

	Sum of Squares	*df*	Mean Square	*F*	*Sig*
Between Groups	37.607	7	5.372	1.236	.284
Within Groups	882.250	203	4.346		
Total	919.858	210			

ANOVA on Pre/Post classification differences

While the classification differences appeared somewhat normally distributed across the sample, they also only nominally maintained that normality in the manipulative assignments. An ANOVA on the classification differences yielded similar results. As with the test score differences, the Levene Statistic to test homogeneity of variances showed no significant variance between the groups as indicated by *p*=.358. Again, however, there was no significant effect between manipulatives on Pre/Post-test classification differences, F(7,203)=1.524, *p* =.161. Therefore, I cannot reject the null hypothesis from the second research question:

H_0 2: There is no significant difference in the classification differences between groups of test subjects using assigned manipulatives.

Table 10 displays the complete results for the ANOVA on classification gains.

Table 10
ANOVA results for classification gains.

	Sum of Squares	*df*	Mean Square	*F*	*Sig*

Between Groups	20.557	7	2.937	1.524	.161
Within Groups	391.130	203	1.927		
Total	411.687	210			

Chapter 5

Conclusion

Discussion

The pilot study conducted in the spring with a control group and the three primary virtual manipulatives revealed potential issues with the approach to fraction comparison used by a significant number of students. Whereas I had anticipated issues with cross products, many more students used decimal conversion and common denominators during both the pretest and the posttest administrations. This led to changes in the initial script emphasizing the need to not change the fractions subject to comparison, and the posting of supplementary instructions to each class in the main project on their Moodle forum after the pretest specifically asking them to avoid the use of algorithmic comparison strategies. Also, about two-thirds of the sample had to be discarded for incomplete or missing paperwork (pretest, posttest, permission, practice material).

Students in the pilot performed the comparison practice activities in class over a two-week period which provided some assurance that the activities were completed using the appropriate tools, but this resulted in a significant impact on class time over the two weeks. The study was redesigned so that the activities became part of a take-home packet for the students to work daily over one week. In addition to saving class time, this also allowed random assignment of the manipulatives within the class sections. Unfortunately, it magnified a major flaw in the implementation of the study- students had little incentive to properly complete the activities

Upon review of the practice activities for the main project, nearly half of the samples were discarded for various reasons- practice items not completed, pictures on practice did not match the tool assigned, use of algorithms to complete the practice, and incomplete or missing paperwork (pretest, posttest, permission, practice material). Even among the ones retained, it was not always possible to verify that students followed the directions.

The manipulative tools have a counter which tracks their usage (Table 11), and the numbers from those tools support the idea that the fraction practice was a possible driver in student improvement as opposed to any tool. Students in the pilot study averaged between two and five tool accesses per student while the students in the main study only averaged between one and three tool accesses per student.

Table 11

Scratch Tool Accesses by Manipulative

Manipulative	Total Times Accessed
Virtual Circle	187*
Virtual Bar	164*
Virtual Number line	194*
Pie/Bar	68
Pie/NL	73
Bar/NL	75
Pie/Bar/NL	85

*Used in both Pilot and Main project

The results of the study were inconclusive with respect to differences between the manipulatives. While the overall sample showed improvement in test score and classification means as indicated by the paired samples T-test, and each manipulative showed improvement in test score (Figure 14) and classification means (Figure 15), the ANOVA analyses showed no significant difference in the improvement in scores from the pretest to the post-test between manipulatives, and no significant difference in the improvement in classification from pretest to posttest between the manipulatives. The overall improvement in mean test scores and mean classification differences leads to the question of whether the manipulatives help generally or if the improvement comes from practicing the comparisons.

Figure 14
Mean Test Improvements

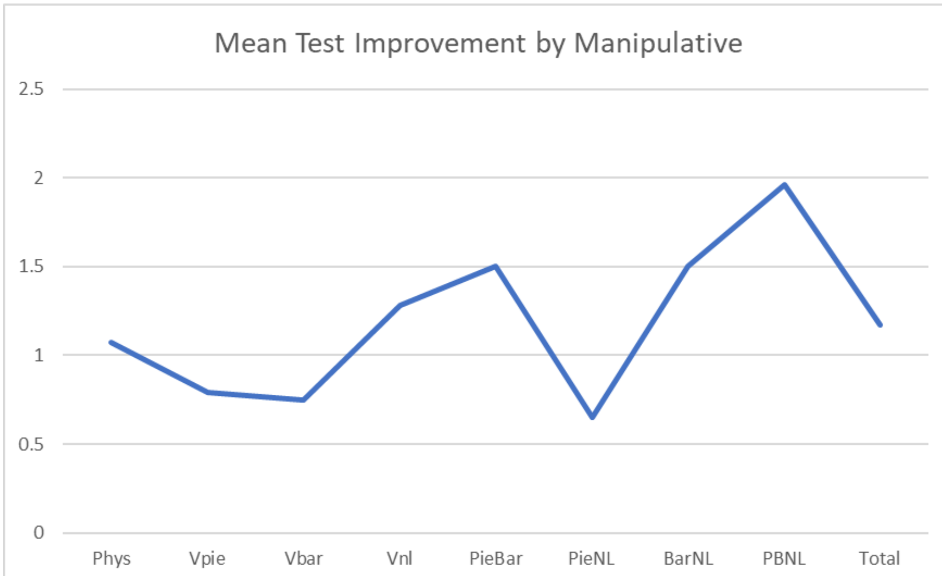
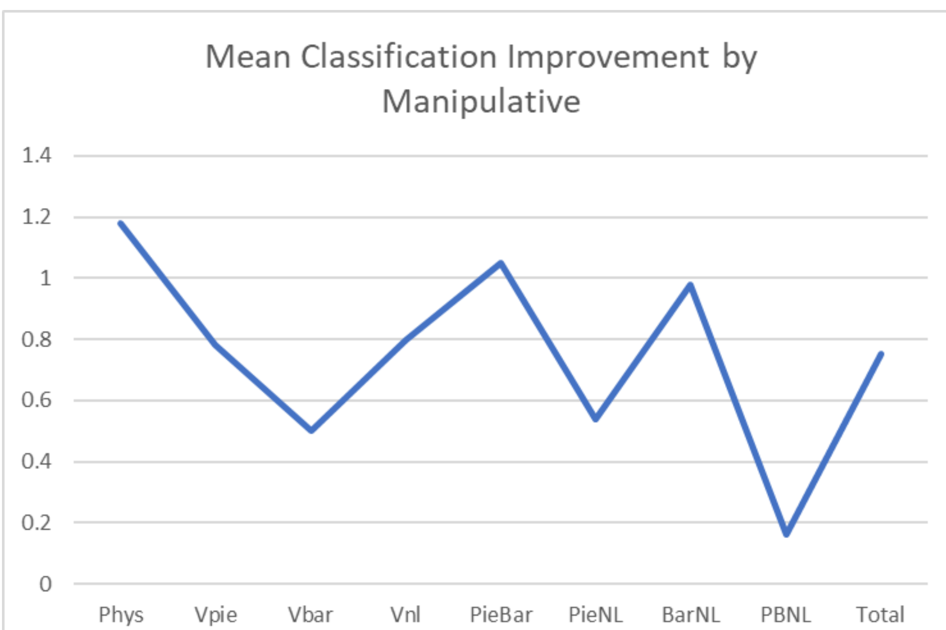


Figure 15
Mean Classification Improvements



Significance of the Study

While I did not see the expected differences between the manipulatives, I was encouraged by the fact that all the manipulative combinations resulted in significant improvements of the variables in the study. The overall improvements have potential implications for mathematics education, mathematics instruction, professional development, and research. In terms of education, if virtual manipulatives have no significant drop off from physical manipulatives, they can be replicated more easily and made more widely available at a potentially lower cost than physical manipulatives. In addition, they do not have small pieces that can get lost (or swallowed).

With current events driving more online instruction, virtual manipulatives provide a natural fit in a remote environment. They can be easily displayed in a remote classroom session, and, while technology can be limiting for underserved populations, virtual manipulatives can be easily distributed where technology is available.

Drawing on the same advantages, virtual manipulatives could become a staple of professional development for teachers. More research on the impacts on early fraction learners would be necessary before fully committing to this avenue as the group of students in this study had all completed high school so they had significant experience with both fractions and their operations. The methods used to teach them comparison specifically are unclear although general practice is to teach common denominators followed by cross products very early in the process.

Future Research

This leaves several unanswered questions. Most obvious, does working practice problems with a manipulative improve performance over just practicing? If students had properly used the manipulatives more consistently, would that have made a difference in the analysis? Does a particular manipulative or combination of manipulatives have more of an effect on any of the types of reasoning identified for this study? Do manipulatives have more impact on fraction learners in elementary school than students that are anywhere from five to eight years removed from initial fraction learning?

I think a better implementation of this study would be to create a control group with no manipulative and conduct it at several different levels of education beginning with elementary students learning about fraction comparison. It might help to have a preconfigured lesson on comparison to accompany the treatment. I found elementary schools to be very protective of their instructional time as I was unable to convince any elementary administrators to assist with my study as it was originally designed. Having a lesson ready to go with the study might convince them that the time will be well spent.

If I were to redesign this study for use at the university level, I would implement it in the teacher education courses during the unit on fractions. Students would take the pretest, and then participate in a lesson on fraction comparison which would include a single homework assignment with 20-25 fraction comparison problems due the next day. Because of the smaller student population, the study would have to span several semesters; four semesters would yield around 150 students for an individual instructor, which would necessitate reconfiguring the groups to the control (no manipulative), a physical fraction circle and the three virtual items and not including the various combinations of manipulatives.

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Appendix A

EDC Instruments and Analysis; Sample Practice Item

Figure A1

EDC Pretest

Comparing Fractions
Pre Assessment

Name____________________

Date________ Class_________

Compare the two fractions provided. Select the choice that shows the relationship between the two fractions.

1) $\frac{5}{12}$ ○ Greater than (>) ○ Less than (<) ○ Equivalent (=) $\frac{3}{5}$	Explain your thinking.
2) $\frac{4}{5}$ ○ Greater than (>) ○ Less than (<) ○ Equivalent (=) $\frac{6}{7}$	Explain your thinking.
3) $\frac{6}{8}$ ○ Greater than (>) ○ Less than (<) ○ Equivalent (=) $\frac{3}{4}$	Explain your thinking.
4) $\frac{7}{9}$ ○ Greater than (>) ○ Less than (<) ○ Equivalent (=) $\frac{3}{5}$	Explain your thinking.

Comparing Fractions
Pre Assessment

Name________________________

Date__________ Class__________

Compare the two fractions provided. Select the choice that shows the relationship between the two fractions.

5) $\frac{3}{7}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{4}{9}$	Explain your thinking.
6) $\frac{7}{8}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{5}{6}$	Explain your thinking.
7) $\frac{4}{7}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{12}{21}$	Explain your thinking.

Figure A2

EDC Posttest

Comparing Fractions
Post Assessment

Name____________________

Date_________ Class_________

Compare the two fractions provided. Select the choice that shows the relationship between the two fractions.

1) $\frac{7}{12}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{5}{8}$	Explain your thinking.
2) $\frac{6}{7}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{8}{9}$	Explain your thinking.
3) $\frac{4}{6}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{2}{3}$	Explain your thinking.
4) $\frac{9}{11}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{5}{7}$	Explain your thinking.

Comparing Fractions
Pre Assessment

Name______________________

Date_________ Class__________

Compare the two fractions provided. Select the choice that shows the relationship between the two fractions.

5) $\frac{2}{5}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{4}{9}$	Explain your thinking.
6) $\frac{9}{10}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{6}{7}$	Explain your thinking.
7) $\frac{3}{7}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{9}{21}$	Explain your thinking.

Figure A3

EDC Test Scoring Guide

Comparing Two Fractions Scoring Guide

Student:	Pre # 1		Pre # 2		Pre # 3		Pre # 4		Pre #5		Pre #6		Pre #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
	Post # 1		Post # 2		Post # 3		Post # 4		Post #5		Post #6		Post #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
Student:	Pre # 1		Pre # 2		Pre # 3		Pre # 4		Pre #5		Pre #6		Pre #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
	Post # 1		Post # 2		Post # 3		Post # 4		Post #5		Post #6		Post #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
Student:	Pre # 1		Pre # 2		Pre # 3		Pre # 4		Pre #5		Pre #6		Pre #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
	Post # 1		Post # 2		Post # 3		Post # 4		Post #5		Post #6		Post #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
Student:	Pre # 1		Pre # 2		Pre # 3		Pre # 4		Pre #5		Pre #6		Pre #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
	Post # 1		Post # 2		Post # 3		Post # 4		Post #5		Post #6		Post #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
Student:	Pre # 1		Pre # 2		Pre # 3		Pre # 4		Pre #5		Pre #6		Pre #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None
	Post # 1		Post # 2		Post # 3		Post # 4		Post #5		Post #6		Post #7		Likelihood?
	Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		Cor M1 M2 M3		M1 M2 M3
	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	Str	Wk	None

Figure A4

Sample Daily Problem Sheet

Day 1 Problems

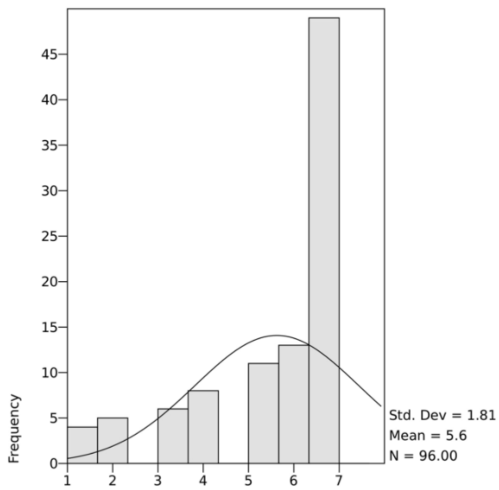
Make a prediction about the comparison, then model the two fractions using the manipulative at the following link:

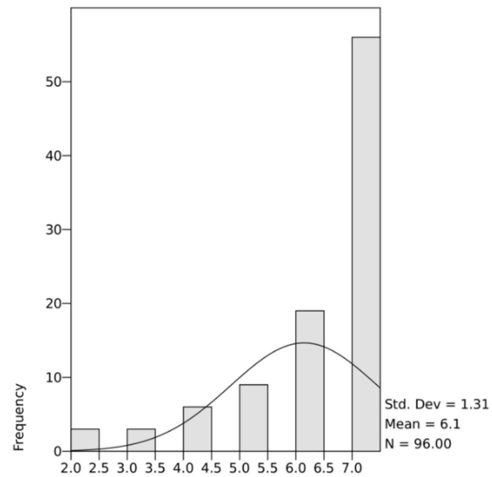
http://bit.ly/PBarNL

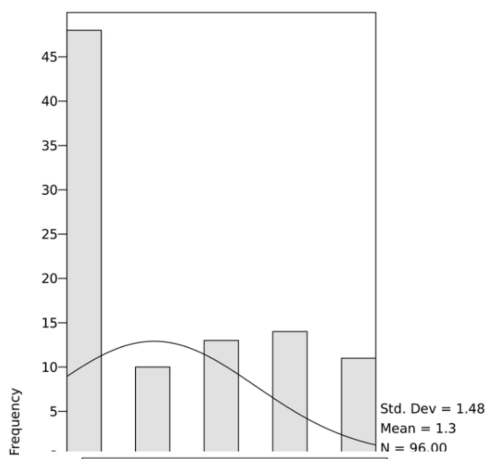
Problem	Explanation
Prediction ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=)	
Comparison $\frac{3}{5}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{5}{8}$	
Prediction ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=)	
Comparison $\frac{3}{4}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{2}{3}$	
Prediction ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=)	
Comparison $\frac{1}{3}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{3}{5}$	
Prediction ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=)	
Comparison $\frac{2}{3}$ ◯ Greater than (>) ◯ Less than (<) ◯ Equivalent (=) $\frac{6}{9}$	

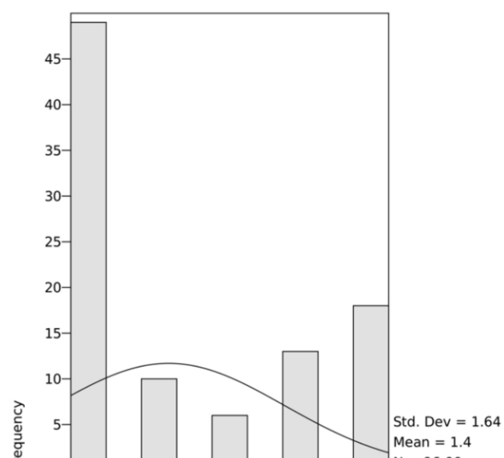
Appendix B

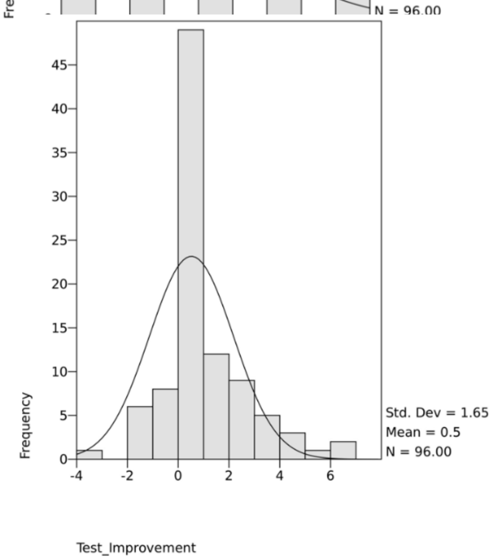
Pilot Study – Histograms

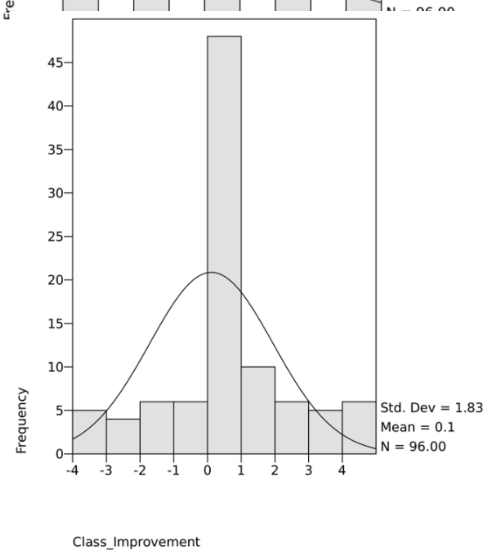






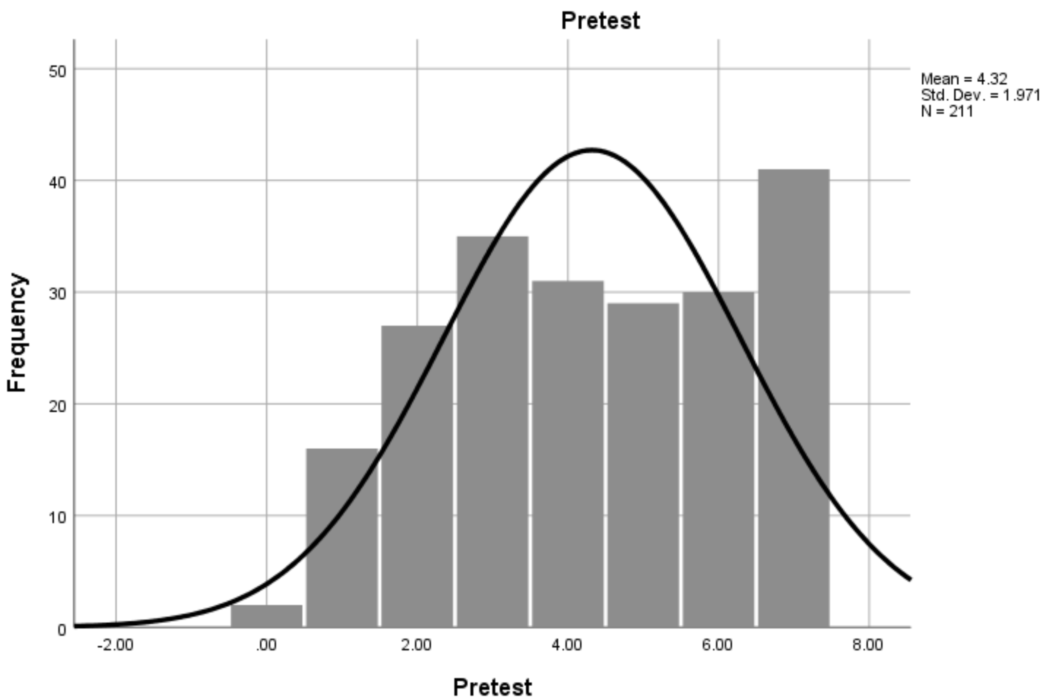


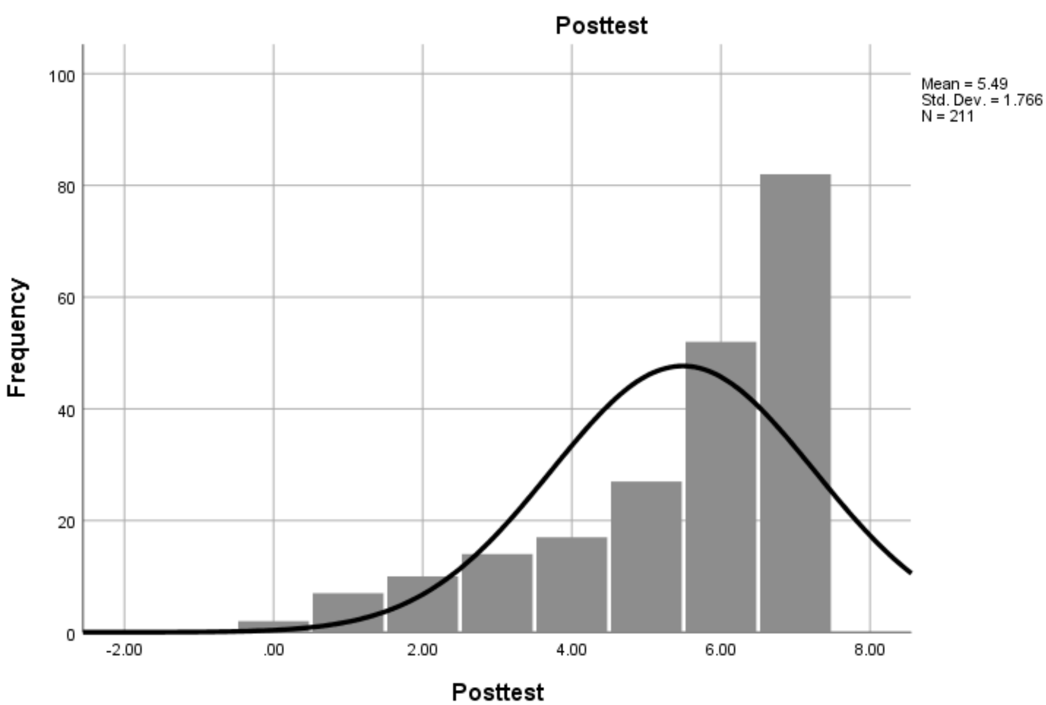


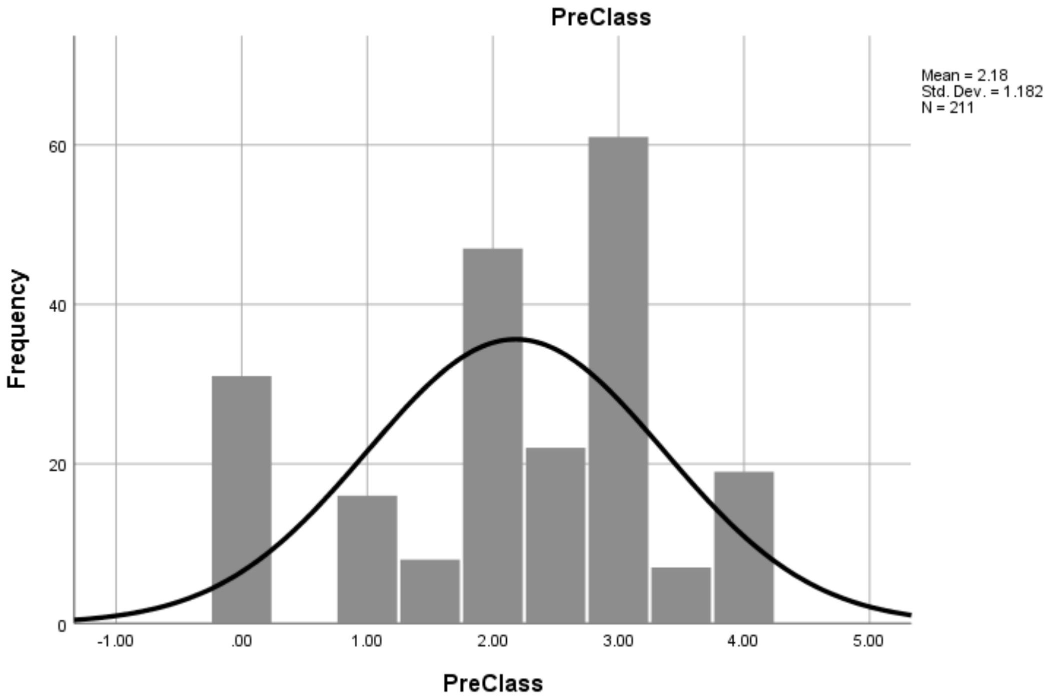


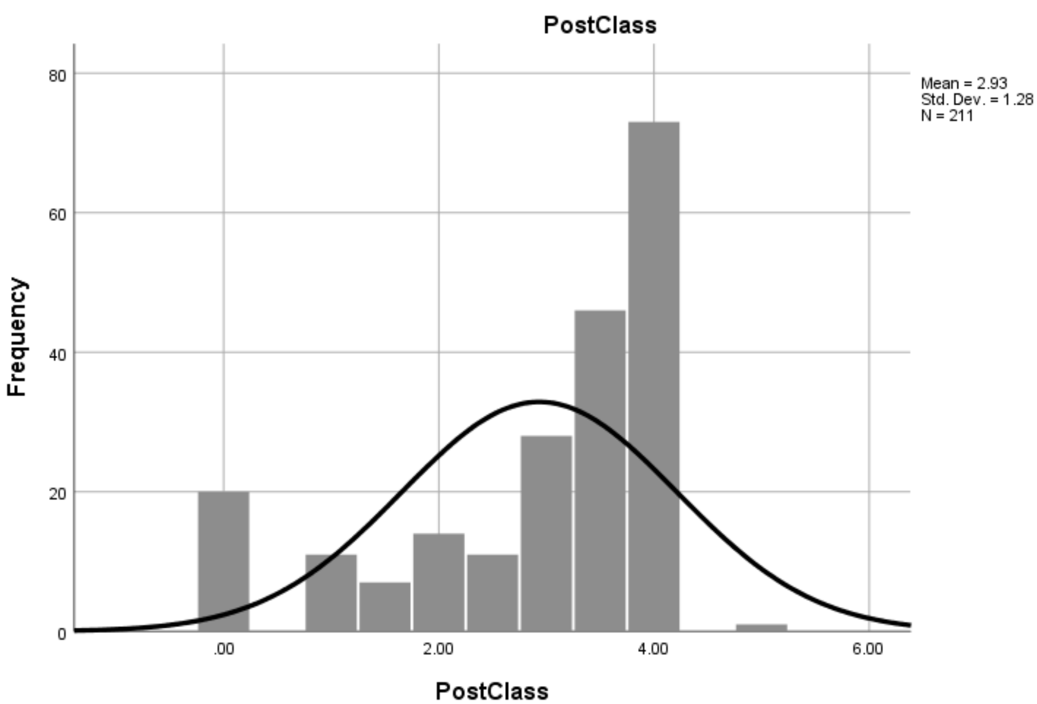
Appendix C

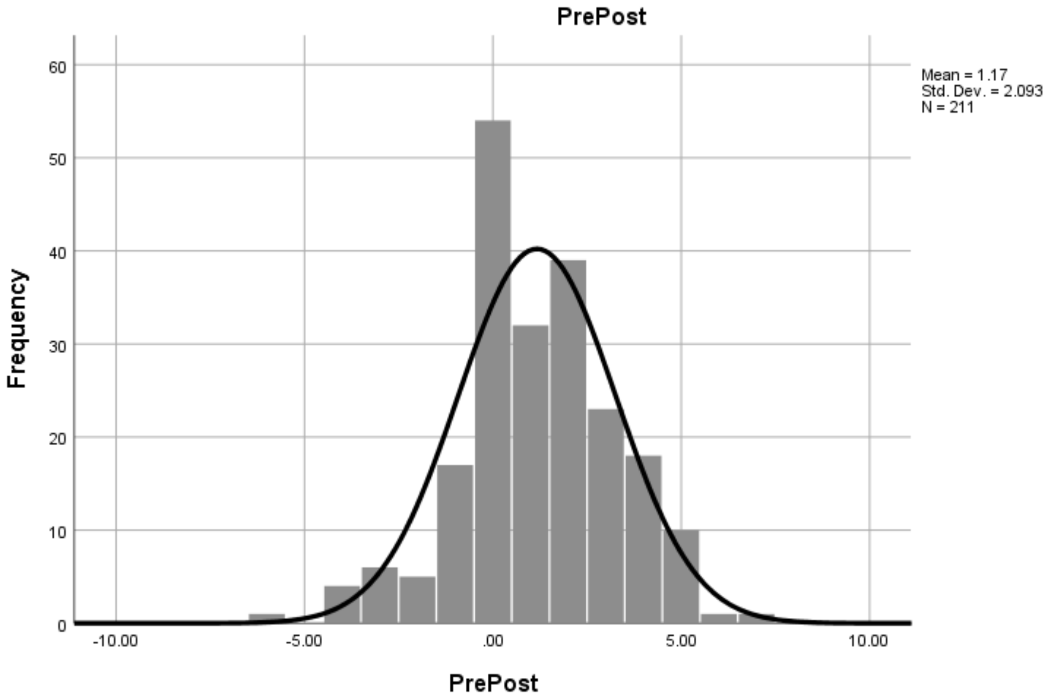
Project Histograms and Boxplots

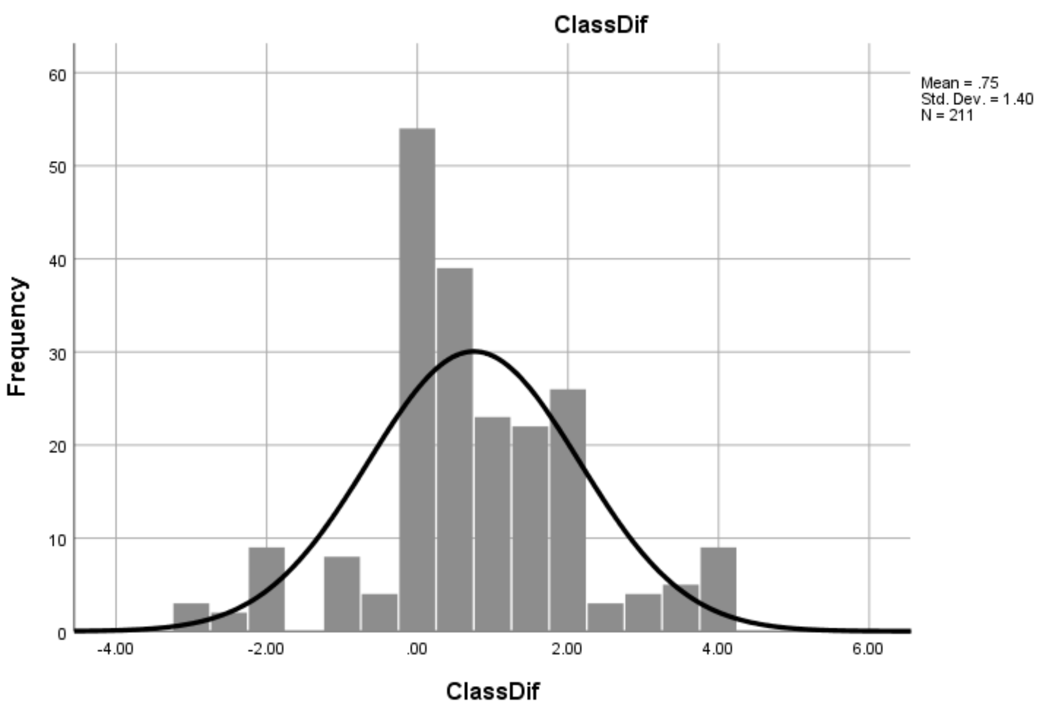


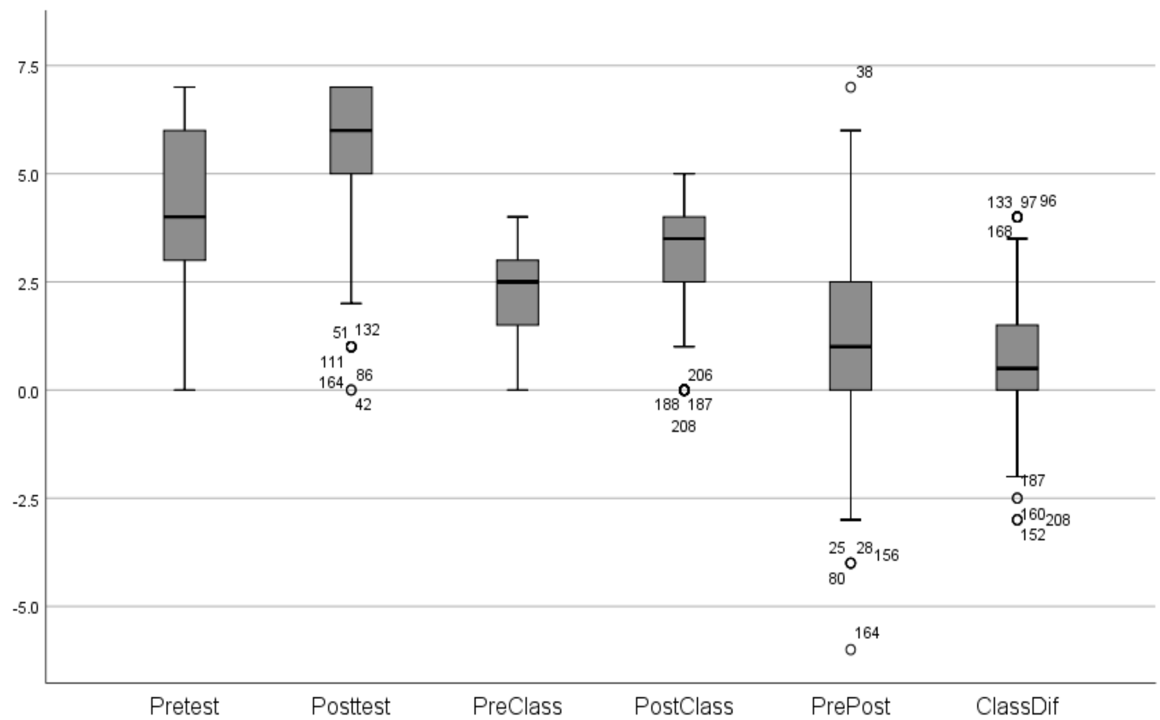



PreClass
Mean = 2.18
Std. Dev. = 1.182
N = 211
Frequency
0
20
40
60
-1.00
.00
1.00
2.00
3.00
4.00
5.00
PreClass


PostClass
Mean = 2.93
Std. Dev. = 1.28
N = 211
Frequency
0
20
40
60
80
.00
2.00
4.00
6.00
PostClass


PrePost
Mean = 1.17
Std. Dev. = 2.093
N = 211
Frequency
0
10
20
30
40
50
60
-10.00
-5.00
.00
5.00
10.00
PrePost

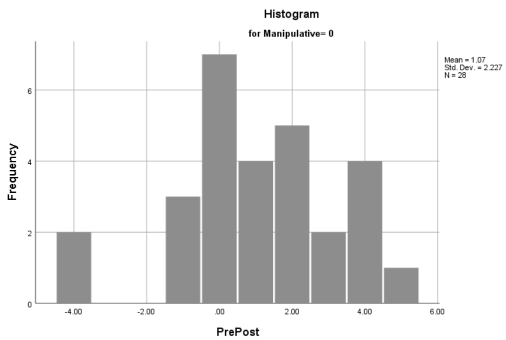

ClassDif
Mean = .75
Std. Dev. = 1.40
N = 211
Frequency
0
10
20
30
40
50
60
-4.00
-2.00
.00
2.00
4.00
6.00
ClassDif

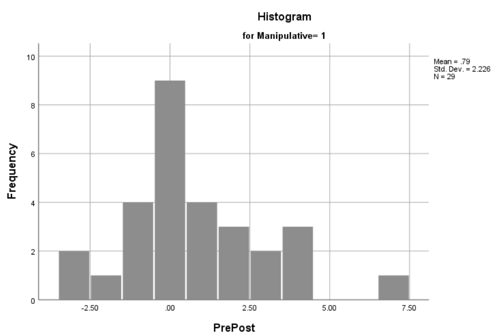

7.5
5.0
2.5
0.0
-2.5
-5.0
Pretest
Posttest
PreClass
PostClass
PrePost
ClassDif
51 132
111 86
164
42
206
188 187
208
38
25 28 156
80
164
133 97 96
168
187
160 208
152

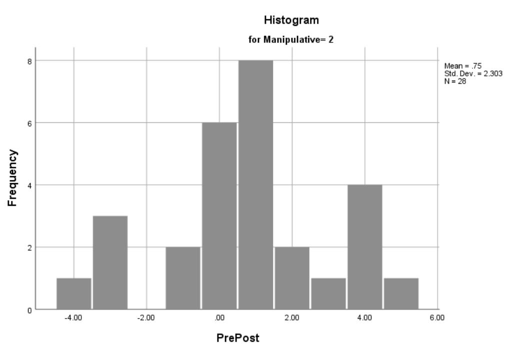
Appendix D

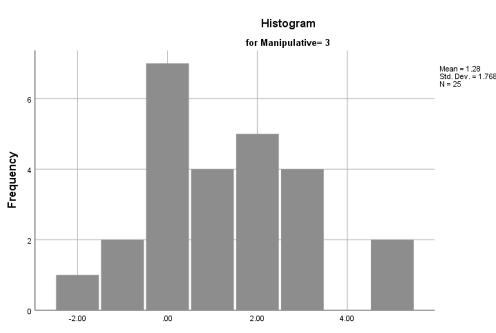
Histograms by Manipulative

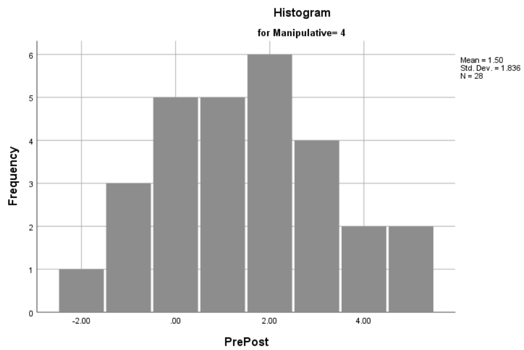
Pre/Post Test Score Differences by Manipulative

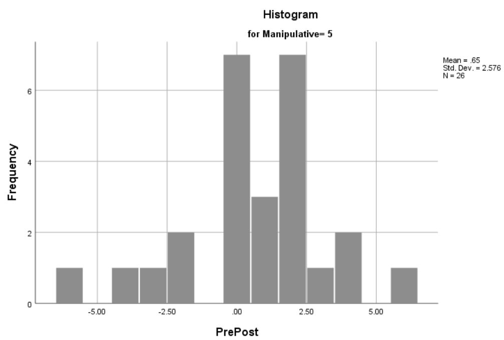


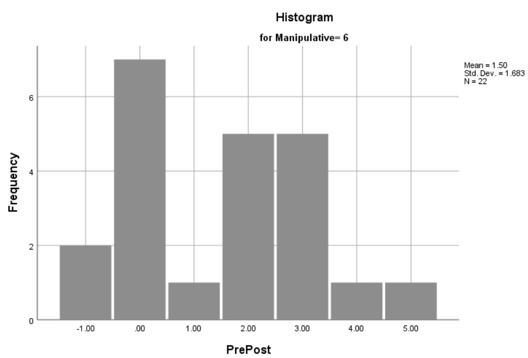


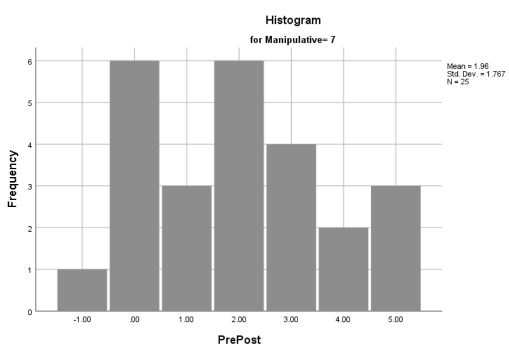




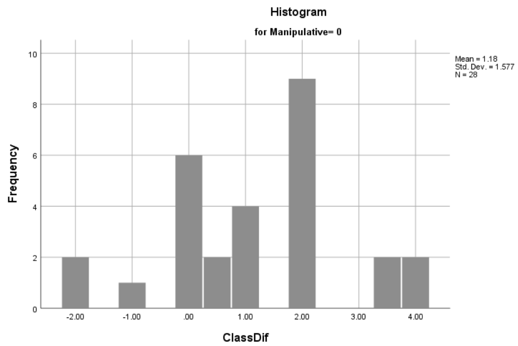

Histogram
for Manipulative= 4
Mean = 1.50
Std. Dev. = 1.836
N = 28
Frequency
PrePost

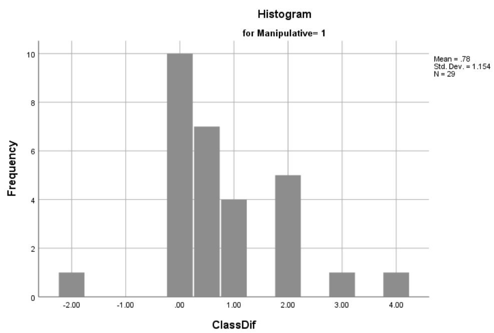

Histogram
for Manipulative= 5
Mean = .65
Std. Dev. = 2.576
N = 26
Frequency
PrePost

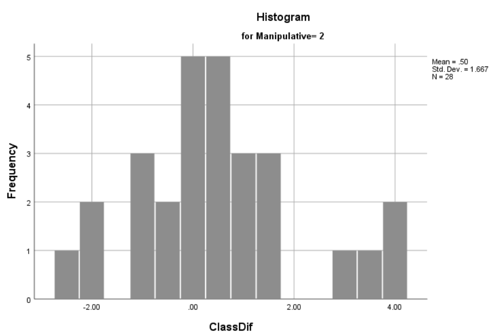

Histogram
for Manipulative= 6
Mean = 1.50
Std. Dev. = 1.683
N = 22
Frequency
PrePost

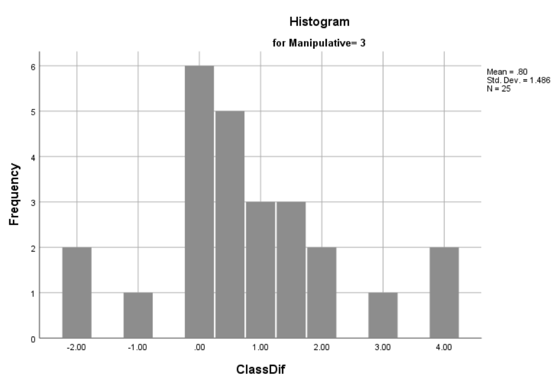

Histogram
for Manipulative= 7
Mean = 1.96
Std. Dev. = 1.767
N = 25
Frequency
PrePost

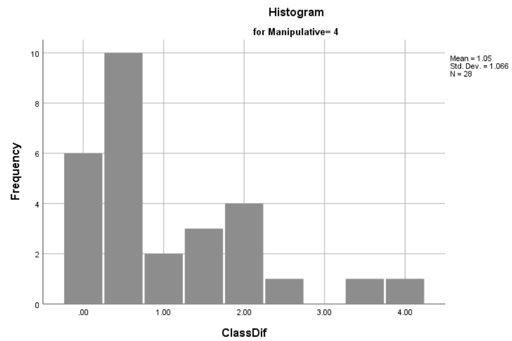
Pre/Post Classification Differences by Manipulative

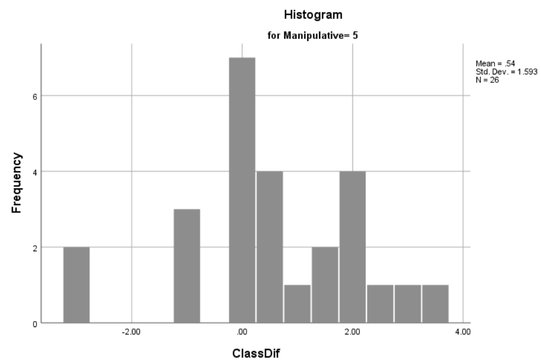


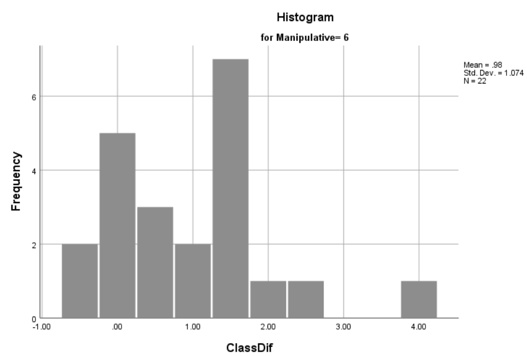


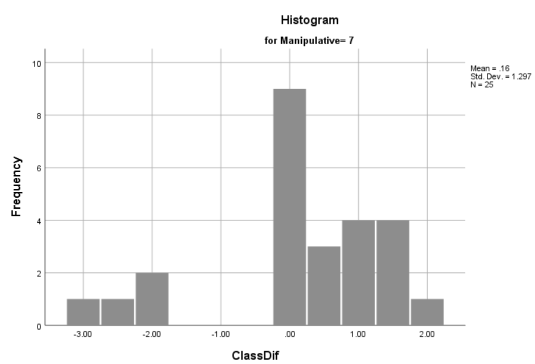





Histogram
for Manipulative= 4
Mean = 1.05
Std. Dev. = 1.066
N = 28
Frequency
ClassDif
0
2
4
6
8
10
.00
1.00
2.00
3.00
4.00


Histogram
for Manipulative= 5
Mean = .54
Std. Dev. = 1.593
N = 26
Frequency
ClassDif
0
2
4
6
-2.00
.00
2.00
4.00


Histogram
for Manipulative= 6
Mean = .98
Std. Dev. = 1.074
N = 22
Frequency
ClassDif
0
2
4
6
-1.00
.00
1.00
2.00
3.00
4.00


Histogram
for Manipulative= 7
Mean = .16
Std. Dev. = 1.297
N = 25
Frequency
ClassDif
0
2
4
6
8
10
-3.00
-2.00
-1.00
.00
1.00
2.00

Appendix E

Assumptions for t-tests and ANOVA

Assumption 1: Independence

Based on the research design, the data was randomly and independently sampled so the assumption is met.

Assumption 2: Scale of Measurement

All of the variables used for the t-tests and ANOVAs have scaled values.

Test	Independent Variable	Dependent Variable	Scale
T-test Equality of Means between manipulatives, Pretest	Manipulative	Pretest score	0-7
T-test Equality of Means between manipulatives, PreClassification	Manipulative	PreClassification	0-4
T-test, test score difference	Pretest Score	Posttest Score	-7 to 7
T-test, classification difference	PreClassification	PostCassification	-4 to 4
ANOVA on test score difference	Manipulative	Test score difference	-7 to 7
ANOVA on classification difference	Manipulative	Classification difference	-4 to 4

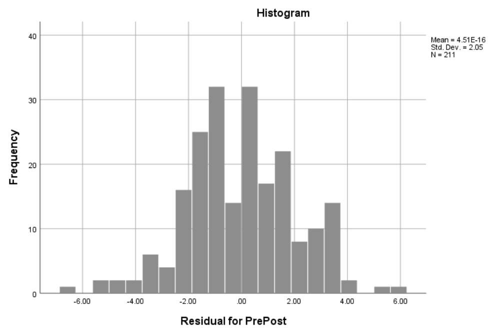
Assumption 3: Normality

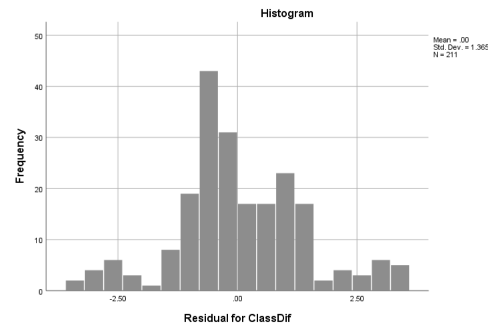
While the histograms and QQ plots for score and classification differences appear fairly normal as do the histograms and QQ plots for their residuals, both the K-S and Shapiro-Wilk tests for normality indicate otherwise.

Variable	Skewness (range -1 to 1)	Kurtosis (range -1 to 1)	K-S	Sig	Shapiro-Wilk	Sig
Posttest score	-1.198	.580	.249	.000	.811	.000
Post Classification	-1.173	.230	.240	.000	.800	.000
Test score difference	-.175	.425	.132	.000	.967	.000

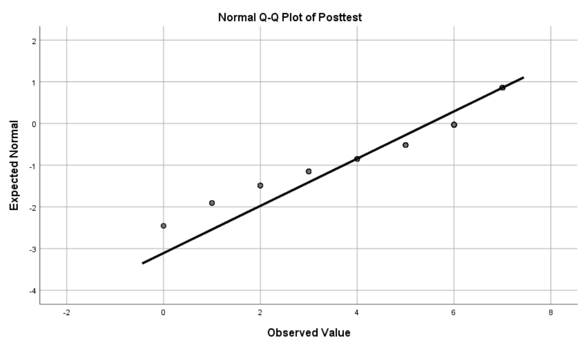
Residual for test score difference	-.032	.336	.052	.200	.992	.317
Classification difference	.080	.685	.173	.000	.947	.000
Residual for classification difference	.143	.544	.081	.000	.974	.001

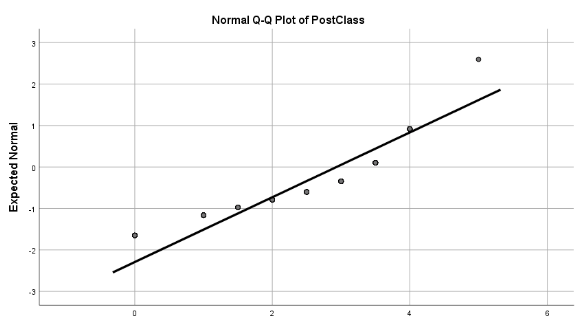
Histograms for Residuals (other histograms included in Appendix C)



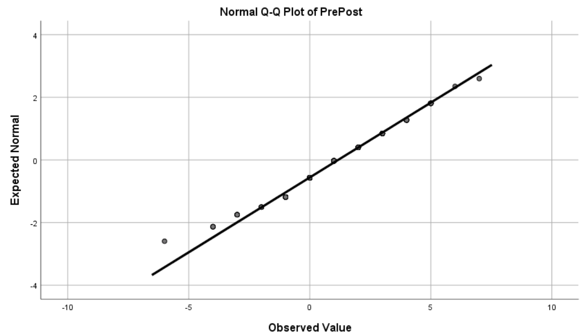


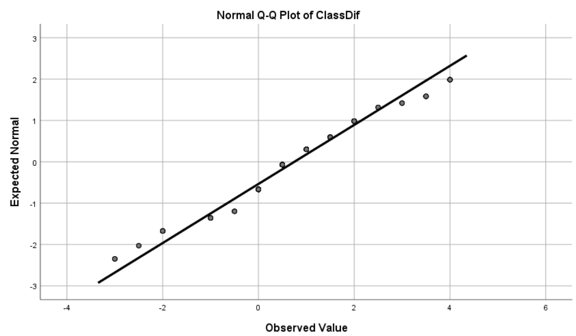
QQ Plots for Post Test/Classification



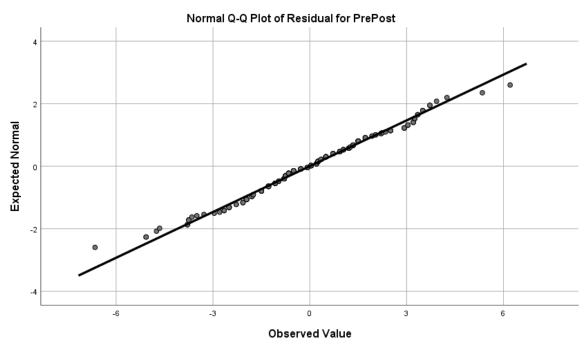


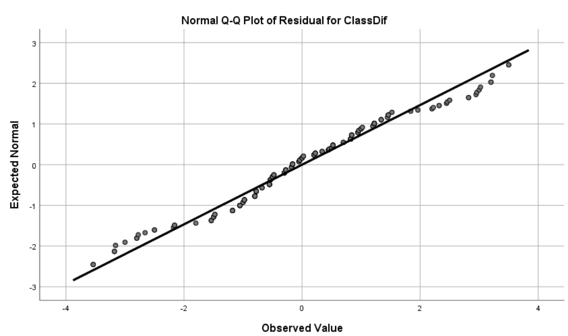
QQ Plots for Test/Classification Differences





QQ Plots for Residuals of Test/Classification Differences





Assumption 4: Homogeneity/Equality of Variance (Independent Sample t-test and ANOVA)

The Levene Statistic for the independent sample t-test on Pretest/Postest Scores shows no significant variance, but for the Preclassification/post-classification pair in the independent sample t-test shows significant variance. The Levene statistic for the two ANOVA tests shows no significant variance.

Test	**Levene Statistic**	**Significance**
Independent Sample t-test on Pretest/Posttest scores	1.013	.423
Independent Sample t-test on Pre/Post classifications	2.177	.038
ANOVA on test score differences	.527	.813
ANOVA on classification differences	1.110	.358