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Construct Local Quasi-Interpolation Operators Using Linear B-Splines

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Abstract

The data interpolating problem is a fundamental problem in data analysis, and B-splines are frequently used as the basis functions for data interpolation. In the real-world applications, the real-time processing is very important. To achieve that, we cannot use any matrix inversion for large amount of data, and we also need to avoid using any global operator. To solve this problem, we develop a new method based on a local quasi-interpolation operator. To construct the local quasi-interpolation operator, we need to factorize the Shoenberg-Whitney matrices for the given data samples. Furthermore, our local quasi-interpolation operator should correspond to a band matrix with the minimum bandwidth, which is critical for the real-time data processing. Finally, we bridge the gap between our local quasi-interpolation operator and a local spline interpolation operator through an impulse interpolation operator using a "blending" method.

Key Words:B-spline/ Cardinal B-spline/ Reproduction and Marsden's Identity/ Shoenberg-Whitney/ Quasi interpolation/ Coefficients of the Marsden's Identities.

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Contents

1 Introduction **5**

2 Preliminaries **10**
2.1 B-splines 10
2.2 Spline evaluations and interpolations 13
2.3 Approximation orders and polynomial reproduction 29
2.3.1 Polynomial reproduction and Marsden's identity 29

3 Shoenberg-Whitney matrices **34**
3.1 General representation for linear Shoenberg-Whitney matrix 34
3.2 Inverses of linear Shoenberg-Whitney matrix for special cases 36
3.3 Inverses of linear Shoenberg-Whitney matrix for the general case . . 44

4 Local Quasi-Interpolating Operators Based-on B-splines **51**
4.1 Framework for local quasi-interpolation 51
4.1.1 Linear operators induced by data points 51
4.1.2 Polynomial-preservation property 52
4.1.3 Blending linear operator 54
4.2 Properties of local quasi-interpolation operators L_y 55
4.2.1 Polynomial-preservation condition for DI-operators 55

5 Local Quasi-Interpolation Operator for Linear B-Splines by Factorization on Shoenberg-Whitney Matrix **63**
5.1 Divided-difference matrices 63
5.2 Matrix factorization with divided-difference matrices 65
5.3 Determine the Band Matrix of Approximate Inverse for a Special Case 73
5.4 Determine the Band Matrix of Approximate Inverse for the General Case 77
5.5 Local Quasi-Interpolating Operator for Linear B-Splines 86

6 Conclusion and Future Research **90**
6.1 Quadratic B-spline - General case 91

6.2 Cubic B-Spline - General Case 97

Chapter 1

Introduction

The word “Spline”, originated from East Anglian dialect, means something elastic, such as a piece of thin wood or metal slat, that was used as a tool in shape design at ancient time. In mathematics, we use the splines to represent a special type of functions - *piecewise-polynomials*, with certain smoothness conditions at the joint points of two adjacent polynomial pieces. The spline functions have many applications in the real-world due to its excellent mathematical properties. There are several industry standards based on the splines. They are an important tool in computer graphics.

In data analysis, the spline functions provide us a powerful tool due to its simplicity and flexibility. A fundamental problem in data analysis is the *data interpolotion problem*, and the spline-based interpolation is the most commonly used method. However, there are still some difficult problems to be solved in data interpolation. When the data size becomes larger and larger, the data processing time becomes slower and slower. If the response time is too long, then those data processing methods would not be practical in many real-world applications.

In the data interpolation problem, when we use the *direct* method, we need to solve a linear system, in which the matrix size corresponds to the data size. Suppose that there are n data points to be interpolated. Then our data interpolation problem involves a matrix of size $n \times n$. If the data interpolating method requires computation of the inverse of this matrix, then the running time complexity function would be in $O(n^3)$, which is not acceptable for large number n. So the first natural question in this situation is: Is it possible to avoid the matrix inverse operation in data interpolation? There is an interesting idea to solve this problem: If one can find a *local quasi-interpolating operator* for the B-splines, then the matrix inverse can be eliminated from the algorithm.

However, constructing a local quasi-interpolating operator is not an easy task. There are two methods for the construction, and they have these properties: 1) The method relies on some special setting between the knots and data samples; 2) The construction procedure is extremely complicated, and it is very hard to implement it. These two "drawbacks" make the methods not very *user-friendly* in the real-world applications. Therefore, to construct a "perfect" local quasi-interpolating operator, we need to fix the above two drawbacks. Specifically, we would like to make our local quasi-interpolating operator have the following properties:

- For any valid knots-samples setting (i.e. satisfying the Shoenberg-Whitney condition), the local quasi-interpolating operator can be defined using explicit formulas.

- The construction procedure is easy to follow, and the performance of the algorithm is fast (in terms of $O(n)$ for the data size n).

The rational behind using the quasi-interpolating operators instead of the *direct* interpolating operators is: We divide the interpolation into two steps: 1) Use a quasi-interpolating operator to *approximate* the data while preserving adequate precision (i.e. certain approximation order); 2) Interpolate the data *exactly* by bridging the gap. This two-step approach has an advantage over the one-step approach, because the room that we can approximate the data in the first step allows us to make the operator local, which is essential for achieving the real-time data processing. Then we apply a special "blending" method to make up the difference part to complete the whole interpolation.

Since it is very difficult to solve the above problem *completely*, in this dissertation, we would like to solve the problem for the *linear* B-splines (the order $m = 2$). The method developed for this *relatively simple* case would establish the foundation for solving the whole problem in the future. Let us give an overview of our method below.

Let $\{B_{k,m}(x)\}_{k=0}^{n-1}$ denote the set of the m-th order B-spline basis functions on the interval $[a, b]$ with the knot sequence $\{t_k\}_{k=-m+1}^{n}$. Let $S_{m,t}$ be the space of spline functions on $[a, b]$ defined as follows

$$S_{m,t} = \left\{ \sum_{j=0}^{n-1} c_j B_{j,m}(x) \,|\, c_j \in \mathbb{R} \text{ for } 0 \leq j \leq n-1 \right\}.$$

The space $S_{m,t}$ is a linear space of dimension n, which will be used to approximate the continuous functions in $C[a, b]$.

When we *interpolate* given data points, we need to find a function in $C[a, b]$ that takes the given values at the given locations. Since there are many different ways to do the interpolation, we want to find a function with good mathematical properties. We start our search in $S_{m,t}$. To find a local linear operator on $S_{m,t}$ that maps the given data to a function in $S_{m,t}$ satisfying the *interpolating* condition, it is extremely hard in general. Then we extend $S_{m,t}$ to a larger subspace of $C[a, b]$, so that we have more freedom to find an interpolating function in it. Specifically, we insert a few appropriate new knots into the original knot sequence, i.e. $t \subset t^*$, then we get a larger spline space, denoted by S_{m,t^*}, i.e. $S_{m,t} \subset S_{m,t^*}$. In other words, we increase the dimension of the initial subspace of $C[a, b]$ to get more flexibility, so that we can derive an appropriate interpolating function $g(x)$ in the larger subspace of $C[a, b]$ without using matrix inverse. That local interpolating linear operator will be a quasi-interpolating operator so as to achieve certain approximation order for the approximation function.

The quasi-interpolating operators, first introduced by De Boor and Fix in [7]. More precisely, the spline approximation operator $Q_m : C[a, b] \to S_{m,t}$ with knot sequence $\{t_k\}$ is called a quasi-interpolating operator if it reproduces polynomials of degree $\leq m - 1$; that is,

$$(Q_m p)(x) = p(x), \quad x \in [a, b]$$

for polynomials $p \in \pi_{m-1}$. This property provides us necessary precision in data processing. Our construction heavily relies on the coefficients of the Marsden's identity, which describe the polynomial reproduction property for the B-splines.

In 1970, Marsden in [13] expressed $(\cdot - y)^\alpha$ in terms of a linear combination of B-splines, which is called Marsden's identity. This identity plays an important role in change of basis procedures and B-spline curve approximation. Moreover this identity is deeply studied and extended in various settings by many researchers. Denote $\{\rho^r_{k,m}, r = 0, 1, ..., m - 1\}$ as the coefficients of the Marsden's Identities given by

$$\rho^0_{k,m} = 1,$$

$$\rho^r_{k,m} = \frac{1}{\binom{m-1}{r}} \sum_{k-m+2 \leq j_1 < j_2 < \ldots < j_r \leq k} t_{j_1} t_{j_2} \ldots t_{j_r}, \quad 1 \leq r \leq m - 1$$

with $0 \leq k \leq n - 1$, and we have

$$x^r = \sum_{k=0}^{n-1} \rho^r_{k,m} B_{k,m}(x), \qquad \text{for } 0 \leq r \leq m - 1.$$

A matrix version criterion of the Marsden's identity provides us a convenient tool in dealing with the linear operators related to the polynomial preservation property. To describe the linear operators used to define the quasi-interpolating operators, we introduce a concept called the *data-induced* (DI) operator, which connects a linear operator to the given data samples. Then a local linear operator corresponds to a band matrix. Thus, to find a local quasi-interpolating operator, we need to find a band matrix that satisfies the above mentioned matrix version criterion of the Marsden's identity.

Another important tool to study the data interpolation is the Shoenberg-Whitney matrix. In the *direct* interpolation problem using the B-splines, the inverse of the Shoenberg-Whitney matrix is needed. Since we want to avoid any matrix inverse operation, because it has two big drawbacks: 1) Computation of the inverse is very time-consuming; 2) The inverse in general is a global matrix which does not support real-time data processing. Therefore, we will use a band matrix to approximate the inverse of the Shoenberg-Whitney matrix under the criterion for preserving certain number of polynomial orders.

To this end, we introduce a concept called *approximate inverse*, for which we use a band matrix to approximate the inverse of some matrix such that their difference is orthogonal to a special subspace. This special subspace is formed from the given data. Then we develop an algorithm to find the approximate inverse of the Shoenberg-Whitney matrix through the matrix factorization technique. Our matrix factorization technique is analogous to the polynomial factorization in this way: When a polynomial takes the zero value at some point, then we can factorize a linear factor from it. In our matrix case, when a matrix is orthogonal to a vector, we can factorize a special matrix factor, which is called a *divided-difference* matrix. We use the divided-difference matrices as the building blocks to construct the approximate inverse of the inverse of the Shoenberg-Whitney matrix.

After we find the approximate inverse of the linear Shoenberg-Whitney matrix, we observe an interesting property: the *duality* property between the linear Shoenberg-Whitney matrix and its approximate inverse. This *duality* property only occurs in the linear B-spline case. In other words, when the spline order $m > 2$, we will not have this property anymore. The reason behind this is: For the linear B-splines, the relationship between the knot sequence $\{t_k\}_{k=-1}^{n}$ and the data samples $\{y_k\}_{k=0}^{n-1}$ has some duality property. Specifically, we have the following conditions;

$$a = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-2} < t_{n-1} = t_n = b,$$

$$a = y_0 < y_1 < y_2 < \cdots < y_{n-2} < y_{n-1} = b,$$

and

$$t_{i-1} < y_i < t_{i+1}, \qquad \text{for} \quad 1 \le i \le n-2,$$

or equivalently,

$$y_{i-1} < t_i < y_{i+1}, \qquad \text{for} \quad 1 \le i \le n-2.$$

If we define $\vec{t} := (t_1, \ldots, t_{n-2})$ for the inner knots, and $\vec{y} := (y_1, \ldots, y_{n-2})$ for the inner samples, we can see that there is some symmetry property between $\vec{t}$ and $\vec{y}$.

This symmetry property also occurs in the linear Shoenberg-Whitney matrix and its approximate inverse. Let $\mathcal{B}_2(\vec{t}, \vec{y})$ be the Shoenberg-Whitney matrix for the linear B-splines, where we treat $\vec{t}$ and $\vec{y}$ two sets of variables in the matrix-valued function $\mathcal{B}_2(\vec{t}, \vec{y})$. We found that $\mathcal{B}_2(\vec{y}, \vec{t})$, switching the positions of $\vec{t}$ and $\vec{y}$ in $\mathcal{B}_2(\vec{t}, \vec{y})$, is an approximate inverse of $\mathcal{B}_2(\vec{t}, \vec{y})$. Here we would like to point out that there are many approximate inverses for $\mathcal{B}_2(\vec{t}, \vec{y})$, and $\mathcal{B}_2(\vec{y}, \vec{t})$ is just one of them. This view is quite natural, because we have the similar view in the general function approximation: One function can be approximated by many different functions, even with the same approximation order. This property makes the computation of the quasi-interpolating operator for the linear B-splines *extremely* easy.

We will organize our presentation in the following chapters as follows. In Chapter 2, we provide the preliminaries for our theory development. In Chapter 3, we obtain the explicit formulas for the inverse of the Shoenberg-Whitney matrix for the general linear B-splines. In Chapter 4, we study the main properties for the quasi-interpolating operators and the data interpolating scheme. In Chapter 5, we apply the matrix factorization technique to find the approximate inverse of the Shoenberg-Whitney matrix for the general linear B-splines. In Chapter 6, we describe some future research problems.

Chapter 2

Preliminaries

2.1 B-splines

Let m be a positive integer and let $t = (t_j)$ be the knot vector or knot sequence, which is a nondecreasing sequence of real numbers of length at least $m+1$ satisfying the following condition

$$t_j > t_{j-m},$$

so that the B-spline defined on this knot sequence is non-vanishing. Then we can define the B-spline functions by the following recursive formulas.

Definition 2.1.1. *The j-th B-spline of order m with knots t (denoted by $B_{j,m,t}(x)$) is defined via*

$$B_{j,m,t}(x) = \frac{x - t_{j-m+1}}{t_j - t_{j-m+1}} B_{j-1,m-1,t}(x) + \frac{t_{j+1} - x}{t_{j+1} - t_{j-m+2}} B_{j,m-1,t}(x) \tag{2.1.1}$$

for all real numbers x, with

$$B_{j,1,t}(x) = \begin{cases} 1, & \text{if } \; t_j \leq x < t_{j+1}, \\ 0, & \text{otherwise.} \end{cases}$$

To simplify the notation $B_{j,m,t}(x)$ slightly, we usually drop the t part, that is, we use $B_{j,m}(x)$ to represent a B-spline function with the underlying knot sequence t.

Example 2.1.2. (B-Splines of order 2) *An application of the recurrence relation* (2.1.1) *gives:*

$$B_{j,2}(x) = \frac{x - t_{j-1}}{t_j - t_{j-1}} B_{j-1,1}(x) + \frac{t_{j+1} - x}{t_{j+1} - t_j} B_{j,1}(x)$$

$$= \begin{cases} \dfrac{x - t_{j-1}}{t_j - t_{j-1}}, & \textit{if} \quad t_{j-1} \le x < t_j, \\[2ex] \dfrac{t_{j+1} - x}{t_{j+1} - t_j}, & \textit{if} \quad t_j \le x < t_{j+1}, \\ 0, & \textit{otherwise.} \end{cases}$$

When we apply the recursive formula (2.1.1), we may encounter the case that $t_{j-m} = t_j$ in evaluating $B_{j-1,m}(x)$, in which we assume that

$$B_{j-1,m,t}(x) = 0 \quad \text{for all } x \text{ when } t_{j-m} = t_j,$$

which corresponds to the vanishing case based on the geometric understanding. This assumption is compatible with the recursive formula (2.1.1).

Notice that a B-spline is determined by its knot sequence, we introduce an alternative notation for the B-spline to reflect this consideration,

$$B_{j,m,t}(x) = B(x|t_{j-m+1}, t_{j-m+2}, \ldots, t_{j+1}).$$

The definition of the B-Splines implies the *translation invariance property*, i.e,

$$B(x + y|t_{j-m+1} + y, \ldots, t_{j+1} + y) = B(x|t_{j-m+1}, \ldots, t_{j+1}), \quad x, y \in \mathbb{R}.$$

It is easy to see that the support of $B_{j,m}(x)$ is $[t_{j-m+1}, t_{j+1}]$, and $B_{j,m}(x) > 0$ on (t_{j-m+1}, t_{j+1}).

The knots do not have to be distinct. If a knot is repeated r times, then we call that the *multiplicity* of this knot is r. The multiplicity of a knot will affect the *smoothness* of the spline at this knot, that is, $B_{j,m}(x) \in C^{d-r}$ in a neighborhood of this knot, where d is the degree of the B-spline, which is $m - 1$. Hence the maximum possible multiplicity of a knot for $B_{j,m}(x)$ is m, in which $B_{j,m}(x)$ is discontinuous at this knot. For example, let $z = t_{j-m+1} = \ldots = t_j < t_{j+1}$, then $B_{j,m}(z) = 1$ and $B_{i,m}(z) = 0$ for $i > j$ or $i < j - 1$.

We can also find the derivative and integral of a B-spline as follows,

$$B'_{i,m}(x) = (m-1)\left(\frac{B_{i-1,m-1}(x)}{t_i - t_{i-m+1}} - \frac{B_{i,m-1}(x)}{t_{i+1} - t_{i-m+2}}\right), \tag{2.1.2}$$

and

$$\frac{m}{t_{i+1} - t_{i-m+1}} \int B_{i,m}(x)dx = 1.$$

The simplest case for the B-splines is the *Cardinal B-Splines*, in which the knots are integers. We denote the cardinal B-Splines of order $m \geq 1$ by

$$N_m(x) = B(x|0, 1, \ldots, m), \quad x \in \mathbb{R}.$$

Then the recurrence relation (2.1.1) for $m \geq 2$ becomes

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1). \tag{2.1.3}$$

It is easy to see that the support of $N_m(x)$ is $[0, m]$ and $N_m(x) > 0$ in $(0, m)$. It also has the *partition of unity* property,

$$\sum_{-\infty}^{\infty} N_m(x-k) = 1, \qquad \text{for all } x \in \mathbb{R}.$$

Other than the recursion formula (2.1.3), $N_m(x)$ can be derived from $N_{m-1}(x)$ by the convolution operation as follows,

$$N_m(x) = (N_{m-1} * N_1)(x) = \int_0^1 N_{m-1}(x-t)dt, \quad m \geq 2. \tag{2.1.4}$$

Based on the relationship between the Fourier transform and the convolution operation, from (2.1.4) we immediately get

$$\hat{N}_m(\omega) = \left(\hat{N}_1(\omega)\right)^m. \tag{2.1.5}$$

It is easy to calculate $\hat{N}_1(\omega)$, which is

$$\hat{N}_1(\omega) = \frac{\sin \omega/2}{\omega/2} e^{-i\omega/2},$$

and (2.1.5) leads to

$$\hat{N}_m(\omega) = \left(\frac{\sin \omega/2}{\omega/2}\right)^m e^{-im\omega/2},$$

which implies that

$$\hat{N}_m(0) = 1.$$

Since

$$\hat{N}_m(\omega) = \int_{-\infty}^{\infty} N_m(x) e^{-i\omega x} dx,$$

we have

$$\hat{N}_m(0) = \int_{-\infty}^{\infty} N_m(x)dx = 1.$$

Equation (2.1.5) can also be written as

$$\hat{N}_m(\omega) = \left(\frac{1-e^{-i\omega}}{i\omega}\right)^m,$$

which is easier to use when we consider

$$\frac{\hat{N}_m(\omega)}{\hat{N}_m(\omega/2)} = \left(\frac{(1-e^{-i\omega})/(i\omega)}{(1-e^{-i\omega/2})/(i\omega/2)}\right)^m = \left(\frac{1+e^{-i\omega/2}}{2}\right)^m.$$

That is,

$$\hat{N}_m(\omega) = P_m(z)\hat{N}_m(\omega/2),$$

where $z = e^{-i\omega/2}$ and

$$P_m(z) = \left(\frac{1+z}{2}\right)^m.$$

Therefore the cardinal B-splines satisfy the following refinement equation

$$N_m(x) = \sum_{j=0}^{m} \frac{\binom{m}{j}}{2^{m-1}} N_m(2x-j).$$

2.2 Spline evaluations and interpolations

Now we want to use the spline functions, (functions generated by the B-splines), to approximate or model the real world functions. We consider the spline functions with the knot sequence (t_j) on the interval $[a,b]$. In order to form a complete basis functions on $[a,b]$, we need to assume that the given knots satisfy the following conditions:

$$a = t_{-m+1} = \cdots = t_0 < t_1 \le t_2 \le \cdots \le t_{n-m} < t_{n-m+1} = \cdots = t_n = b, \quad (2.2.1)$$

and

$$t_{j-m} < t_j, \qquad j = 1, \cdots, n. \quad (2.2.2)$$

This set of B-spline functions satisfy the partition of unity,

$$\sum_{j=0}^{n-1} B_{j,m}(x) = 1, \quad (2.2.3)$$

where $B_{j,m}(x)$ is defined as in (2.1.1). Now, we can define the space of the spline functions on $[a,b]$ with knots $(t_j)_{j=-m+1}^{n}$ satisfying (2.2.1) and (2.2.2) as follows

$$S_{m,t} = \left\{ \sum_{j=0}^{n-1} c_j B_{j,m}(x) | c_j \in \mathbb{R} \text{ for } 1 \leq j \leq n \right\}. \tag{2.2.4}$$

Thus $S_{m,t}$ is a linear space of dimension n.

For a function $f \in S_{m,t}$, we can write it as $f(x) = \sum_{j=0}^{n-1} c_j B_{j,m}(x)$. When we evaluate $f(x)$ at $x = x_0$ with $x_0 \in [t_\mu, t_{\mu+1})$ for $0 \leq \mu \leq n-m$, notice that only those B-splines $B_{j,m}$ with $\mu \leq j \leq \mu+m-1$ may not vanish on x_0, hence $f(x_0) = \sum_{j=\mu}^{\mu+m-1} c_j B_{j,m}(x_0)$.

Let us consider the general evaluation problem of a spline function $f(x) = \sum_{j=0}^{n-1} c_j B_{j,m}(x)$. Since the explicit representation of each B-spline $B_{j,m}(x)$ is complex, we will try to represent the value of $f(x)$ as a product of matrices, which rely on the recursion formula of the B-splines.

Let $\vec{c} = (c_0, \ldots, c_{n-1})^T$. Then we can write

$$f(x) = (B_{0,m}(x),\ \ldots,\ B_{n-1,m}(x))\ \vec{c}. \tag{2.2.5}$$

Next, we will write the vector $(B_{0,m}(x),\ \ldots,\ B_{n-1,m}(x))$ as a product of matrices from the lower order cases.

Assume that the n data samples $\{y_i\}_0^{n-1}$ satisfy:

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b \tag{2.2.6}$$

and

$$t_{i-m+1} < y_i < t_{i+1}, \qquad \text{for} \quad i = 1, \ldots, n-2. \tag{2.2.7}$$

Given a function $f : C[a,b] \to \mathbb{R}$, the spline interpolation operator $S_m : C[a,b] \to S_{m,t}$ satisfies the n interpolation conditions

$$(S_m f)(y_i) = f(y_i),\ i = 0, \ldots, n-1. \tag{2.2.8}$$

Spline interpolation has the advantage over traditional polynomial interpolation (for example, the lagrange and Newton interpolation formulas) that the approximation accuracy may be improved by decreasing the distance between consecutive knots while keeping the polynomial degree $m-1$ relatively low.

Since $\{B_{j,m} : j = 0, \ldots, n-1\}$ forms a basis for the spline space $S_{m,t}$, there exists a spline $S_m f$, defined by

$$(S_m f)(x) = \sum_{j=0}^{n-1} c_j^f B_{j,m}(x), \quad x \in [a,b],$$

that satisfies (2.2.8), if and only if

$$\sum_{j=0}^{n-1} c_j^f B_{j,m}(y_i) = f(y_i), \quad i = 0, \ldots, n-1. \tag{2.2.9}$$

In other words, the vector $c_f := (c_0^f, \ldots, c_{n-1}^f)^T \in \mathbb{R}^n$, where $\mathbb{R}^n$ denotes the n-dimensional real space, is a solution to the matrix equation

$$B_m c_f = f_n,$$

where B_m is an $n \times n$ coefficient matrix of the form

$$B_m := \begin{bmatrix} B_{0,m}(y_0) & \cdots & B_{n-1,m}(y_0) \\ \vdots & & \vdots \\ B_{0,m}(y_{n-1}) & \cdots & B_{n-1,m}(y_{n-1}) \end{bmatrix}, \tag{2.2.10}$$

and $f_n := (f(y_0), \ldots, f(y_{n-1}))^T \in \mathbb{R}^n$. A necessary and sufficient condition for the matrix B_m in (2.2.10) to be invertible is that

$$B_{j,m}(y_j) \neq 0, \quad \text{for} \quad j = 0, \ldots, n-1, \tag{2.2.11}$$

which is the result of the Schoenberg-Whitney theorem [45], and we refer the matrix B_m to *Schoenberg-Whitney matrix.*

Our setting in (2.2.6) and (2.2.7) ensures (2.2.11), thus, we can write

$$(S_m f)(x) = \sum_{j=0}^{n-1} (B_m^{-1} f_n)_j B_{j,m}(x), \quad x \in [a, b],$$

where $(v)_j$ refers to the j-th component of an n-vector v.

In order to calculate the entries of B_m in (2.2.10), we would like to introduce the following notations,

$$\alpha_{i,j}^m := \frac{y_j - t_{i-m+1}}{t_i - t_{i-m+1}}, \qquad \beta_{i,j}^m := \frac{t_{i+1} - y_j}{t_{i+1} - t_{i-m+2}}, \tag{2.2.12}$$

where $\{t_{i-m+1}, \ldots, t_{i+1}\}$ are the knots that define the B-spline $B_{i,m}(x)$ (see 2.1.1). We can easily see a property between $\alpha_{i,j}^m$ and $\beta_{i,j}^m$:

$$\alpha_{i,j}^m + \beta_{i-1,j}^m = 1. \tag{2.2.13}$$

Next we will discuss the B-spline evaluations for three cases: *Linear* ($m = 2$), *Quadratic* ($m = 3$), and *Cubic* ($m = 4$) B-splines.

In the case that $m = 2$, the support of $B_{i,2}(x)$ is $[t_{i-1}, t_{i+1}]$. We consider the following two cases: 1) $y_j \in [t_{i-1}, t_i]$; 2) $y_j \in [t_i, t_{i+1}]$.

1. For $y_j \in [t_{i-1}, t_i]$,
$$B_{i,2}(y_j) := \frac{y_j - t_{i-1}}{t_i - t_{i-1}} = \alpha_{i,j}^2. \tag{2.2.14}$$

2. For $y_j \in [t_i, t_{i+1}]$,
$$B_{i,2}(y_j) := \frac{t_{i+1} - y_j}{t_{i+1} - t_i} = \beta_{i,j}^2. \tag{2.2.15}$$

In order to get a specific view on the knot-data setting, we consider the following special knots and data samples arrangements:

$$\begin{aligned} a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \\ \rightharpoonup t_4 \rightharpoonup (y_4) \rightharpoonup t_5 \rightharpoonup (y_5) \rightharpoonup t_6 \rightharpoonup (y_6) \rightharpoonup b = t_7 = (y_7), \end{aligned} \tag{2.2.16}$$

and we will write the Shronberg-Whitney matrix for the basis functions $\{B_{i,2}(x)\}_{i=0}^{7}$ with respect to this setting.

The Shronberg-Whitney matrix (2.2.10) has the following format:

$$\mathcal{B}_2 := [b_{ji}]_{8\times 8},$$

where $b_{ij} := B_{i,2}(y_j)$. Next, we will calculate $\mathcal{B}_2$ for the non-zero entries in the following pattern:

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{01} & b_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & b_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & b_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{55} & b_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{66} & b_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.2.17}$$

Since there is no data sample between the knots t_3 and t_4, the $\mathcal{B}_2$ matrix is a block-diagonal matrix with two diagonal blocks. Next we verify the *partition of unity* property of B-splines.

Consider the ith row with $1 \le i \le 3$: $\{b_{i-1,i}, b_{i,i}\}$ for the first diagonal block matrix.

- For the knots-data setting: $\{t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i\}$,

$$b_{i-1,i} = B_{i-1,2}(y_i) = \beta^2_{i-1,i}.$$

- For the knots-data setting: $\{t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1}\}$,

$$b_{i,i} = B_{i,2}(y_i) = \alpha^2_{i,i}.$$

Then the sum of all these numbers is:

$$b_{i-1,i} + b_{i,i} = \beta^2_{i-1,i} + \alpha^2_{i,i} = 1,$$

which follows the *partition of unity* property.

Consider the ith row with $4 \leq i \leq 6$: $\{b_{i,i}, b_{i+1,i}\}$ for the second diagonal block matrix in $\mathcal{B}_2$.

- For the knots-data setting: $\{t_{i-1} \rightharpoonup (y_{i-1}) \rightharpoonup t_i \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i+1}\}$

$$b_{i,i} = B_{i,2}(y_i) = \beta^2_{i,i}.$$

- For the knots-data setting: $\{t_i \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i+1} \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+2}\}$

$$b_{i+1,i} = B_{i+1,2}(y_i) = \alpha^2_{i+1,i}.$$

We also have that the sum of all these numbers is:

$$b_{i,i} + b_{i+1,i} = \beta^2_{i,i} + \alpha^2_{i+1,i} = 1,$$

which again satisfies the *partition of unity* property.

For the case that $m = 3$, we consider a general quadratic B-spline $B_{i,3}(x)$ with knots $\{t_{i-2}, t_{i-1}, t_i, t_{i+1}\}$ using the notations in (2.2.12).

1. For $y_j \in [t_{i-2}, t_{i-1}]$,

$$B_{i,3}(y_j) := \frac{(y_j - t_{i-2})^2}{(t_{i-1} - t_{i-2})(t_i - t_{i-2})} = \alpha^2_{i-1,j}\alpha^3_{i,j}. \tag{2.2.18}$$

2. For $y_j \in [t_{i-1}, t_i]$,

$$B_{i,3}(y_j) := \frac{(t_i - y_j)(y_j - t_{i-2})}{(t_i - t_{i-1})(t_i - t_{i-2})} + \frac{(y_j - t_{i-1})(t_{i+1} - y_j)}{(t_i - t_{i-1})(t_i - t_{i-2})}$$

$$= \beta^2_{i-1,j}\alpha^3_{i,j} + \alpha^2_{i,j}\beta^3_{i,j}. \tag{2.2.19}$$

3. For $y_j \in [t_i, t_{i+1}]$,

$$B_{i,3}(y_j) := \frac{(t_{i+1} - y_j)^2}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})} = \beta^2_{i,j}\beta^3_{i,j}. \tag{2.2.20}$$

We consider the following knots and data samples arrangements:

$$\begin{aligned} a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \\ \rightharpoonup t_4 \rightharpoonup (y_5) \rightharpoonup t_5 \rightharpoonup (y_6) \rightharpoonup t_6 \rightharpoonup (y_7) \rightharpoonup b = t_7 = (y_8), \end{aligned} \tag{2.2.21}$$

and we will write the Shronberg-Whitney matrix for the quadratic B-spline basis functions with respect to these knots and data points.

The Shronberg-Whitney matrix has the following format:

$$\mathcal{B}_3 := [B_{ji}]_{9\times 9},$$

where $B_{ij} := B_{i,3}(y_j)$. Next, we will calculate $\mathcal{B}_3$ for the non-zero entries in the following pattern:

$$\mathcal{B}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & B_{11} & B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & B_{22} & B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & B_{33} & B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & B_{44} & B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{45} & B_{55} & B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & B_{66} & B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{67} & B_{77} & B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{9\times 9}. \tag{2.2.22}$$

1. The 2nd row: $\{B_{01}, B_{11}, B_{21}\}$

- With the knot sequence: $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup a \rightharpoonup (\underline{y_1}) \rightharpoonup t_1\}$

$$B_{01} = B_{0,3}(y_1) = \beta^2_{01}\beta^3_{01}.$$

- With the knot sequence: $\{a = t_0 = (y_1) \rightharpoonup a \rightharpoonup (\underline{y_1}) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2\}$

$$B_{11} = B_{1,3}(y_1) = \beta_{01}^2 \alpha_{11}^3 + \alpha_{11}^2 \beta_{11}^3.$$

- With the knot sequence:
$\{a = t_0 = (y_0) \rightharpoonup (\underline{y_1}) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3\}$

$$B_{21} = B_{2,3}(y_1) = \alpha_{11}^2 \alpha_{21}^3.$$

Sum of all these numbers:

$$B_{01} + B_{11} + B_{21} = \beta_{01}^2 \beta_{01}^3 + (\beta_{01}^2 \alpha_{11}^3 + \alpha_{11}^2 \beta_{11}^3) + \alpha_{11}^2 \alpha_{21}^3$$

$$= (\beta_{01}^2 \beta_{01}^3 + \beta_{01}^2 \alpha_{11}^2) + (\alpha_{11}^2 \beta_{11}^3 + \alpha_{11}^2 \alpha_{21}^3) = \beta_{01}^2 + \alpha_{11}^2 = 1.$$

2. The 3rd row: $\{B_{12}, B_{22}, B_{32}\}$

- With the knot sequence: $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (\underline{y_2}) \rightharpoonup t_2\}$

$$B_{12} = B_{1,3}(y_2) = \beta_{12}^2 \beta_{12}^3.$$

- With the knot sequence:
$\{a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (\underline{y_2}) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3\}$

$$B_{22} = B_{2,3}(y_2) = \beta_{12}^2 \alpha_{22}^3 + \alpha_{22}^2 \beta_{22}^3.$$

- With the knot sequence:
$\{t_1 \rightharpoonup (\underline{y_2}) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup t_4\}$

$$B_{32} = B_{3,3}(y_2) = \alpha_{22}^2 \alpha_{32}^3.$$

Sum of all these numbers:

$$B_{12} + B_{22} + B_{32} = \beta_{12}^2 \beta_{12}^3 + (\beta_{12}^2 \alpha_{22}^3 + \alpha_{22}^2 \beta_{22}^3) + \alpha_{22}^2 \alpha_{32}^3$$

$$= (\beta_{12}^2 \beta_{12}^3 + \beta_{12}^2 \alpha_{22}^3) + (\alpha_{22}^2 \beta_{22}^3 + \alpha_{22}^2 \alpha_{32}^3) = \beta_{12}^2 + \alpha_{22}^2 = 1.$$

Now we consider the general case.

The ith row with $1 \leq i \leq 7$: $\{B_{i-1,i}, B_{i,i}, B_{i+1,i}\}$

- With the knot sequence:
  $\{t_{i-3} \rightharpoonup (y_{i-2}) \rightharpoonup t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i\}$
$$B_{i-1,i} = B_{i-1,3}(y_i) = \beta^2_{i-1,i}\beta^3_{i-1,i}.$$
- With the knot sequence:
  $\{t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1}\}$
$$B_{i,i} = B_{i,3}(y_i) = \beta^2_{i-1,i}\alpha^3_{i,i} + \alpha^2_{i,i}\beta^3_{i,i}.$$
- With the knot sequence:
  $\{t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1} \rightharpoonup (y_{i+2}) \rightharpoonup t_{i+2}\}$
$$B_{i+1,i} = B_{i+1,3}(y_i) = \alpha^2_{i,i}\alpha^3_{i+1,i}.$$

Sum of all these numbers:
$$\begin{aligned} &B_{i-1,i} + B_{i,i} + B_{i+1,i} = \beta^2_{i-1,i}\beta^3_{i-1,i} + (\beta^2_{i-1,i}\alpha^3_{i,i} + \alpha^2_{i,i}\beta^3_{i,i}) + \alpha^2_{i,i}\alpha^3_{i+1,i} \\ &= (\beta^2_{i-1,i}\beta^3_{i-1,i} + \beta^2_{i-1,i}\alpha^3_{i,i}) + (\alpha^2_{i,i}\beta^3_{i,i} + \alpha^2_{i,i}\alpha^3_{i+1,i}) = \beta^2_{i-1,i} + \alpha^2_{i,i} = 1. \end{aligned}$$

For the case that $m = 4$, we work on a general cubic B-spline with knots $\{t_{i-3}, t_{i-2}, t_{i-1}, t_i, t_{i+1}\}$:

1. For $x \in [t_{i-3}, t_{i-2}]$,
$$B_{i,4}(x) := \frac{(x - t_{i-3})^3}{(t_{i-2} - t_{i-3})(t_{i-1} - t_{i-3})(t_i - t_{i-3})}. \tag{2.2.23}$$

2. For $x \in [t_{i-2}, t_{i-1}]$,
$$\begin{aligned} B_{i,4}(x) := &\frac{(t_{i-1} - x)(x - t_{i-3})^2}{(t_{i-2} - t_{i-3})(t_{i-2} - t_{i-4})(t_{i-1} - t_{i-4})} + \\ &\frac{(x - t_{i-3})(t_{i-1} - x)(x - t_{i-4})}{(t_{i-2} - t_{i-3})(t_{i-1} - t_{i-3})(t_{i-1} - t_{i-4})} + \frac{(x - t_{i-3})^2(t_i - x)}{(t_{i-2} - t_{i-3})(t_{i-1} - t_{i-3})(t_i - t_{i-3})}. \end{aligned} \tag{2.2.24}$$

3. For $x \in [t_{i-2}, t_{i-1}]$,
$$\begin{aligned} B_{i,4}(x) := &\frac{(t_{i-1} - x)^2(x - t_{i-4})}{(t_{i-1} - t_{i-2})(t_{i-1} - t_{i-3})(t_{i-1} - t_{i-4})} + \\ &\frac{(t_{i-1} - x)(x - t_{i-3})(t_i - x)}{(t_{i-1} - t_{i-2})(t_{i-1} - t_{i-3})(t_i - t_{i-3})} + \frac{(x - t_{i-2})(t_i - x)^2}{(t_{i-1} - t_{i-2})(t_i - t_{i-2})(t_i - t_{i-3})}. \end{aligned} \tag{2.2.25}$$

4. For $x \in [t_{i-1}, t_i]$,

$$B_{i,4}(x) := \frac{(t_i - x)^3}{(t_i - t_{i-1})(t_i - t_{i-2})(t_i - t_{i-3})}. \tag{2.2.26}$$

Then we can represent $B_{i,4}(y_j)$ in the following compact form using the notations (2.2.12):

1. For $y_j \in [t_{i-3}, t_{i-2}]$,

$$B_{i,4}(y_j) := \frac{(y_j - t_{i-3})^3}{(t_{i-2} - t_{i-3})(t_{i-1} - t_{i-3})(t_i - t_{i-3})} = \alpha^2_{i-2,j}\alpha^3_{i-1,j}\alpha^4_{i,j}. \tag{2.2.27}$$

2. For $y_j \in [t_{i-2}, t_{i-1}]$,

$$B_{i,4}(y_j) := \frac{(t_{i-1} - y_j)(y_j - t_{i-3})^2}{(t_{i-1} - t_{i-2})(t_{i-1} - t_{i-3})(t_i - t_{i-3})} +$$

$$\frac{(y_j - t_{i-2})(t_i - y_j)(y_j - t_{i-3})}{(t_{i-1} - t_{i-2})(t_i - t_{i-2})(t_i - t_{i-3})} + \frac{(y_j - t_{i-2})^2(t_{i+1} - y_j)}{(t_{i-1} - t_{i-2})(t_i - t_{i-2})(t_{i+1} - t_{i-2})}$$

$$= \beta^2_{i-2,j}\alpha^3_{i-1,j}\alpha^4_{i,j} + \alpha^2_{i-1,j}\beta^3_{i-1,j}\alpha^4_{i,j} + \alpha^2_{i-1,j}\alpha^3_{i,j}\beta^4_{i,j}. \tag{2.2.28}$$

3. For $y_j \in [t_{i-1}, t_i]$,

$$B_{i,4}(y_j) := \frac{(t_i - y_j)^2(y_j - t_{i-3})}{(t_i - t_{i-1})(t_i - t_{i-2})(t_i - t_{i-3})} +$$

$$\frac{(t_i - y_j)(y_j - t_{i-2})(t_{i+1} - y_j)}{(t_i - t_{i-1})(t_i - t_{i-2})(t_{i+1} - t_{i-2})} + \frac{(y_j - t_{i-1})(t_{i+1} - y_j)^2}{(t_i - t_{i-1})(t_{i+1} - t_{i-1})(t_{i+1} - t_{i-2})}$$

$$= \beta^2_{i-1,j}\beta^3_{i-1,j}\alpha^4_{i,j} + \beta^2_{i-1,j}\alpha^3_{i,j}\beta^4_{i,j} + \alpha^2_{i,j}\beta^3_{i,j}\beta^4_{i,j}. \tag{2.2.29}$$

4. For $y_j \in [t_i, t_{i+1}]$,

$$B_{i,4}(y_j) := \frac{(t_{i+1} - y_j)^3}{(t_{i+1} - t_i)(t_{i+1} - t_{i-1})(t_{i+1} - t_{i-2})} = \beta^2_{i,j}\beta^3_{i,j}\beta^4_{i,j}. \tag{2.2.30}$$

We consider the following knots and data points arrangements:

$$a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4$$
$$\rightharpoonup (y_6) \rightharpoonup t_5 \rightharpoonup (y_7) \rightharpoonup x_6 \rightharpoonup (y_8) \rightharpoonup b = t_7 = (y_9),$$

and we will write the Shronberg-Whitney matrix for the cubic B-spline basis functions with respect to these knots and data points.

The Shronberg-Whitney matrix has the following format:

$$\mathcal{B}_4 := [B_{ji}]_{10\times 10},$$

where $B_{ij} := B_{i,4}(y_j)$. Next, we will calculate $\mathcal{B}_4$ for the non-zero entries in the following pattern:

$$\mathcal{B}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & B_{11} & B_{21} & B_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & B_{22} & B_{32} & B_{42} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & B_{33} & B_{43} & B_{53} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & B_{44} & B_{54} & B_{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{35} & B_{45} & B_{55} & B_{65} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{46} & B_{56} & B_{66} & B_{76} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{57} & B_{67} & B_{77} & B_{87} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{68} & B_{78} & B_{88} & B_{98} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.2.31}$$

1. The 2nd row: $\{B_{01}, B_{11}, B_{21}, B_{31}\}$

- With the knot sequence: $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup a \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1\}$
$$B_{01} = B_{0,4}(y_1) = \beta_{01}^2\beta_{01}^3\beta_{01}^4.$$
- With the knot sequence: $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2\}$
$$B_{11} = B_{1,4}(y_1) = \beta_{01}^2\beta_{01}^3\alpha_{11}^4 + \beta_{01}^2\alpha_{11}^3\beta_{11}^4 + \alpha_{11}^2\beta_{11}^3\beta_{11}^4.$$
- With the knot sequence:
$\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3\}$
$$B_{21} = B_{2,4}(y_1) = \beta_{01}^2\alpha_{11}^3\alpha_{21}^4 + \alpha_{11}^2\beta_{11}^3\alpha_{21}^4 + \alpha_{11}^2\alpha_{21}^3\beta_{21}^4.$$
- With the knot sequence:
$\{a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4\}$
$$B_{31} = B_{3,4}(y_1) = \alpha_{11}^2\alpha_{21}^3\alpha_{31}^4.$$

Sum of all these numbers:

$$
\begin{aligned}
B_{01} + B_{11} + B_{21} + B_{31} = \beta^2_{01}\beta^3_{01}\beta^4_{01} + (\beta^2_{01}\beta^3_{01}\alpha^4_{11} + \beta^2_{01}\alpha^3_{11}\beta^4_{11} + \alpha^2_{11}\beta^3_{11}\beta^4_{11}) + \\
(\beta^2_{01}\alpha^3_{11}\alpha^4_{21} + \alpha^2_{11}\beta^3_{11}\alpha^4_{21} + \alpha^2_{11}\alpha^3_{21}\beta^4_{21}) + \alpha^2_{11}\alpha^3_{21}\alpha^4_{31} \\
= (\beta^2_{01}\beta^3_{01}\beta^4_{01} + \beta^2_{01}\beta^3_{01}\alpha^4_{11}) + (\beta^2_{01}\alpha^3_{11}\beta^4_{11} + \beta^2_{01}\alpha^3_{11}\alpha^4_{21}) + \\
(\alpha^2_{11}\beta^3_{11}\beta^4_{11} + \alpha^2_{11}\beta^3_{11}\alpha^4_{21}) + (\alpha^2_{11}\alpha^3_{21}\beta^4_{21} + \alpha^2_{11}\alpha^3_{21}\alpha^4_{31}) \\
= (\beta^2_{01}\beta^3_{01} + \beta^2_{01}\alpha^3_{11}) + (\alpha^2_{11}\beta^3_{11} + \alpha^2_{11}\alpha^3_{21}) = \beta^2_{01} + \alpha^2_{11} = 1.
\end{aligned}
$$

2. The 3rd row: $\{B_{12}, B_{22}, B_{32}, B_{42}\}$

   - With the knot sequence: $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2\}$
   $$B_{12} = B_{1,4}(y_2) = \beta^2_{12}\beta^3_{12}\beta^4_{12}.$$
   - With the knot sequence:
   $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3\}$
   $$B_{22} = B_{2,4}(y_2) = \beta^2_{12}\beta^3_{12}\alpha^4_{22} + \beta^2_{12}\alpha^3_{22}\beta^4_{22} + \alpha^2_{22}\beta^3_{22}\beta^4_{22}.$$
   - With the knot sequence:
   $\{a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4\}$
   $$B_{32} = B_{3,4}(y_2) = \beta^2_{12}\alpha^3_{22}\alpha^4_{32} + \alpha^2_{22}\beta^3_{22}\alpha^4_{32} + \alpha^2_{22}\alpha^3_{32}\beta^4_{32}.$$
   - With the knot sequence:
   $\{t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4 \rightharpoonup (y_6) \rightharpoonup t_5\}$
   $$B_{42} = B_{4,4}(y_2) = \alpha^2_{22}\alpha^3_{32}\alpha^4_{42}.$$

   Sum of all these numbers:

$$
\begin{aligned}
B_{12} + B_{22} + B_{32} + B_{42} = \beta^2_{12}\beta^3_{12}\beta^4_{12} + (\beta^2_{12}\beta^3_{12}\alpha^4_{22} + \beta^2_{12}\alpha^3_{22}\beta^4_{22} + \alpha^2_{22}\beta^3_{22}\beta^4_{22}) + \\
(\beta^2_{12}\alpha^3_{22}\alpha^4_{32} + \alpha^2_{22}\beta^3_{22}\alpha^4_{32} + \alpha^2_{22}\alpha^3_{32}\beta^4_{32}) + \alpha^2_{22}\alpha^3_{32}\alpha^4_{42} \\
= (\beta^2_{12}\beta^3_{12}\beta^4_{12} + \beta^2_{12}\beta^3_{12}\alpha^4_{22}) + (\beta^2_{12}\alpha^3_{22}\beta^4_{22} + \beta^2_{12}\alpha^3_{22}\alpha^4_{32}) + \\
(\alpha^2_{22}\beta^3_{22}\beta^4_{22} + \alpha^2_{22}\beta^3_{22}\alpha^4_{32}) + (\alpha^2_{32}\alpha^3_{32}\beta^4_{32} + \alpha^2_{22}\alpha^3_{32}\alpha^4_{42}) \\
= (\beta^2_{12}\beta^3_{12} + \beta^2_{12}\alpha^3_{22}) + (\alpha^2_{22}\beta^3_{22} + \alpha^2_{22}\alpha^3_{32}) = \beta^2_{12} + \alpha^2_{22} = 1.
\end{aligned}
$$

3. The 4th row: $\{B_{23}, B_{33}, B_{43}, B_{53}\}$

- With the knot sequence:
  $\{a = t_0 = (y_0) \rightharpoonup a \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3\}$
$$B_{23} = B_{2,4}(y_3) = \beta_{23}^2\beta_{23}^3\beta_{23}^4.$$
- With the knot sequence:
  $\{a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4\}$
$$B_{33} = B_{3,4}(y_3) = \beta_{23}^2\beta_{23}^3\alpha_{33}^4 + \beta_{23}^2\alpha_{33}^3\beta_{33}^4 + \alpha_{33}^2\beta_{33}^3\beta_{33}^4.$$
- With the knot sequence:
  $\{t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4 \rightharpoonup (y_6) \rightharpoonup t_5\}$
$$B_{43} = B_{4,4}(y_3) = \beta_{23}^2\alpha_{33}^3\alpha_{43}^4 + \alpha_{33}^2\beta_{33}^3\alpha_{43}^4 + \alpha_{33}^2\alpha_{43}^3\beta_{43}^4.$$
- With the knot sequence:
  $\{t_2 \rightharpoonup (y_3) \rightharpoonup t_3 \rightharpoonup (y_4) \rightharpoonup (y_5) \rightharpoonup t_4 \rightharpoonup (y_6) \rightharpoonup t_5 \rightharpoonup (y_7) \rightharpoonup t_6\}$
$$B_{53} = B_{5,4}(y_3) = \alpha_{33}^2\alpha_{43}^3\alpha_{53}^4.$$

Sum of all these numbers:

$$\begin{aligned}
B_{23} + B_{33} + B_{43} + B_{53} = {} & \beta_{23}^2\beta_{23}^3\beta_{23}^4 + (\beta_{23}^2\beta_{23}^3\alpha_{33}^4 + \beta_{23}^2\alpha_{33}^3\beta_{33}^4 + \alpha_{33}^2\beta_{33}^3\beta_{33}^4) + \\
& (\beta_{23}^2\alpha_{33}^3\alpha_{43}^4 + \alpha_{33}^2\beta_{33}^3\alpha_{43}^4 + \alpha_{33}^2\alpha_{43}^3\beta_{43}^4) + \alpha_{33}^2\alpha_{43}^3\alpha_{53}^4 \\
= {} & (\beta_{23}^2\beta_{23}^3\beta_{23}^4 + \beta_{23}^2\beta_{23}^3\alpha_{33}^4) + (\beta_{23}^2\alpha_{33}^3\beta_{33}^4 + \beta_{23}^2\alpha_{33}^3\alpha_{43}^4) + \\
& (\alpha_{33}^2\beta_{33}^3\beta_{33}^4 + \alpha_{33}^2\beta_{33}^3\alpha_{43}^4) + (\alpha_{33}^2\alpha_{43}^3\beta_{43}^4 + \alpha_{33}^2\alpha_{43}^3\alpha_{53}^4) \\
= {} & (\beta_{23}^2\beta_{23}^3 + \beta_{23}^2\alpha_{33}^3) + (\alpha_{33}^2\beta_{33}^3 + \alpha_{33}^2\alpha_{43}^3) = \beta_{23}^2 + \alpha_{33}^2 = 1.
\end{aligned}$$

Now we consider the general case.

1. The ith row with $1 \le i \le 4$: $\{B_{i-1,i}, B_{i,i}, B_{i+1,i}, B_{i+2,i}\}$

- With the knot sequence:
  $\{t_{i-4} \rightharpoonup (y_{i-3}) \rightharpoonup t_{i-3} \rightharpoonup (y_{i-2}) \rightharpoonup t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i\}$
$$B_{i-1,i} = B_{i-1,4}(y_i) = \beta_{i-1,i}^2\beta_{i-1,i}^3\beta_{i-1,i}^4.$$
- With the knot sequence:
  $\{t_{i-3} \rightharpoonup (y_{i-2}) \rightharpoonup t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1}\}$
$$B_{ii} = B_{i,4}(y_i) = \beta_{i-1,i}^2\beta_{i-1,i}^3\alpha_{ii}^4 + \beta_{i-1,i}^2\alpha_{ii}^3\beta_{ii}^4 + \alpha_{ii}^2\beta_{ii}^3\beta_{ii}^4.$$

- With the knot sequence:
  $\{t_{i-2} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1} \rightharpoonup (y_{i+2}) \rightharpoonup t_{i+2}\}$

$$B_{i+1,i} = B_{i+1,4}(y_i) = \beta^2_{i-1,i}\alpha^3_{ii}\alpha^4_{i+1,i} + \alpha^2_{ii}\beta^3_{ii}\alpha^4_{i+1,i} + \alpha^2_{ii}\alpha^3_{i+1,i}\beta^4_{i+1,i}.$$

- With the knot sequence:
  $\{t_{i-1} \rightharpoonup (\underline{y_i}) \rightharpoonup t_i \rightharpoonup (y_{i+1}) \rightharpoonup t_{i+1} \rightharpoonup (y_{i+2}) \rightharpoonup t_{i+2} \rightharpoonup (y_{i+3}) \rightharpoonup t_{i+3}\}$

$$B_{i+2,i} = B_{i+2,4}(y_i) = \alpha^2_{ii}\alpha^3_{i+1,i}\alpha^4_{i+2,i}.$$

Sum of all these numbers:

$$B_{i-1,i}+B_{ii}+B_{i+1,i}+B_{i+2,i} = \beta^2_{i-1,i}\beta^3_{i-1,i}\beta^4_{i-1,i}+(\beta^2_{i-1,i}\beta^3_{i-1,i}\alpha^4_{ii}+\beta^2_{i-1,i}\alpha^3_{ii}\beta^4_{ii}+\alpha^2_{ii}\beta^3_{ii}\beta^4_{ii})+$$

$$(\beta^2_{i-1,i}\alpha^3_{ii}\alpha^4_{i+1,i} + \alpha^2_{ii}\beta^3_{ii}\alpha^4_{i+1,i} + \alpha^2_{ii}\alpha^3_{i+1,i}\beta^4_{i+1,i}) + \alpha^2_{ii}\alpha^3_{i+1,i}\alpha^4_{i+2,i}$$
$$= (\beta^2_{i-1,i}\beta^3_{i-1,i}\beta^4_{i-1,i} + \beta^2_{i-1,i}\beta^3_{i-1,i}\alpha^4_{ii}) + (\beta^2_{i-1,i}\alpha^3_{ii}\beta^4_{ii} + \beta^2_{i-1,i}\alpha^3_{ii}\alpha^4_{i+1,i})+$$
$$(\alpha^2_{ii}\beta^3_{ii}\beta^4_{ii} + \alpha^2_{ii}\beta^3_{ii}\alpha^4_{i+1,i}) + (\alpha^2_{ii}\alpha^3_{i+1,i}\beta^4_{i+1,i} + \alpha^2_{ii}\alpha^3_{i+1,i}\alpha^4_{i+2,i})$$
$$= (\beta^2_{i-1,i}\beta^3_{i-1,i} + \beta^2_{i-1,i}\alpha^3_{ii}) + (\alpha^2_{ii}\beta^3_{ii} + \alpha^2_{ii}\alpha^3_{i+1,i}) = \beta^2_{i-1,i} + \alpha^2_{ii} = 1.$$

2. The ith row with $5 \le i \le 8$: $\{B_{i-2,i}, B_{i-1,i}, B_{i,i}, B_{i+1,i}\}$

- With the knot sequence:
  $\{t_{i-5} \rightharpoonup (y_{i-3}) \rightharpoonup t_{i-4} \rightharpoonup (y_{i-2}) \rightharpoonup t_{i-3} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-2} \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i-1}\}$

$$B_{i-2,i} = B_{i-2,4}(y_i) = \beta^2_{i-2,i}\beta^3_{i-2,i}\beta^4_{i-2,i}.$$

- With the knot sequence:
  $\{t_{i-4} \rightharpoonup (y_{i-2}) \rightharpoonup t_{i-3} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-2} \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i-1} \rightharpoonup (y_{i+1}) \rightharpoonup t_i\}$

$$B_{i-1,i} = B_{i-1,4}(y_i) = \beta^2_{i-2,i}\beta^3_{i-2,i}\alpha^4_{i-1,i}+\beta^2_{i-2,i}\alpha^3_{i-1,i}\beta^4_{i-1,i}+\alpha^2_{i-1,i}\beta^3_{i-1,i}\beta^4_{i-1,i}.$$

- With the knot sequence:
  $\{t_{i-3} \rightharpoonup (y_{i-1}) \rightharpoonup t_{i-2} \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i-1} \rightharpoonup (y_{i+1}) \rightharpoonup t_i \rightharpoonup (y_{i+2}) \rightharpoonup t_{i+1}\}$

$$B_{i,i} = B_{i,4}(y_i) = \beta^2_{i-2,i}\alpha^3_{i-1,i}\alpha^4_{i,i} + \alpha^2_{i-1,i}\beta^3_{i-1,i}\alpha^4_{i,i} + \alpha^2_{i-1,i}\alpha^3_{i,i}\beta^4_{i,i}.$$

- With the knot sequence:
  $\{t_{i-2} \rightharpoonup (\underline{y_i}) \rightharpoonup t_{i-1} \rightharpoonup (y_{i+1}) \rightharpoonup t_i \rightharpoonup (y_{i+2}) \rightharpoonup t_{i+1} \rightharpoonup (y_{i+3}) \rightharpoonup t_{i+2}\}$

$$B_{i+1,i} = B_{i+1,4}(y_i) = \alpha^2_{i-1,i}\alpha^3_{i,i}\alpha^4_{i+1,i}.$$

Sum of all these numbers:

$$
\begin{aligned}
& B_{i-2,i} + B_{i-1,i} + B_{i,i} + B_{i+1,i} \\
= {} & \beta^2_{i-2,i}\beta^3_{i-2,i}\beta^4_{i-2,i} + (\beta^2_{i-2,i}\beta^3_{i-2,i}\alpha^4_{i-1,i} + \beta^2_{i-2,i}\alpha^3_{i-1,i}\beta^4_{i-1,i} + \alpha^2_{i-1,i}\beta^3_{i-1,i}\beta^4_{i-1,i}) \\
& +(\beta^2_{i-2,i}\alpha^3_{i-1,i}\alpha^4_{i,i} + \alpha^2_{i-1,i}\beta^3_{i-1,i}\alpha^4_{i,i} + \alpha^2_{i-1,i}\alpha^3_{i,i}\beta^4_{i,i}) + \alpha^2_{i-1,i}\alpha^3_{i,i}\alpha^4_{i+1,i} \\
= {} & (\beta^2_{i-2,i}\beta^3_{i-2,i}\beta^4_{i-2,i} + \beta^2_{i-2,i}\beta^3_{i-2,i}\alpha^4_{i-1,i}) + (\beta^2_{i-2,i}\alpha^3_{i-1,i}\beta^4_{i-1,i} + \beta^2_{i-2,i}\alpha^3_{i-1,i}\alpha^4_{i,i}) + \\
& (\alpha^2_{i-1,i}\beta^3_{i-1,i}\beta^4_{i-1,i} + \alpha^2_{i-1,i}\beta^3_{i-1,i}\alpha^4_{i,i}) + (\alpha^2_{i-1,i}\alpha^3_{i,i}\beta^4_{i,i} + \alpha^2_{i-1,i}\alpha^3_{i,i}\alpha^4_{i+1,i}) \\
= {} & (\beta^2_{i-2,i}\beta^3_{i-2,i} + \beta^2_{i-2,i}\alpha^3_{i-1,i}) + (\alpha^2_{i-1,i}\beta^3_{i-1,i} + \alpha^2_{i-1,i}\alpha^3_{i,i}) = \beta^2_{i-2,i} + \alpha^2_{i-1,i} = 1.
\end{aligned}
$$

In order to find the inverse for the Shoenberg-Whitney matrix for $m = 2$, we need the following theorem from [17].

Theorem 2.2.1. *Let B be an n-by-n tridiagonal matrix as in*

$$
B := \begin{bmatrix} d_1 & a_1 & & & 0 \\ b_2 & d_2 & \ddots & & \vdots \\ & \ddots & \ddots & a_{n-2} & \vdots \\ & & b_{n-1} & d_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{2.2.32}
$$

and $c_i \neq 0$, for $i = 1, 2, \cdots, n-1$. Then B is invertible and $det(B) = \prod_{i=1}^{n-1} c_i$. If we denote $B^{-1} = [\alpha_{ij}]$, then we have

$$
\alpha_{ii} = \begin{cases} 1, & \text{for } i = n, \\ \dfrac{1}{c_{n-1}}, & \text{for } i = n-1, \\ \dfrac{1}{c_i} + y_i z_i \alpha_{i+1,i+1}, & \text{for } i = n-2, n-3, \cdots, 1 \end{cases}
$$

and

$$
\alpha_{ij} = \begin{cases} -y_i\alpha_{i+1,j}, & \text{for } i < j < n-1, \\ -z_j\alpha_{i,j+1}, & \text{for } j < i < n-1, \\ 0, & \text{else}, \end{cases}
$$

where $y_i = \dfrac{a_i}{c_i}$, $z_i = \dfrac{b_{i+1}}{c_i}$.

In order to estimate the computing cost of B_m (in Chapter 4), we represent each row of B_m as products of some matrices. For easier discussion, we can assume that $x \in [t_\mu, t_{\mu+1})$.

For the linear B-splines, $m = 2$. When we evaluate $f(x)$, we notice that the only linear B-Splines that are non-zero on $[t_\mu, t_{\mu+1})$ are $B_{\mu,2}$ and $B_{\mu+1,2}$, and their basis vector representation is

$$(B_{\mu,2}(x), B_{\mu+1,2}(x)) = \left(\frac{t_{\mu+1} - x}{t_{\mu+1} - t_\mu}, \ \frac{x - t_\mu}{t_{\mu+1} - t_\mu} \right).$$

For $m = 3$, to evaluate $f(x)$ when $x \in [t_\mu, t_{\mu+1})$, we only need to use 3 B-splines: $\{B_{j,3}\}_{j=\mu}^{\mu+2}$. Apply the recursion formula (2.1.1), we have

$$\begin{aligned}
B_{\mu,3}(x) &= \frac{x - t_{\mu-2}}{t_\mu - t_{\mu-2}} B_{\mu-1,2}(x) + \frac{t_{\mu+1} - x}{t_{\mu+1} - t_{\mu-1}} B_{\mu,2}(x) \\
B_{\mu+1,3}(x) &= \frac{x - t_{\mu-1}}{t_{\mu+1} - t_{\mu-1}} B_{\mu,2}(x) + \frac{t_{\mu+2} - x}{t_{\mu+2} - t_\mu} B_{\mu+1,2}(x) \\
B_{\mu+2,3}(x) &= \frac{x - t_\mu}{t_{\mu+2} - t_\mu} B_{\mu+1,2}(x) + \frac{t_{\mu+3} - x}{t_{\mu+3} - t_{\mu+1}} B_{\mu+2,2}(x),
\end{aligned}$$

which can be represented as

$$(B_{\mu,3}, \quad B_{\mu+1,3}, \quad B_{\mu+2,3}) = (B_{\mu,2}, \quad B_{\mu+1,2}) \begin{bmatrix} \dfrac{t_{\mu+1} - x}{t_{\mu+1} - t_{\mu-1}} & \dfrac{x - t_{\mu-1}}{t_{\mu+1} - t_{\mu-1}} & 0 \\ 0 & \dfrac{t_{\mu+2} - x}{t_{\mu+2} - t_\mu} & \dfrac{x - t_\mu}{t_{\mu+2} - t_\mu} \end{bmatrix}$$

due to the fact that $B_{\mu-1,2}(x) = B_{\mu+2,2}(x) = 0$ because $x \in [t_\mu, t_{\mu+1})$ is outside the supports of these two B-splines.

For the cubic B-spline case where $m = 4$, we can get the basis vector representation by the similar approach,

$$(B_{\mu,4},\quad B_{\mu+1,4},\quad B_{\mu+2,4},\quad B_{\mu+3,4}) = (B_{\mu,3},\quad B_{\mu+1,3},\quad B_{\mu+2,3})$$

$$\times \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-2}} & \frac{x-t_{\mu-2}}{t_{\mu+1}-t_{\mu-2}} & 0 & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+2}-t_{\mu-1}} & 0 \\ 0 & 0 & \frac{t_{\mu+3}-x}{t_{\mu+3}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+3}-t_{\mu}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+1}-t_{\mu}} \end{bmatrix} \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu-1}} & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+2}-t_{\mu}} \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu-2}} & \frac{x-t_{\mu-2}}{t_{\mu+1}-t_{\mu-2}} & 0 & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu-1}} & \frac{x-t_{\mu-1}}{t_{\mu+2}-t_{\mu-1}} & 0 \\ 0 & 0 & \frac{t_{\mu+3}-x}{t_{\mu+3}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+3}-t_{\mu}} \end{bmatrix}$$

We can summarize the above analysis for the general case. To evaluate $f(x)$ for $x \in [t_\mu, t_{\mu+1})$, we only need to consider the vector $(B_{\mu,m},\ \ldots,\ \ B_{\mu+m-1,m})$. In order to represent this vector as a product of matrices, we define the matrices $R_k^\mu(x)$ by:

$$R_k^\mu(x) = \begin{bmatrix} \frac{t_{\mu+1}-x}{t_{\mu+1}-t_{\mu+1-k}} & \frac{x-t_{\mu-1}}{t_{\mu+1}-t_{\mu+1-k}} & 0 & \ldots & 0 \\ 0 & \frac{t_{\mu+2}-x}{t_{\mu+2}-t_{\mu+2-k}} & \frac{x-t_{\mu+2-k}}{t_{\mu+2}-t_{\mu+2-k}} & \ldots & 0 \\ \ldots & & & & \\ & \ldots & 0 & \frac{t_{\mu+k}-x}{t_{\mu+k}-t_{\mu}} & \frac{x-t_{\mu}}{t_{\mu+k}-t_{\mu}} \end{bmatrix}. \tag{2.2.33}$$

Then

$$(B_{\mu,m},\ \ldots,\ B_{\mu+m-1,m}) = R_1(x)R_2(x)\ldots R_{m-1}(x) \tag{2.2.34}$$

Therefore $f(x)$ can be represented by:

$$f(x) = R_1(x)R_2(x)\ldots R_{m-1}(x)\,\vec{c}, \qquad \text{for } x \in [t_\mu, t_{\mu+1}). \tag{2.2.35}$$

2.3 Approximation orders and polynomial reproduction

2.3.1 Polynomial reproduction and Marsden's identity

Since x^k is a special function in the space $S_{m,t}$ for $0 \le k \le m-1$, we expect to have an expression for x as follows

$$x = \sum_{j=0}^{n-1} p_j B_{j,m}(x), \quad \text{for } x \in [a,b], \tag{2.3.1}$$

where $\{p_j\}$'s are the real coefficients. If (2.3.1) is true, we can take derivatives on both sides and get

$$1 = \sum_{j=0}^{n-1} p_j \frac{d}{dx} B_{j,m}(x). \tag{2.3.2}$$

From equations (2.1.2) and (2.3.2), we get

$$1 = (m-1)\sum_{j=0}^{n-1} p_j \left(\frac{B_{j-1,m-1}(x)}{t_{j-1}-t_{j-m}} - \frac{B_{j,m-1}(x)}{t_j - t_{j-m+1}} \right), \tag{2.3.3}$$

which can be simplified as

$$1 = (m-1)\sum_{j=0}^{n-1} \frac{p_{j+1}-p_j}{t_j - t_{j-m+1}} B_{j,m-1}(x). \tag{2.3.4}$$

By the *partition of unity* property, we get

$$p_{j+1} - p_j = \frac{t_{j+1} - t_{j-m+2}}{m-1}. \tag{2.3.5}$$

We can find p_j by (2.3.1) and (2.3.5)

$$p_j = \frac{t_{j-m+2} + \cdots + t_j}{m-1}. \tag{2.3.6}$$

Therefore (2.3.1) can be represented as

$$x = \sum_{j=0}^{n-1} \frac{t_{j-m+2} + \cdots + t_j}{m-1} B_{j,m}(x). \tag{2.3.7}$$

In order to represent x^k as a linear combination of $B_{j,m}(x)$ for $0 \le k \le m-1$, we need a more powerful tool. To this end, we introduce the dual polynomial of the B-Spline $B_{j,m}$, which is defined by :

$$\begin{cases} \rho_{j,1}(y) = 1 \\ \rho_{j,m}(y) = (y - t_{j-m+2})(y - t_{j-m+3}) \dots (y - t_j), \quad m \ge 2. \end{cases}$$

Furthermore, we define the dual vector of $\mathbf{B}_m = (B_{\mu,m}, ..., B_{\mu+m-1,m})^T$ on the interval $[t_\mu, t_{\mu+1})$ by

$$\vec{\rho}_m = \vec{\rho}_m(y) = (\rho_{\mu,m}(y), ..., \rho_{\mu+m-1,m}(y))^T. \tag{2.3.8}$$

We have the following property that is crucial in deriving the polynomial reproduction in B-splines,

$$R_{m-1}(x)\vec{\rho}_m(y) = (y-x)\vec{\rho}_{m-1}(y), \qquad \text{for } m \ge 2, \quad \text{and } x, y \in \mathbb{R}, \tag{2.3.9}$$

where $R_{m-1}(x)$ is defined as in (2.2.33). We apply (2.3.9) recursively $m-1$ times and get

$$R_1(x_1) \cdots R_{m-1}(x_{m-1})\vec{\rho}_m(y) = (y - x_1) \cdots (y - x_{m-1}), \tag{2.3.10}$$

for all real numbers $x_1, x_2, ..., x_{m-1}$ and y with $t_\mu < t_{\mu+1}$. We also have the following property: For $m \ge 2$ and $x, z \in \mathbb{R}$, then

$$R_{m-1}(z)R_m(x) = R_{m-1}(x)R_m(z). \tag{2.3.11}$$

Now we consider the spline space $S_{m,t}$ defined in (2.2.4) on the interval $[a, b]$. We will derive the B-spline representation for x^k with $0 \le k \le m-1$ by the properties we have listed above.

1. Taking $x_1 = x_2 = ... = x_{m-1} = x$ in (2.3.10), we get

$$R_1(x)...R_{m-1}(x)\vec{\rho}_m(y) = (y-x)^{m-1}. \tag{2.3.12}$$

2. Then (2.2.34) implies that

$$(y-x)^{m-1} = \mathbf{B}_m(x)^T \vec{\rho}_m(y) = \sum_{j=\mu}^{\mu+m-1} \rho_{j,m}(y) B_{j,m}(x), \tag{2.3.13}$$

provided that $x \in [t_\mu, t_{\mu+1})$.

Equation (2.3.13) is the so-called *Marsden's Identity.* We can use it to write explicit B-spline representations for the powers $1, x, x^2, ..., x^{m-1}$ using the basis functions $\{B_{j,m}(x)\}_{j=0}^{n-1}$ on the interval $[a, b]$ as follows:

$$1 = \sum_{j=0}^{n-1} B_{j,m}(x), \quad \text{for} \quad m \geq 1, \tag{2.3.14}$$

$$x = \sum_{j=0}^{n-1} t^*_{j,m} B_{j,m}(x), \quad \text{for} \quad m \geq 2, \tag{2.3.15}$$

$$x^2 = \sum_{j=0}^{n-1} t^{**}_{j,m} B_{j,m}(x), \quad \text{for} \quad m \geq 3 \tag{2.3.16}$$

on the interval $[a, b]$ where

$$t^*_{j,m} = \frac{1}{m-1}(t_{j-m+2} + \ldots + t_j),$$
$$t^{**}_{j,m} = \frac{1}{\binom{m-1}{2}} \sum_{i=j-m+2}^{j-1} \sum_{k=i+1}^{j} t_i t_k.$$

And more generally, for $r = 0, 1, 2, ...m - 1$ we have

$$x^r = \sum_{j=0}^{n-1} \rho^r_{j,m} B_{j,m}(x), \qquad \text{for } x \in [a, b], \tag{2.3.17}$$

where $\rho^r_{j,m}$ are the symmetric polynomials given by:

$$\rho^r_{j,m} = \frac{1}{\binom{m-1}{r}} \sum t_{j_1} t_{j_2} \ldots t_{j_r} \tag{2.3.18}$$

and the sum is over all integers $j_1, j_2, \ldots, j_r$ such that $j - m + 2 \leq j_1 < \ldots < j_r \leq j$ and the total number of terms is $\binom{m-1}{r}$.

Example 2.3.1. *In the cubic case, the explicit representations of the Marsden's* *identities are given below,*

$$1 = \sum_{j=0}^{n-1} B_{j,4}(x),$$

$$x = \frac{1}{3}\sum_{j=0}^{n-1}(t_{j-2} + t_{j-1} + t_j)\, B_{j,4}(x),$$

$$x^2 = \frac{1}{3}\sum_{j=0}^{n-1}(t_{j-2}t_{j-1} + t_{j-2}t_j + t_{j-1}t_j) \quad B_{j,4}(x),$$

$$x^3 = \sum_{j=0}^{n-1} t_{j-2}t_{j-1}t_j \;\; B_{j,4}(x)$$

on $[a, b]$*. see [11].*

When we consider the cardinal B-splines, that is, $N_m(x)$ as defined in (2.1.3), we have the following property.

Theorem 2.3.2. *[3] For any polynomial* $p(x)$ *with degree up to* $m - 1$*,*

$$\sum_{k=-\infty}^{\infty} p(k)N_m(x-k) = \sum_{k=0}^{m-1} N_m(k)p(x-k). \tag{2.3.19}$$

Proof. By Marsden's Identity, for $r = 0, 1, 2, ..., m-1$, we have

$$x^r = \sum_{j=-\infty}^{\infty} \rho^r_{j,m} N_m(x-j), \tag{2.3.20}$$

where

$$\rho^r_{j,m} = \frac{1}{\binom{m-1}{r}} \sum_{j+1 \le j_1 < j_2 < \ldots < j_r \le j+m-1} t_{j_1}t_{j_2}\ldots t_{j_r} \tag{2.3.21}$$

Then, we get

$$\sum_{k=-\infty}^{\infty} k^r N_m(x-k) = \sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \rho^r_{j,m} N_m(k-j)N_m(x-k).$$

By change of variable $s = k - j$, we get

$$\begin{aligned}\sum_{k=-\infty}^{\infty} k^r N_m(x-k) &= \sum_{s=-\infty}^{\infty}\sum_{j=-\infty}^{\infty} \rho^r_{j,m} N_m(s)N_m(x-j-s) \\ &= \sum_{s=-\infty}^{\infty}\Big(\sum_{j=-\infty}^{\infty} \rho^r_{j,m} N_m(x-j-s)\Big)N_m(s).\end{aligned}$$

Therefore, using (2.3.20), we obtain

$$\sum_{k=-\infty}^{\infty} k^r N_m(x-k) = \sum_{s=-\infty}^{\infty} (x-s)^r N_m(s). \tag{2.3.22}$$

Hence, we get

$$\sum_{k=-\infty}^{\infty} p(k) N_m(x-k) = \sum_{k=-\infty}^{\infty} N_m(k) p(x-k). \tag{2.3.23}$$

Since $\text{supp}\{N_m(x)\} = [0, m]$, (2.3.23) is equivalent to (2.3.19). $\square$

Chapter 3

Inverses of Shoenberg-Whitney Matrices for Linear B-Splines

For the general case of the Shoenberg-Whitney matrix $\mathcal{B}_m$, it is very hard to calculate $\mathcal{B}_m^{-1}$. But for the linear case where $m = 2$, it is doable to calculate $\mathcal{B}_2^{-1}$ in an explicit form. Next, we will calculate it in a step-by-step approach.

3.1 General representation for linear Shoenberg-Whitney matrix

Assume that we are given n sample points: $\{y_k\}_{k=0}^{n-1}$ spread on the interval $[a, b]$ with the following condition:

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b. \tag{3.1.1}$$

To do the linear B-spline interpolation on these sample points, we construct a set of linear B-splines $\{B_{i,2}(x)\}_{i=0}^{n-1}$ using the knots $\{t_k\}_{k=-1}^{n}$ with the following form:

$$a = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-2} < t_{n-1} = t_n = b, \tag{3.1.2}$$

where the basis function $B_{i,2}(x)$ is constructed from the knots: $\{t_{i-1}, t_i, t_{i+1}\}$ for $i = 0, \ldots, n-1$. Furthermore, the Shoenberg-Whitney condition must be satisfied, that is,

$$t_{i-1} < y_i < t_{i+1}, \quad \text{for } 1 \le i \le n-2, \tag{3.1.3}$$

which implies that

$$y_{i-1} < t_i < y_{i+1}, \quad \text{for } 1 \le i \le n-2. \tag{3.1.4}$$

With the above setting, we shall derive the *unified form* of the Shoenberg-Whitney matrix, whose general form is given by

$$\mathcal{B}_2 := \begin{bmatrix} B_{0,2}(y_0) & \cdots & B_{n-1,2}(y_0) \\ B_{0,2}(y_1) & \cdots & B_{n-1,2}(y_1) \\ \vdots & \cdots & \vdots \\ B_{0,2}(y_{n-1}) & \cdots & B_{n-1,2}(y_{n-1}) \end{bmatrix}_{n\times n}. \tag{3.1.5}$$

Many of the entries in matrix $\mathcal{B}_2$ are zero. Specifically, when $j \geq i+2$ for $0 \leq i \leq n-3$, the (i,j) entry (counting from zero) of $\mathcal{B}_2$ is $B_{j,2}(y_i)$. Since the knots for $B_{j,2}(x)$ are $\{t_{j-1}, t_j, t_{j+1}\}$ and $j-2 \geq i$, by (3.1.4) and (3.1.2), we have $y_i < t_{i+1} \leq t_{j-1}$, that is, $y_i \notin [t_{j-1}, t_{j+1}]$, which implies that $B_{j,2}(y_i) = 0$.

On the other hand, when $i \geq j+2$, by (3.1.4) and (3.1.1), we have $t_{j+1} < y_{j+2} \leq y_i$, that is, $y_i \notin [t_{j-1}, t_{j+1}]$, which implies that $B_{j,2}(y_i) = 0$. Thus we conclude that $\mathcal{B}_2$ matrix in (3.1.5) is a tridiagonal matrix. Specifically, we can write (3.1.5) as

$$\mathcal{B}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ B_{0,2}(y_1) & B_{1,2}(y_1) & B_{2,2}(y_1) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & B_{n-3,2}(y_{n-2}) & B_{n-2,2}(y_{n-2}) & B_{n-1,2}(y_{n-2}) \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n\times n}. \tag{3.1.6}$$

Furthermore, this tridiagonal matrix has a special property:

$$B_{i-1,2}(y_i)B_{i+1,2}(y_i) = 0. \tag{3.1.7}$$

That means, either $B_{i-1,2}(y_i) = 0$ or $B_{i+1,2}(y_i) = 0$, which depends on $y_i \in (t_{i-1}, t_i)$ or $y_i \in (t_i, t_{i+1})$. In particular, when $y_i = t_i$, we have $B_{i-1,2}(y_i) = B_{i+1,2}(y_i) = 0$, and $B_{i,2}(y_i) = 1$.

In order to handle the uncertainty in (3.1.6) in a controlled way, we introduce a set of indicator variables $\{\sigma_i\}_{i=1}^{n-2}$ as follows,

$$\sigma_i = \begin{cases} 1 & \text{if} \quad y_i \in (t_{i-1}, t_i) \\ 0 & \text{if} \quad y_i \in [t_i, t_{i+1}). \end{cases} \tag{3.1.8}$$

With these $(n-2)$ indicator variables, there are 2^{n-2} different choices for the $(n-2)$-tuple $(\sigma_1, \ldots, \sigma_{n-2})$, with each corresponds to one specific knots-data setting.

Now we can write explicit expressions for $B_{i-1,2}(y_i), B_{i,2}(y_i), B_{i+1,2}(y_i)$ as follows,

$$\begin{cases} B_{i-1,2}(y_i) = \sigma_i \dfrac{t_i - y_i}{t_i - t_{i-1}}, \\ B_{i,2}(y_i) = \sigma_i \dfrac{y_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i) \dfrac{t_{i+1} - y_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(y_i) = (1-\sigma_i) \dfrac{y_i - t_i}{t_{i+1} - t_i}. \end{cases} \tag{3.1.9}$$

Next, we consider several special cases which can help us find the general $\mathcal{B}_2^{-1}$ gradually.

3.2 Inverses of linear Shoenberg-Whitney matrix for special cases

We need the following lemma that is a special case of the theorem [17] for our first case.

Lemma 3.2.1. *Let G be an upper bi-diagonal matrix with the following form*

$$G = \begin{bmatrix} b_1 & c_1 & \cdots & \cdots & \cdots & 0 \\ 0 & b_2 & c_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & c_{n-1} \\ 0 & \cdots & \cdots & \cdots & \cdots & b_n \end{bmatrix}_{n\times n} . \tag{3.2.1}$$

Then $G^{-1} = (\beta_{i,j})$ with

$$\beta_{i,j} = \begin{cases} 0 & \text{if} \quad i > j; \\ \dfrac{1}{b_i} & \text{if} \quad i = j; \\ (-1)^{i+j} \dfrac{c_i \cdots c_{j-1}}{b_i \cdots b_j} & \text{if} \quad i < j. \end{cases} \tag{3.2.2}$$

Proof: First we denote $G = (a_{i,j})$ where

$$a_{i,j} = \begin{cases} b_i & \text{if} \quad i = j; \\ c_i & \text{if} \quad j = i+1; \\ 0 & \text{otherwise.} \end{cases}$$

In order to show that $G^{-1} = (\beta_{i,j})$, we let $L = (\beta_{i,j})$, and it is equivalent to show that $GL = I_n$.

Now we assume that $GL = (c_{i,j})$, which gives us $c_{i,j} = \sum_{k=1}^{n} a_{i,k}\beta_{k,j}$. We use the following three steps to verify that $(c_{i,j})$ is the identity matrix.

- Verify that $c_{i,i} = 1$ for $i = 1, 2, \ldots, n$.

$$c_{i,i} = \sum_{k=1}^{i-1} a_{i,k}\beta_{k,i} + a_{i,i}\beta_{i,i} + \sum_{k=i+1}^{n} a_{i,k}\beta_{k,i} = 0 + b_i\left(\frac{1}{b_i}\right) + 0 = 1.$$

- Verify that $c_{i,j} = 0$ for $i > j$.

$$c_{i,j} = \sum_{k=1}^{i-1} a_{i,k}\beta_{k,j} + a_{i,i}\beta_{i,j} + \sum_{k=i+1}^{n} a_{i,k}\beta_{k,j} = 0.$$

- Verify that $c_{i,j} = 0$ for $i < j$.

  First we consider the case that $i + 1 < j$. We have

$$\begin{aligned}
c_{i,j} &= \sum_{k=1}^{j-1} a_{i,k}\beta_{k,j} + a_{i,j}\beta_{j,j} + \sum_{k=j+1}^{n} a_{i,k}\beta_{k,j} = \sum_{k=1}^{j-1} a_{i,k}B_{k,j} + 0 + 0 \\
&= \sum_{k=1}^{i-1} a_{i,k}\beta_{k,j} + a_{i,i}\beta_{i,j} + \sum_{k=i+1}^{j} a_{i,k}\beta_{k,j} \\
&= 0 + b_i\frac{(-1)^{i+j}c_i\cdots c_{j-1}}{b_i\cdots b_j} + a_{i,i+1}\beta_{i+1,j} \\
&= \frac{(-1)^{i+j}c_i\cdots c_{j-1}}{b_{i+1}\cdots b_j} + c_i\frac{(-1)^{i+1+j}c_{i+1}\cdots c_{j-1}}{b_{i+1}\cdots b_j} = 0.
\end{aligned}$$

  Second we consider the case that $i + 1 = j$. We still have

$$\begin{aligned}
c_{i,j} &= \sum_{k=1}^{i-1} a_{i,k}\beta_{k,j} + a_{i,i}\beta_{i,j} + \sum_{k=i+1}^{j} a_{i,k}\beta_{k,j} \\
&= b_i(-1)\frac{c_i}{b_ib_{i+1}} + c_i\left(\frac{1}{b_{i+1}}\right) = 0.
\end{aligned}$$

Thus, we complete the proof. $\square$

Case I: $\sigma_i = 0$ for $i = 1, \ldots, n-2$.

In this case, we have that $y_i \in [t_i, t_{i+1})$ for all $i = 1, \ldots, n-2$, and (3.1.9) becomes

$$\begin{cases} B_{i-1,2}(y_i) = 0, \\ B_{i,2}(y_i) = \dfrac{t_{i+1} - y_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(y_i) = \dfrac{y_i - t_i}{t_{i+1} - t_i}. \end{cases} \tag{3.2.3}$$

$\mathcal{B}_2$ becomes an upper triangular matrix in the form of

$$B_2^I = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \dfrac{t_2 - y_1}{t_2 - t_1} & \dfrac{y_1 - t_1}{t_2 - t_1} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & \dfrac{t_3 - y_2}{t_3 - t_2} & \dfrac{y_2 - t_2}{t_3 - t_2} & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \dfrac{t_{n-1} - y_{n-2}}{t_{n-1} - t_{n-2}} & \dfrac{y_{n-2} - t_{n-2}}{t_{n-1} - t_{n-2}} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{3.2.4}$$

We introduce another notation to make the expression of B_2^I a little simpler,

$$\eta_{i,j} = \frac{t_i - y_j}{t_i - t_{i-1}}. \tag{3.2.5}$$

Then B_2^I becomes

$$B_2^I = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \eta_{2,1} & 1 - \eta_{2,1} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & \eta_{3,2} & 1 - \eta_{3,2} & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \eta_{n-1,n-2} & 1 - \eta_{n-1,n-2} \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{3.2.6}$$

In order to represent $\left(B_2^I\right)^{-1}$ in a concise way, we need another notation,

$$\xi_{i,j} = \left(\frac{y_{i-1} - t_{i-1}}{t_i - y_{i-1}}\right)\left(\frac{y_i - t_i}{t_{i+1} - y_i}\right) \cdots \left(\frac{y_{j-1} - t_{j-1}}{t_j - y_{j-1}}\right). \tag{3.2.7}$$

Then by **Lemma 3.2.1** we can write

$$\left(B_2^I\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{t_2-t_1}{y_1-t_1}\xi_{2,2} & -\frac{t_3-t_2}{y_2-t_2}\xi_{2,3} & \frac{t_4-t_3}{y_3-t_3}\xi_{2,4} & \cdots & (-1)^{n+1}\frac{t_{n-1}-t_{n-2}}{y_{n-2}-t_{n-2}}\xi_{2,n-1} & \vdots \\ \vdots & \ddots & \frac{t_3-t_2}{y_2-t_2}\xi_{3,3} & -\frac{t_4-t_3}{y_3-t_3}\xi_{3,4} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \frac{t_4-t_3}{y_3-t_3}\xi_{4,4} & \ddots & \frac{t_{n-1}-t_{n-2}}{y_{n-2}-t_{n-2}}\xi_{n-3,n-1} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\frac{t_{n-1}-t_{n-2}}{y_{n-2}-t_{n-2}}\xi_{n-2,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \frac{t_{n-1}-t_{n-2}}{y_{n-2}-t_{n-2}}\xi_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix},$$

which can be further simplified as

$$\left(B_2^I\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \xi_{2,2} & -\xi_{2,3} & \xi_{2,4} & \cdots & (-1)^{n+1}\xi_{2,n-1} & \vdots \\ \vdots & \ddots & \xi_{3,3} & -\xi_{3,4} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \xi_{4,4} & \ddots & \xi_{n-3,n-1} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\xi_{n-2,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \xi_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \Lambda_2,$$

where

$$\Lambda_2 = \begin{bmatrix} 1 & & & & & \\ & \frac{1}{\eta_{2,1}} & & & & \\ & & \frac{1}{\eta_{3,2}} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{\eta_{n-1,n-2}} & \\ & & & & & 1 \end{bmatrix}_{n\times n}.$$

Case II: $\sigma_i = 1$ for $i = 1, \dots, n-2$.

In this case, we have that $y_i \in (t_{i-1}, t_i)$ for all $i = 1, \dots, n-2$, and (3.1.9) becomes

$$\begin{cases} B_{i-1,2}(y_i) = \dfrac{t_i - y_i}{t_i - t_{i-1}}, \\ B_{i,2}(y_i) = \dfrac{y_i - t_i}{t_i - t_{i-1}}, \\ B_{i+1,2}(y_i) = 0. \end{cases} \tag{3.2.8}$$

$\mathcal{B}_2$ becomes a lower bi-diagonal matrix in the form of

$$B_2^{II} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \dfrac{t_1 - y_1}{t_1 - t_0} & \dfrac{y_1 - t_0}{t_1 - t_0} & 0 & \cdots & \cdots & \cdots & \vdots \\ 0 & \dfrac{t_2 - y_2}{t_2 - t_1} & \dfrac{y_2 - t_1}{t_2 - t_1} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & 0 & \dfrac{t_{n-2} - y_{n-2}}{t_{n-2} - t_{n-3}} & \dfrac{y_{n-2} - t_{n-3}}{t_{n-2} - t_{n-3}} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n \times n} \tag{3.2.9}$$

Notice that the transpose of B_2^{II} is an upper bi-diagonal matrix, we take following expressions

$$\begin{aligned} B_2^{II} &= \begin{bmatrix} b_0 & 0 & \cdots & \cdots & \cdots & 0 \\ c_1 & b_1 & 0 & 0 & \cdots & \vdots \\ 0 & c_2 & b_2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & c_{n-1} & b_{n-1} \end{bmatrix} \\ &= \begin{bmatrix} b_0 & c_1 & \cdots & \cdots & \cdots & 0 \\ 0 & b_1 & c_2 & 0 & \cdots & \vdots \\ 0 & 0 & b_2 & c_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_{n-2} & c_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & b_{n-1} \end{bmatrix}^T =: F_2^T, \end{aligned}$$

where

$$b_i = \begin{cases} \dfrac{y_i - t_{i-1}}{t_i - t_{i-1}}, & \text{if} \quad 1 \leqslant i \leqslant n-2, \\ 1, & \text{if} \quad i = 0 \quad \text{or} \quad i = n-1, \end{cases} \tag{3.2.10}$$

and

$$c_i = \begin{cases} \dfrac{t_i - y_i}{t_i - t_{i-1}}, & \text{if} \quad 1 \leqslant i \leqslant n-2, \\ 0, & \text{if} \quad i = n-1. \end{cases} \tag{3.2.11}$$

Observe that $\left(B_2^{II}\right)^{-1} = (F_2^T)^{-1} = (F_2^{-1})^T$, by **Lemma 3.2.1**, we get

$$\left(B_2^{II}\right)^{-1} = [\alpha_{i,j}]_{n\times n},$$

where

$$\alpha_{i,j} = \begin{cases} 0, & \text{if} \quad i < j, \\ \dfrac{1}{b_i}, & \text{if} \quad i = j, \\ (-1)^{i+j}\dfrac{c_j \cdots c_{i-1}}{b_j \cdots b_i}, & \text{if} \quad i > j. \end{cases} \tag{3.2.12}$$

To further simplify the expressions in (3.2.12), we introduce the notation $\nu_{i,j}$ as follows,

$$\nu_{i,j} = \left(\frac{t_j - y_{j+1}}{y_{j+1} - t_{j-1}}\right)\left(\frac{t_{j-1} - y_j}{y_j - t_{j-2}}\right)\cdots\left(\frac{t_{i-1} - y_i}{y_i - t_{i-2}}\right). \tag{3.2.13}$$

With (3.2.10), (3.2.11), and (3.2.13), we can simplify (3.2.12) to the following form

$$\alpha_{i,j} = \begin{cases} 0, & \text{if} \quad i < j, \\ \dfrac{t_{i-1} - t_{i-2}}{y_i - t_{i-2}}, & \text{if} \quad i = j, \\ (-1)^{i+j}\left(\dfrac{t_{j-1} - t_{j-2}}{y_j - t_{j-2}}\right)\nu_{i,j}, & \text{if} \quad i > j. \end{cases} \tag{3.2.14}$$

Thus, we can write $\left(B_2^{II}\right)^{-1}$ as

$$\left(B_2^{II}\right)^{-1} =$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -\nu_{2,1} & \dfrac{t_1 - t_0}{y_2 - t_0} & 0 & \ddots & \ddots & \ddots & \vdots \\ \nu_{3,1} & -\left(\dfrac{t_1 - t_0}{y_2 - t_0}\right)\nu_{3,2} & & \ddots & \ddots & \vdots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ (-1)^n\nu_{n-1,1} & (-1)^{n+1}\left(\dfrac{t_1 - t_0}{y_2 - t_0}\right)\nu_{n-1,2} & \cdots & 0 & -\left(\dfrac{t_{n-3} - t_{n-4}}{y_{n-2} - t_{n-4}}\right)\nu_{n-1,n-2} & \dfrac{t_{n-2} - t_{n-3}}{t_{n-1} - y_{n-3}} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n\times n},$$

which can be further represented as

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ -\nu_{2,1} & \nu_{2,2} & 0 & \ddots & \ddots & \ddots & \vdots \\ \nu_{3,1} & -\nu_{3,2} & & \ddots & \ddots & \vdots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ (-1)^n\nu_{n-1,1} & (-1)^{n+1}\nu_{n-1,2} & \cdots & 0 & -\nu_{n-1,n-2} & \nu_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n\times n} \Lambda_2^*,$$

where

$$\Lambda_2^* = \begin{bmatrix} 1 & & & & & \\ & \dfrac{1}{1-\eta_{1,2}} & & & & \\ & & \dfrac{1}{1-\eta_{2,3}} & & & \\ & & & \ddots & & \\ & & & & \dfrac{1}{1-\eta_{n-2,n-1}} & \\ & & & & & 1 \end{bmatrix}_{n\times n} .$$

Case III: Assume that n is even, and $\sigma_1 = \sigma_3 = \cdots = \sigma_{n-3} = 0$ and $\sigma_2 = \sigma_4 = \cdots = \sigma_{n-2} = 1$.

In this case, we have that $y_{2k-1} \in [t_{2k-1}, t_{2k})$ for all $k = 1, 2, \ldots, \dfrac{n}{2} - 1$, and (3.1.9) becomes

$$\begin{cases} B_{2k-2,2}(y_{2k-1}) = 0, \\ B_{2k-1,2}(y_{2k-1}) = \dfrac{t_{2k} - y_{2k-1}}{t_{2k} - t_{2k-1}}, \\ B_{2k+1,2}(y_{2k-1}) = \dfrac{y_{2k-1} - t_{2k-1}}{t_{2k} - t_{2k-1}}. \end{cases} \tag{3.2.15}$$

We also have that $y_{2k} \in (t_{2k-1}, t_{2k})$ for $k = 1, 2, \ldots, \dfrac{n}{2} - 1$, and (3.1.9) becomes

$$\begin{cases} B_{2k-1,2}(y_{2k}) = \dfrac{t_{2k} - y_{2k}}{t_{2k} - t_{2k-1}}, \\ B_{2k,2}(y_{2k}) = \dfrac{y_{2k} - t_{2k-1}}{t_{2k} - t_{2k-1}}, \\ B_{2k+1,2}(y_{2k}) = 0. \end{cases} \tag{3.2.16}$$

$\mathcal{B}_2$ becomes a block diagonal matrix in the form of

$$
B_2^{III} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 \\
0 & \frac{t_2 - y_1}{t_2 - t_1} & \frac{y_1 - t_1}{t_2 - t_1} & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \frac{t_2 - y_2}{t_2 - t_1} & \frac{y_2 - t_1}{t_2 - t_1} & 0 & \ddots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & 0 & \frac{t_4 - y_3}{t_4 - t_3} & \frac{y_3 - t_3}{t_4 - t_3} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \frac{t_4 - y_4}{t_4 - t_3} & \frac{y_4 - t_3}{t_4 - t_3} & 0 & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \ddots & 0 & \ddots & \ddots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \ddots & \frac{t_{n-2} - y_{n-3}}{t_{n-2} - t_{n-3}} & \frac{y_{n-3} - t_{n-3}}{t_{n-2} - t_{n-3}} & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \ddots & \frac{t_{n-2} - y_{n-2}}{t_{n-2} - t_{n-3}} & \frac{y_{n-2} - t_{n-3}}{t_{n-2} - t_{n-3}} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1
\end{bmatrix}
\tag{3.2.17}
$$

Denote

$$
D_j = \begin{bmatrix}
\frac{t_j - y_{j-1}}{t_j - t_{j-1}} & \frac{y_{j-1} - t_{j-1}}{t_j - t_{j-1}} \\
\frac{t_j - y_j}{t_j - t_{j-1}} & \frac{y_j - t_{j-1}}{t_j - t_{j-1}}
\end{bmatrix}.
\tag{3.2.18}
$$

Then we can write (3.2.17) as

$$
B_2^{III} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & D_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & D_4 & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & D_{n-2} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}_{n\times n}.
\tag{3.2.19}
$$

Thus the inverse of B_2^{III} can be written as

$$
\left(B_2^{III}\right)^{-1} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & D_2^{-1} & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & D_4^{-1} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & D_{n-2}^{-1} & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{bmatrix}_{n\times n}.
\tag{3.2.20}
$$

To make the calculation of D_j^{-1} easier, we use the notation $\eta_{i,j}$ as in (3.2.5), and get

$$D_j = \begin{bmatrix} \eta_{j,j-1} & 1-\eta_{j,j-1} \\ \eta_{j,j} & 1-\eta_{j,j} \end{bmatrix}, \quad j \in \{2, 4, \cdots, n-2\}.$$

Then we get

$$D_j^{-1} = \frac{1}{\eta_{j,j-1}-\eta_{j,j}} \begin{bmatrix} 1-\eta_{j,j} & \eta_{j,j-1}-1 \\ -\eta_{j,j} & \eta_{j,j-1} \end{bmatrix} = \begin{bmatrix} \dfrac{y_j - t_{j-1}}{y_j - y_{j-1}} & \dfrac{t_{j-1} - y_{j-1}}{y_j - y_{j-1}} \\ \\ \dfrac{y_j - t_j}{y_j - y_{j-1}} & \dfrac{t_j - y_{j-1}}{y_j - y_{j-1}} \end{bmatrix}.$$

Now we can write $\left(B_2^{III}\right)^{-1}$ explicitly as follows,

$$\left(B_2^{III}\right)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{y_2-t_1}{y_2-y_1} & \frac{t_1-y_1}{y_2-y_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \frac{y_2-t_2}{y_2-y_1} & \frac{t_2-y_1}{y_2-y_1} & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \frac{y_4-t_3}{y_4-y_3} & \frac{t_3-y_3}{y_4-y_3} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{y_4-t_4}{y_4-y_3} & \frac{t_4-y_3}{y_4-y_3} & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{y_{n-2}-t_{n-3}}{y_{n-2}-y_{n-3}} & \frac{t_{n-3}-y_{n-3}}{y_{n-2}-y_{n-3}} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{y_{n-2}-t_{n-2}}{y_{n-2}-y_{n-3}} & \frac{t_{n-2}-y_{n-3}}{y_{n-2}-y_{n-3}} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n\times n} \tag{3.2.21}$$

3.3 Inverses of linear Shoenberg-Whitney matrix for the general case

In order to study the general case of the Shoenberg-Whitney matrix, we start with the following special case first, because it can be used as the building block of the general case.

Case IV: Assume that $\sigma_1 = \sigma_2 = \cdots = \sigma_k = 0$ and $\sigma_{k+1} = \cdots = \sigma_{n-2} = 1$ for $1 \le k \le n-3$.

In this case, we have that $y_i \in [t_i, t_{i+1})$ for $i = 1, \dots, k$, and (3.1.9) becomes

$$\begin{cases} B_{i-1,2}(y_i) = 0, \\ B_{i,2}(y_i) = \dfrac{t_{i+1} - y_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(y_i) = \dfrac{y_i - t_i}{t_{i+1} - t_i}. \end{cases}$$

We also have that $y_i \in (t_{i-1}, t_i)$ for $i = k+1, \dots, n-2$, and (3.1.9)becomes

$$\begin{cases} B_{i-1,2}(y_i) = \dfrac{t_i - y_i}{t_i - t_{i-1}}, \\ B_{i,2}(y_i) = \dfrac{y_i - t_{i-1}}{t_i - t_{i-1}}, \\ B_{i+1,2}(y_i) = 0. \end{cases}$$

Then $\mathcal{B}_2$ has the form of

$$B_2^{IV} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{t_2 - y_1}{t_2 - t_1} & \frac{y_1 - t_1}{t_2 - t_1} & 0 & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & \frac{t_k - y_{k-1}}{t_k - t_{k-1}} & \frac{y_{k-1} - t_{k-1}}{t_k - t_{k-1}} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{t_{k+1} - y_k}{t_{k+1} - t_k} & \frac{y_k - t_k}{t_{k+1} - t_k} & \cdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{t_{k+1} - y_{k+1}}{t_{k+1} - t_k} & \frac{y_{k+1} - t_k}{t_{k+1} - t_k} & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \cdots & \frac{t_{n-2} - y_{n-2}}{t_{n-2} - t_{n-3}} & \frac{y_{n-2} - t_{n-3}}{t_{n-2} - t_{n-3}} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n \times n} \tag{3.3.1}$$

With the $\eta_{i,j}$ notation as in (3.2.5), we can write (3.3.1) as

$$B_2^{IV} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & \eta_{21} & 1 - \eta_{21} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & \eta_{k+1,k} & 1 - \eta_{k+1,k} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \eta_{k+1,k+1} & 1 - \eta_{k+1,k+1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \eta_{n-2,n-2} & 1 - \eta_{n-2,n-2} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}_{n \times n} \tag{3.3.2}$$

In order to write a general form for the inverse of (3.3.2), we define a special tri-diagonal matrix from the given independent variables $u_1, \ldots, u_p, v_1, \ldots, v_q$ with $p \geq 1$ and $q \geq 1$ as follows,

$$\Omega(\vec{u}, \vec{v}) = \begin{bmatrix} u_1 & 1-u_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & u_2 & 1-u_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & u_p & 1-u_p & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_1 & 1-v_1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & v_2 & 1-v_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & v_q & 1-v_q \end{bmatrix}_{m\times m}, \tag{3.3.3}$$

where we denote $\vec{u} := (u_1, \ldots, u_p)$ and $\vec{v} := (v_1, \ldots, v_q)$ and $m = p + q$. In the following lemma, we give the formula for $\Omega^{-1}(\vec{u}, \vec{v})$.

Lemma 3.3.1. *Given a p-vector $\vec{u} := (u_1, \ldots, u_p)$ and a q-vector $\vec{v} := (v_1, \ldots, v_q)$ that satisfy the conditions: $p \geq 1, q \geq 1$ and $p+q = m$, define a tridiagonal matrix $\Omega(\vec{u}, \vec{v})$ as in (3.3.3). In order to make $\Omega(\vec{u}, \vec{v})$ invertible, we require that*

$$u_i \neq 0 \quad \text{for } 1 \leq i \leq p, \quad u_p \neq v_1, \text{ and } \quad v_j \neq 1 \quad \text{for } 2 \leq j \leq q. \tag{3.3.4}$$

Then we can write $\Omega^{-1}(\vec{u}, \vec{v})$ as $[\alpha_{ij}]$, where the non-zero entries in $\{\alpha_{ij}\}$ are given by the following formulas:

- *For the diagonal entries, we have*

$$\alpha_{ii} = \frac{1}{u_i}, \quad \text{for } 1 \leq i \leq p-1, \quad \alpha_{pp} = \frac{1-v_1}{u_p - v_1}, \tag{3.3.5}$$

  and

$$\alpha_{p+1,p+1} = \frac{u_p}{u_p - v_1}, \quad \alpha_{jj} = \frac{1}{1 - v_{j-p}} \quad \text{for } p+2 \leq j \leq m. \tag{3.3.6}$$

- *For the lower triangular entries $\alpha_{i,j}$ with $p \leq j < i \leq m$, we represent them in the following general formulas, for $r = 1, \ldots, q-1$:*

$$\alpha_{p+r,p} = -\frac{v_1}{u_p}\,\alpha_{p+r,p+1}, \quad \alpha_{p+r+1,p+1} = -\frac{u_p\, v_2}{u_p - v_1}\,\alpha_{p+r+1,p+2}, \tag{3.3.7}$$

  and for $p+2 \leq j \leq m-r$,

$$\alpha_{j+r,j} = -\frac{v_{j-p+1}}{1 - v_{j-p}}\,\alpha_{j+r,j+1}. \tag{3.3.8}$$

- *For the upper triangular entries, for $s = 1, \dots, q$,*

$$\forall\, 1 \le i \le p, \qquad \alpha_{i,i+s} = \Big(1 - \frac{1}{u_i}\Big)\alpha_{i+1,i+s}, \tag{3.3.9}$$

  and for $t = q+1, \dots, m-1$,

$$\forall\, 1 \le i \le m-t, \qquad \alpha_{i,i+t} = \Big(1 - \frac{1}{u_i}\Big)\alpha_{i+1,i+t}. \tag{3.3.10}$$

With **Lemma 3.3.1**, we can represent $\big(B_2^{IV}\big)^{-1}$ as $\Omega^{-1}(\vec{u}, \vec{v})$ with $\vec{u} := (\eta_{21}, \dots, \eta_{k+1,k})$ and a q-vector $\vec{v} := (\eta_{k+1,k+1}, \dots, \eta_{n-2,n-2})$.

Case V: (The general case)

Notice that σ_k can only be 0 or 1 for $1 \le k \le n-2$, we can formulate the general case in the following format:

Assume that $\sigma_1 = \cdots = \sigma_{k_1} = 0, \sigma_{k_1+1} = \cdots = \sigma_{k_2} = 1, \cdots, \sigma_{k_{s-1}+1} = \cdots = \sigma_{k_s} = 1 + 2\left\lfloor \frac{s}{2} \right\rfloor - s$. (That means, when s is even, $\sigma_{k_s} = 1$; when s is odd, $\sigma_{k_s} = 0$.)

Note: In order to make the discussion a little easier, we allow $k_1 = 0$. When it happens, we have $\sigma_1 = \cdots = \sigma_{k_2} = 1$ and we treat the part $\sigma_1 = \cdots = \sigma_{k_1} = 0$ empty.

Under this assumption, $\mathcal{B}_2$ in 3.1.6 can be written as a block-diagonal matrix as follows,

$$B_2^V := \begin{bmatrix} \mathcal{B}_2^1 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{B}_2^2 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \mathcal{B}_2^{\lceil s/2 \rceil - 1} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{B}_2^{\lceil s/2 \rceil} \end{bmatrix}, \tag{3.3.11}$$

where $\mathcal{B}_2^1$ has different structures based on the value of k_1, and $\mathcal{B}_2^{\lceil s/2 \rceil}$ has different structures based on the parity of s.

More specifically, when $k_1 > 0$, we have a tridiagonal matrix

$$\mathcal{B}_2^1 := \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & B_{1,2}(y_1) & B_{2,2}(y_1) & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & B_{k_1,2}(y_{k_1}) & B_{k_1+1,2}(y_{k_1}) & 0 & 0 \\ \vdots & \ddots & \ddots & B_{k_1,2}(y_{k_1+1}) & B_{k_1+1,2}(y_{k_1+1}) & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & B_{k_2-1,2}(y_{k_2}) & B_{k_2,2}(y_{k_2}) \end{bmatrix}. \quad (3.3.12)$$

When $k_1 = 0$, its structure becomes a lower bidiagonal matrix

$$\mathcal{B}_2^1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ B_{0,2}(y_1) & B_{1,2}(y_1) & 0 & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & B_{k_2-1,2}(y_{k_2}) & B_{k_2,2}(y_{k_2}) \end{bmatrix}. \quad (3.3.13)$$

For $\mathcal{B}_2^{\lceil s/2 \rceil}$, when s is even, we have a tridiagonal matrix

$$\mathcal{B}_2^{\lceil s/2 \rceil} :=$$

$$\begin{bmatrix} B_{k_{s-2},2}(y_{k_{s-2}}) & B_{k_{s-2}+1,2}(y_{k_{s-2}}) & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{k_{s-1},2}(y_{k_{s-1}}) & B_{k_{s-1}+1,2}(y_{k_{s-1}}) & 0 & 0 & 0 \\ 0 & 0 & B_{k_{s-1},2}(y_{k_{s-1}+1}) & B_{k_{s-1}+1,2}(y_{k_{s-1}+1}) & 0 & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \cdots & B_{k_s-1,2}(y_{k_s}) & B_{k_s,2}(y_{k_s}) & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}. \quad (3.3.14)$$

When s is odd, we have an upper bidiagonal matrix

$$B_2^{\lceil s/2 \rceil} := \begin{bmatrix} B_{k_{s-1},2}(y_{k_{s-1}}) & B_{k_{s-1}+1,2}(y_{k_{s-1}}) & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & B_{k_s,2}(y_{k_s}) & B_{k_s+1,2}(y_{k_s}) \\ 0 & \cdots & 0 & 1 \end{bmatrix}. \quad (3.3.15)$$

For the interior matrix-block cases $2 \leq j \leq \lceil s/2 \rceil - 1$, we have a tridiagonal matrix

$$\mathcal{B}_2^j :=$$

$$
\begin{bmatrix}
B_{k_{2(j-1)}+1,2}(y_{k_{2(j-1)}+1}) & B_{k_{2(j-1)}+2,2}(y_{k_{2(j-1)}+1}) & 0 & 0 & 0 & 0 \\
0 & \ddots & \ddots & 0 & 0 & 0 \\
\ddots & 0 & B_{k_{2j-1},2}(y_{k_{2j-1}}) & B_{k_{2j-1}+1,2}(y_{k_{2j-1}}) & 0 & 0 \\
\ddots & 0 & B_{k_{2j-1},2}(y_{k_{2j-1}+1}) & B_{k_{2j-1}+1,2}(y_{k_{2j-1}+1}) & 0 & \vdots \\
\ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\ddots & \ddots & \ddots & \cdots & B_{k_{2j}-1,2}(y_{k_{2j}}) & B_{k_{2j},2}(y_{k_{2j}})
\end{bmatrix}. \tag{3.3.16}
$$

Since each diagonal block matrix $\mathcal{B}_2^j$ in (3.3.11) is invertible for $1 \leq j \leq \lceil s/2 \rceil$, we can easily write the inverse of B_2^V as follows,

$$
\left(B_2^V\right)^{-1} := \begin{bmatrix}
(\mathcal{B}_2^1)^{-1} & 0 & 0 & 0 & 0 \\
0 & (\mathcal{B}_2^2)^{-1} & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \left(\mathcal{B}_2^{\lceil s/2 \rceil - 1}\right)^{-1} & 0 \\
0 & 0 & 0 & 0 & \left(\mathcal{B}_2^{\lceil s/2 \rceil}\right)^{-1}
\end{bmatrix}. \tag{3.3.17}
$$

Explicit formulas for $\left(\mathcal{B}_2^j\right)^{-1}$ can also be given by the result of *Case IV*.

The explicit formulas for $\left(\mathcal{B}_2^j\right)^{-1}$ with $1 \leq j \leq \lceil s/2 \rceil$ in (3.3.17) can be given in the following three cases:

- For $j = 1$, when $k_1 = 0$, we choose $\vec{u_1} := (1)$ and $\vec{v_1} := (\eta_{01}, \ldots, \eta_{k_2-1,k_2})$. Then $(\mathcal{B}_2^1)^{-1} = \Omega^{-1}(\vec{u_1}, \vec{v_1})$.

  When $k_1 > 0$, we take $\vec{u_1} := (\eta_{11}, \ldots, \eta_{k_1,k_1})$ and $\vec{v_1} := (\eta_{k_1,k_1+1}, \ldots, \eta_{k_2-1,k_2})$. Then
$$
\left(\mathcal{B}_2^1\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega^{-1}(\vec{u_1}, \vec{v_1}) \end{bmatrix}.
$$

- For $j = \lceil s/2 \rceil$, when s is even, let $\vec{u_s} := (\eta_{k_{s-2},k_{s-2}}, \ldots, \eta_{k_{s-1},k_{s-1}})$ and $\vec{v_s} := (\eta_{k_{s-1},k_{s-1}+1}, \ldots, \eta_{k_s-1,k_s})$. Then
$$
\left(\mathcal{B}_2^{\lceil s/2 \rceil}\right)^{-1} = \begin{bmatrix} \Omega^{-1}(\vec{u_s}, \vec{v_s}) & 0 \\ 0 & 1 \end{bmatrix}.
$$

  When s is odd, let $\vec{u_s} := (\eta_{k_{s-1},k_{s-1}}, \ldots, \eta_{k_s,k_s})$ and $\vec{v_s} := (1)$. Then $\left(\mathcal{B}_2^{\lceil s/2 \rceil}\right)^{-1} = \Omega^{-1}(\vec{u_s}, \vec{v_s})$.

- For $2 \leq j \leq \lceil s/2 \rceil - 1$, we note by $\vec{u_j} := (\eta_{k_{2(j-1)}+1, k_{2(j-1)}+1}, \ldots, \eta_{k_{2j-1}, k_{2j-1}})$ and $\vec{v_j} := (\eta_{k_{2j-1}, k_{2j-1}+1}, \ldots, \eta_{k_{2j}-1, k_{2j}})$. Then $\left(\mathcal{B}_2^j\right)^{-1} = \Omega^{-1}(\vec{u_j}, \vec{v_j})$.

With these formulas, *Case V* covers all the possible cases for the linear Shoenberg-Whitney matrices.

Chapter 4

Local Quasi-Interpolating Operators Based-on B-splines

4.1 Framework for local quasi-interpolation

Linear operators from the local quasi-interpolation are very useful in data analysis, and the tool of B-splines is very powerful in many application problems. In this chapter, we will describe the general framework for the local quasi-interpolating operators from the B-splines.

4.1.1 Linear operators induced by data points

Here we would like to use the linear operator way to describe the method. Since the space $C[a,b]$ is infinite-dimensional, we would like to consider its finite-dimensional subspaces to approximate functions in $C[a,b]$. The spline space $S_{m,t}$ defined as in (2.2.4) for certain knot sequence $t=(t_j)$ satisfying (2.2.1) and (2.2.2) is a good choice in this situation.

Our linear operators are data dependent, that means they rely on a set of data samples. Given a set of data samples $\{y_i\}_{i=0}^{n-1}$ that satisfy the conditions (2.2.6) and (2.2.7) with respect to the n-dimensional spline space $S_{m,t}$, for any $n\times n$ matrix L, we define a linear operator $L_y : C[a,b]\to S_{m,t}$ as follows

$$(L_y f)(x) = \sum_{k=0}^{n-1} (L\vec{f_y})_k B_{k,m}(x), \tag{4.1.1}$$

where

$$\vec{f_y} = [f(y_0),\cdots,f(y_{n-1})]^T. \tag{4.1.2}$$

We call L_y the linear operator *induced* by the data samples $\{y_i\}_{i=0}^{n-1}$ from a given $n \times n$ matrix L, or we refer L_y as a *DI-operator*, which means a data-induced operator.

4.1.2 Polynomial-preservation property

Now we would like to know how well that $L_y f$ approximates f on $[a, b]$. Let us consider a subspace of $C[a, b]$, that is $C^{(m)}[a, b]$, which is dense in $C[a, b]$. Take any $f_0(x) \in C^{(m)}[a, b]$. We want to estimate $|f_0(x) - L_y f_0(x)|$ on $[a, b]$. Notice that

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b.$$

Denote Δy the length of the longest subinterval $[y_j, y_{j+1}]$, that is,

$$\Delta y = \max_{0 \le j \le n-2} |y_{j+1} - y_j|.$$

For any $x \in [a, b]$, there exists some j with $0 \le j \le n-2$, such that $y_j \le x \le y_{j+1}$. Since $x \in [y_j, y_{j+1}]$, by Taylor's theorem, we have

$$f_0(x) = f_0(y_j) + f_0'(y_j)(x - y_j) + \cdots + \frac{f_0^{(m-1)}(y_j)}{(m-1)!}(x - y_j)^{m-1} + R_m(x),$$

where

$$R_m(x) = \frac{f_0^{(m)}(\xi)}{(m)!}(x - y_j)^m, \quad \text{for some } \xi \in [y_j, y_{j+1}].$$

Denote $p_0(x)$ the polynomial (which is an approximation to $f_0(x)$) as

$$p_0(x) = f_0(y_j) + f_0'(y_j)(x - y_j) + \cdots + \frac{f_0^{(m-1)}(y_j)}{(m-1)!}(x - y_j)^{m-1}. \tag{4.1.3}$$

Thus, for any $f_0(x) \in C^{(m)}[a, b]$, we can find a polynomial $p_0(x) \in \pi_{m-1}$, such that

$$|f_0(x) - p_0(x)| \le C_0 |\Delta y|^m \max_{a \le x \le b} |f_0^{(m)}(x)|, \tag{4.1.4}$$

for some positive constant C_0.

Notice that L_y is a bounded linear operator from $C[a, b]$ to $S_{t,m}$, that is, for any $f(x), g(x) \in C[a, b]$, we have

$$|L_y f(x) - L_y g(x)| \le C(n) \|L_y\| \max_{a \le x \le b} |f(x) - g(x)|. \tag{4.1.5}$$

Now we can estimate $|f_0(x) - L_y f_0(x)|$ for any $f_0(x) \in C^{(m)}[a, b]$. In fact, we can find a polynomial $p_0(x) \in \pi_{m-1}$ as in (4.1.3), which satisfies (4.1.4). We can write

$$\begin{aligned}|f_0(x) - L_y f_0(x)| \leq &|f_0(x) - p_0(x)| + |p_0(x) - L_y p_0(x)| \\ &+ |L_y p_0(x) - L_y f_0(x)|.\end{aligned}$$

We have

$$\begin{aligned}&|f_0(x) - p_0(x)| + |L_y p_0(x) - L_y f_0(x)| \\ &\qquad \leq (1 + C(n)\|L_y\|) \max_{a \leq x \leq b} |f_0(x) - p_0(x)| \\ &\qquad \leq C_0 \, (1 + C(n)\|L_y\|) \, |\Delta y|^m \max_{a \leq x \leq b} |f_0^{(m)}(x)|.\end{aligned}$$

But we still need to deal with the term $|p_0(x) - L_y p_0(x)|$. It is desirable to have the property

$$L_y p_0(x) = p_0(x), \qquad \text{for all} \quad p_0(x) \in \pi_{m-1}, \tag{4.1.6}$$

then we have

$$|f_0(x) - L_y f_0(x)| \leq C_0 \, (1 + C(n)\|L_y\|) \, |\Delta y|^m \max_{a \leq x \leq b} |f_0^{(m)}(x)|. \tag{4.1.7}$$

In other words, if the operator L_y has the *polynomial preservation* property as in (4.1.6), then $L_y f_0(x)$ can approximate $f_0(x)$ well in the sense of (4.1.7). Therefore, when we construct an operator L_y, we would like it to have the *polynomial preservation* property. In fact, it is the so-called *quasi-interpolation* operator, which is defined as follows.

Definition. A bounded linear operator Q on $C[a, b]$ is called a *quasi-interpolation* operator if it preserves polynomials as follows,

$$(Qp)(x) = p(x), \quad p \in \pi_{m-1}. \tag{4.1.8}$$

We would like to make our *data-samples-induced* linear operator L_y a quasi-interpolation operator. Furthermore, in order to make this operator support efficient computation, we would like it to be a *local* operator in the sense that the corresponding matrix L is a band matrix. Hence we need the following concept: *local quasi-interpolation operator induced by data samples through B-splines.*

Definition. Given a set of data samples $\{y_i\}_{i=0}^{n-1}$ that satisfy the conditions (2.2.6) and (2.2.7) with respect to the n-dimensional B-spline space $S_{m,t}$, let $L_y : C[a, b] \to S_{m,t}$ be a linear operator defined as in (4.1.1). If L is an $n \times n$ *band* matrix, and

$$(L_y p)(x) = p(x), \quad \text{for all } p \in \pi_{m-1}, \tag{4.1.9}$$

then we call L_y a *local quasi-interpolation operator induced by data samples* $\{y_i\}_{i=0}^{n-1}$ *through B-splines.*

4.1.3 Blending linear operator

The quasi-interpolation operator defined above has the desired approximation property, unfortunately it does not interpolate the given data. We would like to have a local linear operator that has both the polynomial-preservation property and the interpolatory property. A method in Chui [3] solves this problem in a simple way:

Making corrections through an impulse interpolation operator

Notice that our quasi-interpolator Q does not interpolate the data exactly, we need to make some small corrections to make up the differences. To this end, we insert a few appropriate new knots and get a larger spline space, denoted by S_{m,t^*}, i.e. $S_{m,t} \subset S_{m,t^*}$ (with respect to $t \subset t^*$). In S_{m,t^*}, we can choose a set of special interpolating B-splines $\{\xi_{k,m}(x)\}_{k=0}^{n-1}$ with the property: Each $\xi_{k,m}(x)$ interpolates one of the data points y_k and its support is the interval between two adjacent knots in the t sequence that covers y_k. Then we have the property

$$\xi_{k,m}(y_j) = \delta_{kj}, \qquad \text{for} \quad 0 \leq k, j \leq n-1,$$

where δ is the Kronecker's delta notation. We call $\{\xi_{k,m}(x)\}_{k=0}^{n-1}$ the *impulse* interpolating functions.

Now we can define our impulse interpolation operator $R_m : C[a, b] \to S_{m,t^*}$ as follows,

$$(R_m f)(x) := \sum_{k=0}^{n-1} f(y_k)\xi_{k,m}(x), \tag{4.1.10}$$

which obviously satisfies the data interpolating property

$$(R_m f)(y_k) = f(y_k), \qquad k = 0, \ldots, n-1. \tag{4.1.11}$$

Next we define the following "blending" operator $P : C[a, b] \to S_{m,t^*}$ as

$$P := R_m + Q - R_m Q \tag{4.1.12}$$

This operator possesses the polynomial-preservation property. In deed, for any $p \in \pi_{m-1}$, we have

$$\begin{aligned} (Pp)(x) &= (R_m p)(x) + (Qp)(x) - (R_m Qp)(x) \\ &= (R_m p)(x) + p(x) - (R_m p)(x) \\ &= p(x). \end{aligned}$$

P satisfies the data interpolatory property as well. In fact, we look at the function values at the data points y_j, $j = 0, \ldots, n-1$, then we have

$$\begin{aligned} (Pf)(y_j) &= (R_m f)(y_j) + (Qf)(y_j) - (R_m Qf)(y_j) \\ &= f(y_j) + (Qf)(y_j) - (Qf)(y_j) \\ &= f(y_j). \end{aligned}$$

If we take Q as a local linear operator L_y: $C[a,b] \to S_{m,t}$, and R_m is another local linear operator: $C[a,b] \to S_{m,t^*}$. Therefore, P is a local linear operator: $C[a,b] \to S_{m,t^*}$, which satisfies our requirements for data analysis. Our next challenge is: How to construct a local quasi-interpolation operator L_y without any matrix inverse.

4.2 Properties of local quasi-interpolation operators L_y

4.2.1 Polynomial-preservation condition for DI-operators

Let L_y be a DI-operator as defined in (4.1.1). We want to know under what condition that L_y will preserve polynomials in π_{m-1}. To this end, we need to use the Marsden's identities. We use $\rho^r_{0,m}, \ldots, \rho^r_{n-1,m}$ to denote the Marsden's coefficients that satisfy

$$x^r = \sum_{k=0}^{n-1} \rho^r_{k,m} B_{k,m}(x), \quad \text{for} \quad 0 \le r \le m-1. \tag{4.2.1}$$

Proposition 4.2.1. *Given a set of data samples $\{y_i\}_{i=0}^{n-1}$ that satisfy the conditions (2.2.6) and (2.2.7) with respect to the n-dimensional B-spline space $S_{m,t}$, let $L_y : C[a,b] \to S_{m,t}$ be a DI-operator as in* (4.1.1) *with associated $n \times n$ matrix L. If L satisfies the equation:*

$$L \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho^r_{0,m} \\ \rho^r_{1,m} \\ \vdots \\ \rho^r_{n-1,m} \end{bmatrix}, \quad \textit{for} \quad r = 0, 1, \cdots, m-1, \tag{4.2.2}$$

then

$$(L_y p)(x) = p(x), \quad \textit{for all} \quad p \in \pi_{m-1}, \tag{4.2.3}$$

where $\rho^r_{0,m}, \ldots, \rho^r_{n-1,m}$ are the Marsden's coefficients defined as in (4.2.1).

Proof. Let us consider the building blocks of the polynomials in π_{m-1}, that is, the monomials $\eta^r(x) := x^r$ for $0 \le r \le m-1$. Since $\vec{\eta}_y^{\,r} = \left[y_0^r, \cdots, y_{n-1}^r\right]^T$ by (4.1.2), and from (4.2.2), we get

$$L\vec{\eta}_y^{\,r} = L \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,m}^r \\ \rho_{1,m}^r \\ \vdots \\ \rho_{n-1,m}^r \end{bmatrix},$$

which leads to

$$\left(L\vec{\eta}_y^{\,r}\right)_k = \rho_{k,m}^r, \quad \text{for} \quad 0 \le r \le m-1. \tag{4.2.4}$$

Thus,

$$L_y(\eta^r)(x) = \sum_{k=0}^{n-1} \rho_{k,m}^r B_{k,m}(x) = \eta^r(x), \tag{4.2.5}$$

which results in (4.2.3) by the *linearity* of the operator L_y. □

Based on Proposition 4.2.1, we are interested in those matrices that satisfy the condition (4.2.2). The next proposition gives us another matrix with this property.

Proposition 4.2.2. *Let $\{B_{i,m}(x)\}_{i=0}^{n-1}$ be the B-splines on $[a,b]$ with knots satisfying (2.2.1) and (2.2.2). Given a set of data points $\{y_i\}_{i=0}^{n-1}$ that satisfy the conditions (2.2.6) and (2.2.7), let B_m be the Shoenberg-Whitney matrix , i.e.*

$$B_m := \begin{bmatrix} B_{0,m}(y_0) & \cdots & B_{n-1,m}(y_0) \\ \vdots & & \vdots \\ B_{0,m}(y_{n-1}) & \cdots & B_{n-1,m}(y_{n-1}) \end{bmatrix}_{n\times n} \tag{4.2.6}$$

and let $\{\rho_{i,m}^r\}_{i=0}^{n-1}$ be the Marsden's coefficients for x^r with $0 \le r \le m-1$. Then

$$B_m \begin{bmatrix} \rho_{0,m}^r \\ \vdots \\ \rho_{n-1,m}^r \end{bmatrix} = \begin{bmatrix} y_0^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} \quad \textit{for} \quad r = 0, 1, \cdots, m-1. \tag{4.2.7}$$

Proof. By the Marsden's Identity, we have for $0 \le r \le m-1$

$$\begin{aligned} x^r &= \sum_{i=0}^{n-1} \rho_{i,m}^r B_{j,m}(x) \\ &= \begin{bmatrix} \rho_{0,m}^r & \cdots & \rho_{n-1,m}^r \end{bmatrix} \begin{bmatrix} B_{0,m}(x) \\ \vdots \\ B_{n-1,m}(x) \end{bmatrix}. \end{aligned}$$

Thus, for $j = 0, 1, \cdots, n-1$, we get

$$y_j^r = \begin{bmatrix} \rho_{0,m}^r & \cdots & \rho_{n-1,m}^r \end{bmatrix} \begin{bmatrix} B_{0,m}(y_j) \\ \vdots \\ B_{n-1,m}(y_j) \end{bmatrix}.$$

By putting them together, we have

$$\begin{bmatrix} y_0^r & \cdots & y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,m}^r & \cdots & \rho_{n-1,m}^r \end{bmatrix} \begin{bmatrix} B_{0,m}(y_0) & \cdots & B_{n-1,m}(y_{n-1}) \\ \vdots & & \vdots \\ B_{0,m}(y_0) & \cdots & B_{n-1,m}(y_{n-1}) \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} y_0^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} B_{0,m}(y_0) & \cdots & B_{n-1,m}(y_0) \\ \vdots & & \vdots \\ B_{0,m}(y_{n-1}) & \cdots & B_{n-1,m}(y_{n-1}) \end{bmatrix} \begin{bmatrix} \rho_{0,m}^r \\ \vdots \\ \rho_{n-1,m}^r \end{bmatrix},$$

which gives (4.2.7) and we complete the proof. □

Remark 4.2.3. *We can write equation* (4.2.7) *as*

$$\begin{bmatrix} 1 & y_0 & \cdots & y_0^r \\ 1 & y_1 & \cdots & y_1^r \\ \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & \cdots & y_{n-1}^r \end{bmatrix}_{n\times(r+1)} = B_m \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^r \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^r \\ \vdots & \vdots & & \vdots \\ 1 & \rho_{n-1,m}^1 & \cdots & \rho_{n-1,m}^r \end{bmatrix}_{n\times(r+1)}, \tag{4.2.8}$$

with $0 \le r \le m-1$, *or equivalently,*

$$B_m^{-1} \begin{bmatrix} 1 & y_0 & \cdots & y_0^r \\ 1 & y_1 & \cdots & y_1^r \\ \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & \cdots & y_{n-1}^r \end{bmatrix}_{n\times(r+1)} = \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^r \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^r \\ \vdots & \vdots & & \vdots \\ 1 & \rho_{n-1,m}^1 & \cdots & \rho_{n-1,m}^r \end{bmatrix}_{n\times(r+1)} \tag{4.2.9}$$

with $0 \le r \le m-1$.

Remark 4.2.4. *In view of (4.2.9), we notice that* B_m^{-1} *is a full matrix in general, which could not give us a local linear operator. To find a local quasi-interpolating operator, we need to find a band matrix* L_y *(determined by the given data samples* $\{y_i\}_{i=0}^{n-1}$*), such that*

$$L_y \begin{bmatrix} 1 & y_0 & \cdots & y_0^{m-1} \\ 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & \cdots & y_{n-1}^{m-1} \end{bmatrix}_{n\times m} = \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^{m-1} \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \rho_{n-1,m}^1 & \cdots & \rho_{n-1,m}^{m-1} \end{bmatrix}_{n\times m}. \tag{4.2.10}$$

In Chapter 5, we will use a special matrix factorization method to find this $n \times n$ band matrix L_y.

The following result gives us the estimate of computation cost for calculating the Shoenberg-Whitney matrix.

Proposition 4.2.5. *To calculate the Shoenberg-Whitney matrix $\mathcal{B}_m$, we need at most $2m(m-1)n$ multiplications or divisions and $\frac{5}{2}m(m-1)n$ additions or subtractions, which means that its complexity function is in $O(n)$ or the algorithm is linear with respect to the spline order m.*

Proof. To calculate the Shoenberg-Whitney matrix $\mathcal{B}_m$, we need to calculate $B_{i,m}(y_j)$ for $0 \le i, j \le n-1$. We use the recurrence formula

$$B_{i,m}(y_j) = \frac{y_j - t_{i-m+1}}{t_i - t_{i-m+1}} B_{i-1,m-1}(y_j) + \frac{t_{i+1} - y_j}{t_{i+1} - t_{i-m+2}} B_{i,m-1}(y_j) \tag{4.2.11}$$

for $0 \le i, j \le n-1$, where the $(m-1)$th order B-spline basis functions $\{B_{i,m-1}(x)\}_{i=0}^{n-2}$ are defined on the knots:

$$\{\underbrace{a, \ldots, a}_{m-1}, t_1, t_2, \cdots, t_{n-m}, \underbrace{b, \ldots, b}_{m-1}\}. \tag{4.2.12}$$

We write (4.2.11) in the form of matrix product as follows,

$$B_{i,m}(y_j) = \begin{bmatrix} \dfrac{y_j - t_{i-m+1}}{t_i - t_{i-m+1}} & \dfrac{t_{i+1} - y_j}{t_{i+1} - t_{i-m+2}} \end{bmatrix} \begin{bmatrix} B_{i-1,m-1}(y_j) \\ B_{i,m-1}(y_j) \end{bmatrix} \tag{4.2.13}$$

for $0 \le i, j \le n-1$. Similarly, we have

$$B_{i,m-1}(y_j) = \begin{bmatrix} \dfrac{y_j - t_{i-m+2}}{t_i - t_{i-m+2}} & \dfrac{t_{i+1} - y_j}{t_{i+1} - t_{i-m+3}} \end{bmatrix} \begin{bmatrix} B_{i-1,m-2}(y_j) \\ B_{i,m-2}(y_j) \end{bmatrix} \tag{4.2.14}$$

for $0 \le i \le n-2$ and for $0 \le j \le n-1$. Let

$$\alpha_{i,m}^j := \frac{y_j - t_{i-m+1}}{t_i - t_{i-m+1}} \quad \text{and} \quad \beta_{i,m}^j := \frac{t_{i+1} - y_j}{t_{i+1} - t_{i-m+2}}. \tag{4.2.15}$$

Thus, (4.2.13) and (4.2.14) can be written as

$$B_{i,m}(y_j) = \begin{bmatrix}\alpha^j_{i,m} & \beta^j_{i,m}\end{bmatrix} \begin{bmatrix} B_{i-1,m-1}(y_j) \\ B_{i,m-1}(y_j) \end{bmatrix} \tag{4.2.16}$$

and

$$B_{i,m-1}(y_j) = \begin{bmatrix}\alpha^j_{i,m-1} & \beta^j_{i,m-1}\end{bmatrix} \begin{bmatrix} B_{i-1,m-2}(y_j) \\ B_{i,m-2}(y_j) \end{bmatrix}. \tag{4.2.17}$$

It follows from (4.2.16) and (4.2.17) that

$$B_{i,m}(y_j) = \begin{bmatrix}\alpha^j_{i,m} & \beta^j_{i,m}\end{bmatrix} \begin{bmatrix} \begin{bmatrix}\alpha^j_{i-1,m-1} & \beta^j_{i-1,m-1}\end{bmatrix} \begin{bmatrix} B_{i-2,m-2}(y_j) \\ B_{i-1,m-2}(y_j) \end{bmatrix} \\ \begin{bmatrix}\alpha^j_{i,m-1} & \beta^j_{i,m-1}\end{bmatrix} \begin{bmatrix} B_{i-1,m-2}(y_j) \\ B_{i,m-2}(y_j) \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix}\alpha^j_{i,m} & \beta^j_{i,m}\end{bmatrix} \begin{bmatrix} \alpha^j_{i-1,m-1} & \beta^j_{i-1,m-1} & 0 \\ 0 & \alpha^j_{i,m-1} & \beta^j_{i,m-1} \end{bmatrix} \begin{bmatrix} B_{i-2,m-2}(y_j) \\ B_{i-1,m-2}(y_j) \\ B_{i,m-2}(y_j) \end{bmatrix}.$$

Now we consider the general case for the $(m-r)$th order B-spline basis functions $\{B_{i,m-r}(x)\}_{i=0}^{n-r-1}$ that are defined on the knots:

$$\{\underbrace{a, \ldots, a}_{m-r}, t_1, t_2, \cdots, t_{n-m}, \underbrace{b, \ldots, b}_{m-r}\} \tag{4.2.18}$$

for $0 \le r \le m-1$. Then we have the general version for (4.2.16) as

$$B_{i,m-r}(y_j) = \begin{bmatrix}\alpha^j_{i,m-r} & \beta^j_{i,m-r}\end{bmatrix} \begin{bmatrix} B_{i-1,m-r-1}(y_j) \\ B_{i,m-r-1}(y_j) \end{bmatrix} \tag{4.2.19}$$

for $0 \le r \le m-2$. Denote $R_r(y_j)$ as the $(r+1) \times (r+2)$ matrix

$$R^i_r(y_j) = \begin{bmatrix} \alpha^j_{i-r,m-r} & \beta^j_{i-r,m-r} & 0 & \cdots & 0 \\ 0 & \alpha^j_{i-r+1,m-r} & \beta^j_{i-r+1,m-r} & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \alpha^j_{i,m-r} & \beta^j_{i,m-r} \end{bmatrix} \tag{4.2.20}$$

for $0 \le r \le m-2$. Then, we get

$$B_{i,m}(y_j) = R^i_0(y_j) R^i_1(y_j) \cdots R^i_{m-2}(y_j) \begin{bmatrix} B_{i-(m-1),1}(y_j) \\ \vdots \\ B_{i-1,1}(y_j) \\ B_{i,1}(y_j) \end{bmatrix}. \tag{4.2.21}$$

Here we take that

$$B_{i,m-r}(x) = 0 \qquad \text{for} \quad i < 0 \text{ or } i > n-r-1. \tag{4.2.22}$$

To compute $\mathcal{B}_m$ using (4.2.21), we start with $\{B_{i,1}(y_j)\}$ for $0 \le i \le n-m$ and $0 \le j \le n-1$ with respect to the knot sequence

$$\{a, t_1, t_2, \cdots, t_{n-m}, b\}, \tag{4.2.23}$$

and we take $t_0 = a$ and $t_{n-m+1} = b$ as usual. By (2.2.7), we can see that for $1 \le j \le n-2$, y_j could be in any of the following m intervals,

$$(t_{j-m+1}, t_{j-m+2}),\ [t_{j-m+2}, t_{j-m+3}),\ \cdots,\ [t_j, t_{j+1}). \tag{4.2.24}$$

In order to locate the interval $[t_k, t_{k+1}]$ that covers y_j, we use the following notation

$$\zeta_j = \max_{t_i \le y_j}\{i\}. \tag{4.2.25}$$

For each j with $1 \le j \le n-2$, y_j falls into the interval $[t_{\zeta_j}, t_{\zeta_j+1})$. (4.2.24) implies that

$$j - m + 1 \le \zeta_j < j + 1. \tag{4.2.26}$$

Thus, we have for $1 \le j \le n-2$,

$$[B_{i-(m-1),1}(y_j), \cdots, B_{i,1}(y_j)]^T = 0, \quad \text{for} \quad i \le \zeta_j - 1 \text{ or } i \ge \zeta_j + m; \tag{4.2.27}$$

and for $\zeta_j \le i \le \zeta_j + m - 1$,

$$[B_{i-(m-1),1}(y_j), \cdots, B_{i,1}(y_j)]^T = [\underbrace{0, \ldots, 0}_{\zeta_j+m-1-i},\ 1,\ \underbrace{0, \ldots, 0}_{i-\zeta_j}]^T. \tag{4.2.28}$$

For $j = 0$, we have that $\zeta_0 = 0$ and for $0 \le i \le m-1$,

$$[B_{i-(m-1),1}(a), \cdots, B_{i,1}(a)]^T = [\underbrace{0, \ldots, 0}_{m-i-1},\ 1,\ \underbrace{0, \ldots, 0}_{i}]^T; \tag{4.2.29}$$

and for $j = n-1$, we have that $\zeta_{n-1} = n-m$ and for $n-m \le i \le n-1$,

$$[B_{i-(m-1),1}(b), \cdots, B_{i,1}(b)]^T = [\underbrace{0, \ldots, 0}_{n-i-1},\ 1,\ \underbrace{0, \ldots, 0}_{i+m-n}]^T. \tag{4.2.30}$$

Notice that (4.2.29) is a special case of (4.2.28), but (4.2.30) is not a special case of (4.2.28) due to the way we define $B_{n-m+1,1}(x)$ at $x = b$, which is different from the way we define $B_{i,1}(x)$ at $x = t_i$ for $0 \le i \le n-m-1$. With (4.2.21), (4.2.28) and (4.2.30), we can find $B_{i,m}(y_j)$ for all $0 \le i, j \le n-1$.

Next we will show that the total number of multiplications (or divisions) and additions (or subtractions) for computing $\mathcal{B}_m$ is bounded by some constant multiple (as a function of m) of n. To this end, we need to count the number of multiplications (or divisions) and additions (or subtractions) for computing $B_{i,m}(y_j)$ using (4.2.21).

From the structure of $\mathcal{B}_m$ as in (2.2.10), for each fixed j, $[B_{i,m}(y_j)]_{i=0}^{n-1}$ is the j-th row of $\mathcal{B}_m$. The 1st row and n-th row are given by

$$\begin{bmatrix}1 & 0 & \cdots & 0\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}0 & \cdots & 0 & 1\end{bmatrix},$$

respectively, due to the assumption that $y_0 = a$ and $y_{n-1} = b$ from (2.2.6).

Now we calculate the j-th row for $1 \leq j \leq n-2$. For each i in the range $\zeta_j \leq i \leq \zeta_j + m - 1$, we start with the vector

$$[\underbrace{0,\ldots,0}_{\zeta_j+m-1-i},\ 1,\ \underbrace{0,\ldots,0}_{i-\zeta_j}]^T$$

as in (4.2.28), which we multiply with $R_{m-2}(y_j)$ from the right based on (4.2.21). Notice that

$$R^i_{m-2}(y_j) = \begin{bmatrix} \alpha^j_{i-(m-2),2} & \beta^j_{i-(m-2),2} & 0 & \cdots & 0 \\ 0 & \alpha^j_{i-(m-3),2} & \beta^j_{i-(m-3),2} & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \alpha^j_{i,2} & \beta^j_{i,2} \end{bmatrix}_{(m-1)\times m}.$$

When $i = \zeta_j$ and $i = \zeta_j + m - 1$, we have

$$R^{\zeta_j}_{m-2}(y_j) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{m\times 1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta^j_{\zeta_j,2} \end{bmatrix}_{(m-1)\times 1} \tag{4.2.31}$$

and

$$R^{\zeta_j+m-1}_{m-2}(y_j) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1} = \begin{bmatrix} \alpha^j_{\zeta_j+1,2} \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{(m-1)\times 1}. \tag{4.2.32}$$

When $\zeta_j + 1 \leq i \leq \zeta_j + m - 2$, we have

$$R^i_{m-2}(y_j)\ [\underbrace{0,\ldots,0}_{\zeta_j+m-1-i},\ 1,\ \underbrace{0,\ldots,0}_{i-\zeta_j}]^T = [\underbrace{0,\ldots,0}_{\zeta_j+m-i-2},\ \beta^j_{\zeta_j+1,2},\ \alpha^j_{\zeta_j+2,2},\ \underbrace{0,\ldots,0}_{i-\zeta_j-1}]^T. \tag{4.2.33}$$

It is easy to see that in this matrix multiplication, we need at most 2 divisions and 2 subtractions. Next we consider the following matrix multiplication

$$R^i_{m-3}(y_j)[\underbrace{0,\ldots,0}_{\zeta_j+m-i-2},\ \beta^j_{\zeta_j+1,2},\ \alpha^j_{\zeta_j+2,2},\ \underbrace{0,\ldots,0}_{i-\zeta_j-1}]^T.$$

Notice that

$$R^i_{m-3}(y_j) = \begin{bmatrix} \alpha^j_{i-(m-3),3} & \beta^j_{i-(m-3),3} & 0 & \cdots & 0 \\ 0 & \alpha^j_{i-(m-4),3} & \beta^j_{i-(m-4),3} & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & \alpha^j_{i,3} & \beta^j_{i,3} \end{bmatrix}_{(m-2)\times(m-1)}.$$

Since the counting for precise number of multiplications (or divisions) and additions (or subtractions) could be very complicated, here we just do a very generous counting but still meet our goal for $O(n)$ complexity. We rewrite (4.2.21) in the form of

$$B_{i,m}(y_j) = R^i_0(y_j)R^i_1(y_j)\cdots R^i_{m-2}(y_j)b^i_{0,j} \tag{4.2.34}$$

where $b^i_{0,j}$ is a column vector with size $m\times 1$. Now we define a sequence of vectors $b^i_{r,j}$ for $1\le r\le m-1$ as follows,

$$b^i_{r,j} = R^i_{m-r-1}(y_j)b^i_{r-1,j}. \tag{4.2.35}$$

Thus $b^i_{r,j}$ is a column vector of size $(m-r)\times 1$. When we do the counting, we allow that all the components of $b^i_{r,j}$ could be nonzero.

Now we do the counting for the number of multiplications (or divisions) and additions (or subtractions) on the matrix multiplication $R^i_{m-r-1}(y_j)b^i_{r-1,j}$ for $1\le r\le m-1$. There are $(m-r)$ rows. For each row, there are only two nonzero entries in $R^i_{m-r-1}(y_j)$. Hence our counting has this result: 2 divisions, 2 multiplications, 4 subtractions, and 1 addition for the multiplication of one row of $R^i_{m-r-1}(y_j)$ with $b^i_{r-1,j}$. Therefore to calculate $R^i_{m-r-1}(y_j)b^i_{r-1,j}$, we need at most $4(m-r)$ multiplications or divisions and $5(m-r)$ additions or subtractions. To get the total number of operations, we find the summation

$$\sum_{r=1}^{m-1}(m-r) = 1+2+\cdots+(m-1) = \frac{1}{2}m(m-1).$$

We conclude that to calculate $\mathcal{B}_m$, we need at most $2m(m-1)n$ multiplications or divisions and $\frac{5}{2}m(m-1)n$ additions or subtractions, which means that its complexity function is in $O(n)$ or the algorithm is linear, and we complete the proof. $\square$

Chapter 5

Local Quasi-Interpolation Operator for Linear B-Splines by Factorization on Shoenberg-Whitney Matrix

5.1 Divided-difference matrices

Define the *difference* matrix of *order* k as

$$D_k = \begin{bmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & 0 & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{k\times(k+1)}. \tag{5.1.1}$$

We will consider m difference matrices: D_k for $k = n-1, n-2, \ldots, n-m$.

Then we define a sequence of *scaling matrices* as the diagonal matrices with respect to the data points: $\{y_i\}_0^{n-1}$ as follows,

$$G_i = \begin{bmatrix} \dfrac{1}{y_i - y_0} & & 0 \\ & & \\ 0 & \ddots & \dfrac{1}{y_{n-1} - y_{n-i-1}} \end{bmatrix}_{(n-i)\times(n-i)} \quad \text{for } i = 1, 2, \cdots, m. \tag{5.1.2}$$

With (5.1.1) and (5.1.2), we can define m *divided-difference* matrices as follows,

$$E_i = G_i D_{n-i}, \quad i = 1, 2, \cdots, m. \tag{5.1.3}$$

Proposition 5.1.1. *Given a set of data points $\{y_i\}_{i=0}^{n-1}$ that satisfy the conditions (2.2.6) and (2.2.7), the m* divided-difference *matrices $E_1, E_2, \dots, E_m$ defined in* (5.1.3) *have the following property:*

$$E_m E_{m-1} \cdots E_1 \begin{bmatrix} 1 & y_0 & \cdots & y_0^{m-1} \\ 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & \cdots & y_{n-1}^{m-1} \end{bmatrix} = 0. \tag{5.1.4}$$

Proof. Denote

$$M_y = \begin{bmatrix} 1 & y_0 & \cdots & y_0^{m-1} \\ 1 & y_1 & \cdots & y_1^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & y_{n-1} & \cdots & y_{n-1}^{m-1} \end{bmatrix}. \tag{5.1.5}$$

We will apply the divided-difference matrices $E_1, \dots, E_m$ on M_y in (5.1.5) one by one. To see the results clearly, we need to use the standard *divided difference* operation, which is defined as follows: Given a polynomial $f(x)$, for $l \geq 1$,

$$f[y_l, y_{l+1}, \cdots, y_{l+j}] := \frac{f[y_{l+1}, \cdots, y_{l+j}] - f[y_l, \cdots, y_{l+j-1}]}{y_{l+j} - y_l} \tag{5.1.6}$$

and

$$f[y_l] = f(y_l). \tag{5.1.7}$$

Here we need to use the following property of the divided-difference operation: When $f(x)$ is a polynomial of degree j, then $f[y_l, y_{l+1}, \cdots, y_{l+j}]$ is a nonzero constant (the leading coefficient of $f(x)$); when the degree of $f(x)$ is less than j, then $f[y_l, y_{l+1}, \cdots, y_{l+j}] = 0$.

Denote $p_r(x) = x^r$ for $r = 0, 1, \dots, m-1$. For $j = 1$, we have

$$E_1 M_y = \begin{bmatrix} 0 & 1 & \dfrac{y_0^2 - y_1^2}{y_0 - y_1} & \cdots & \dfrac{y_0^{m-1} - y_1^{m-1}}{y_0 - y_1} \\ \vdots & \vdots & & \vdots & \\ 0 & 1 & \dfrac{y_{n-2}^2 - y_{n-1}^2}{y_{n-2} - y_{n-1}} & \cdots & \dfrac{y_{n-2}^{m-1} - y_{n-1}^{m-1}}{y_{n-2} - y_{n-1}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & p_2[y_0, y_1] & \cdots & p_{m-1}[y_0, y_1] \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & p_2[y_{n-2}, y_{n-1}] & \cdots & p_{m-1}[y_{n-2}, y_{n-1}] \end{bmatrix}.$$

By the definition of the divided difference operation, we can easily see that

$$E_j \cdots E_1 M_y =$$

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & p_{j+1}[y_0, \cdots, y_j] & \cdots & p_{m-1}[y_0, \cdots, y_j] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & p_{j+1}[y_{n-j-1}, \cdots, y_{n-1}] & \cdots & p_{m-1}[y_{n-j-1}, \cdots, y_{n-1}] \end{bmatrix}. \tag{5.1.8}$$

In particular, when $j = m$, all the columns at the right-hand-side of (5.1.8) are zero, thus we get (5.1.4), and complete the proof. □

5.2 Matrix factorization with divided-difference matrices

Assume that $\{y_i\}_0^{n-1}$ and $\{t_i\}_{-m+1}^{n}$ satisfy the Shoenberg-Whitney condition. In order to factorize the general case of $\mathcal{B}_2$, we need to develop a general factorization theory.

Lemma 5.2.1. *: Let $A = [a_{ij}]_{1\le i,j\le n}$ be an $n \times n$ matrix which satisfies*

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_n = 0. \tag{5.2.1}$$

Then, there exists an $n \times (n-1)$matrix X such that

$$A = XD_1, \tag{5.2.2}$$

where D_1 is the difference matrix, given by

$$D_1 = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(n-1)\times n.} \tag{5.2.3}$$

Furthermore, we can write X explicitly as follows,

$$X = \left[\sum_{k=1}^{j} a_{ik} \right]_{1\le i\le n, 1\le j\le n-1.} \tag{5.2.4}$$

Proof. The condition (5.2.1) leads to

$$\sum_{k=1}^{n} a_{ik} = 0, \quad \text{for } i = 1, 2, \cdots, n. \tag{5.2.5}$$

Since D_1 is of full rank, we can easily find its pseudo-inverse as follows,

$$D_1^+ = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times(n-1).} \tag{5.2.6}$$

Notice that

$$D_1 D_1^+ = I_{n-1}, \tag{5.2.7}$$

which inspires us to consider the following form

$$X := AD_1^+, \tag{5.2.8}$$

which has the explicit representation as follows,

$$X = \left[\sum_{k=1}^{j} a_{ik} \right]_{1\le i\le n, 1\le j\le n-1.} \tag{5.2.9}$$

Next we shall show that X satisfies the condition (5.2.2). A straightforward calculation results in

$$XD_1 = \begin{bmatrix} a_{11} & \cdots & a_{1,n-1} & -\sum_{k=1}^{n-1} a_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & \cdots & a_{i,n-1} & -\sum_{k=1}^{n-1} a_{ik} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & -\sum_{k=1}^{n-1} a_{nk} \end{bmatrix}. \tag{5.2.10}$$

The equality (5.2.5) implies that

$$a_{in} = -\sum_{k=1}^{n-1} a_{ik}, \quad \text{for } i = 1, 2, \ldots, n, \tag{5.2.11}$$

which allows us to simplify the matrix at the right-hand-side of (5.2.10), which is exactly A. Hence, we complete the proof. $\square$

Definition 5.2.2. *Let $A = [a_{ij}]$ be an $m \times n$ matrix with the following properties: All the entries of its first row and last row are zero, that is,*

$$a_{1j} = a_{mj} = 0, \quad \textit{for } j = 1, 2, \ldots, n.$$

Then we call it zero-row-ending *matrix.*

Next, we have a *zero-row-ending* version of **Lemma 5.2.1**.

Lemma 5.2.3. *Let $A = [a_{ij}]$ be an $m \times n$* zero-row-ending *matrix such that*

$$A \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_n = 0. \tag{5.2.12}$$

Then, there exists an $m \times (n-1)$ matrix X such that

$$A = XD_1, \tag{5.2.13}$$

where

$$D_1 = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(n-1)\times n.} \tag{5.2.14}$$

Furthermore, X is also a zero-row-ending *matrix. More specifically, if we write $X = [x_{ij}]$, then we have*

$$x_{ij} = \begin{cases} 0 & \textit{if } i = 1 \textit{ or } m \\ \sum_{k=1}^{j} a_{ik} & \textit{for } 2 \le i \le m-1 \textit{ and } 1 \le j \le n-1. \end{cases} \tag{5.2.15}$$

Proof. Since A is a *zero-row-ending* matrix, we have

$$a_{1j} = a_{mj} = 0, \quad \text{for } j = 1, 2, \ldots, n. \tag{5.2.16}$$

From (5.2.12), we get

$$\sum_{k=1}^{n} a_{ik} = 0 \tag{5.2.17}$$

for $i = 2, \cdots, m-1$. Next we just need to verify that X with the form of (5.2.15) satisfies (5.2.13).

To this end, we write X explicitly based on (5.2.15) and get

$$X = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ a_{21} & a_{21} + a_{22} & \cdots & a_{21} + \cdots + a_{2,n-1} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m-1,1} & a_{m-1,1} + a_{m-1,2} & \cdots & a_{m-1,1} + \cdots + a_{m-1,n-1} \\ 0 & 0 & \cdots & 0 \end{bmatrix}. \tag{5.2.18}$$

It follows from (5.2.18) and (5.2.14) that

$$XD_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & \cdots & a_{2,n-1} & -\sum_{j=1}^{n-1} a_{2j} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{m-1,1} & a_{m-1,2} & \cdots & a_{m-1,n-1} & -\sum_{j=1}^{n-1} a_{m-1,j} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{5.2.19}$$

(5.2.17) implies that

$$-\sum_{j=1}^{n-1} a_{ij} = a_{in}, \quad \text{for } i = 2, \ldots, m-1. \tag{5.2.20}$$

Thus the right-hand-side of (5.2.19) is exactly A, and we complete the proof. □

Proposition 5.2.4. *Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a, b]$ with the endpoints interpolating property: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions on the interval $[a, b]$ defined on the knots $\vec{t} := \{t_i\}_{-m+1}^{n}$. Assume that $\vec{t}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{t}$ and $\vec{y}$. There exists an $n \times (n-1)$* zero-row-ending *matrix $\tilde{X}_1$, such that*

$$B_m - I = \tilde{X}_1 E_1, \quad (m \geq 1). \tag{5.2.21}$$

Proof. By the *partition of unity* property of B-splines, we have

$$B_m \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_n, \tag{5.2.22}$$

which leads to

$$(B_m - I)\begin{bmatrix}1\\ \vdots\\ 1\end{bmatrix}_n = 0. \tag{5.2.23}$$

We also notice that the matrix $B_m - I$ is an $n \times n$ *zero-row-ending* matrix. By Lemma 5.2.3, there exists an $n \times (n-1)$ *zero-row-ending* matrix X such that

$$(B_m - I) = XD_1. \tag{5.2.24}$$

Observe that

$$XD_1 = XG_1^{-1}G_1D_1 = XG_1^{-1}E_1.$$

By taking $\tilde{X}_1 = XG_1^{-1}$, which is a *zero-row-ending* matrix obviously, we obtain (5.2.21) immediately from (5.2.24), and complete the proof. □

Proposition 5.2.5. *Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a, b]$ with the endpoints interpolating property: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions $\{B_{i,m}(x)\}_{i=0}^{n-1}$ on the interval $[a, b]$ defined on the knots $\vec{t} := \{t_i\}_{-m+1}^{n}$. Assume that $\vec{t}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{t}$ and $\vec{y}$. For any $n \times (n-1)$* zero-row-ending *matrix $\tilde{X}_1$, there exists an $n \times (n-1)$ band matrix $\tilde{B}_2$ of bandwidth 2, such that*

$$B_m\tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2E_2, \quad (m \geq 2) \tag{5.2.25}$$

for some $n \times (n-2)$ zero-row-ending *matrix $\tilde{X}_2$.*

Proof. When n is even, we take $\tilde{B}_2$ in the following form

$$\tilde{B}_2 = \begin{bmatrix}
0 & 0\,\ddots & \ddots & \ddots & \ddots & \ddots & 0\\
0 & \tilde{b}_2 & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\
\vdots & \cdots & 0 & \tilde{b}_{\frac{n}{2}} & \cdots & \cdots & \vdots\\
\vdots & \cdots & & \tilde{b}_{\frac{n}{2}+1} & 0 & \ddots & \vdots\\
\vdots & \ddots & & \ddots & \ddots & \ddots & \vdots\\
\vdots & \cdots & & \ddots & \ddots & \tilde{b}_{n-1} & 0\\
0 & \cdots & \ddots & & \ddots & \ddots & 0
\end{bmatrix}_{n\times(n-1).} \tag{5.2.26}$$

When n is odd, we take

$$\tilde{B}_2 = \begin{bmatrix} 0 & 0\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \tilde{b}_2 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & 0 & \tilde{b}_{\frac{n-1}{2}} & \cdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \ddots & 0 & \tilde{b}_{\frac{n+1}{2}} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \tilde{b}_{\frac{n+3}{2}} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \cdots & & \ddots & \ddots & \ddots & \tilde{b}_{n-1} & 0 \\ 0 & \cdots & \ddots & \ddots & \ddots & \ddots & 0 & 0 \end{bmatrix}_{n\times(n-1).} \tag{5.2.27}$$

Obviously (5.2.26) and (5.2.27) are band matrix of bandwidth 2. To determine the parameters $\{\tilde{b}_2, \ldots, \tilde{b}_{n-1}\}$, we need

$$(B_m\tilde{B}_2 + \tilde{X}_1)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{(n-1)\times 1} = 0. \tag{5.2.28}$$

Denote $\tilde{X}_1 = [x_{ij}]_{1\le i\le n, 1\le j\le n-1}$. Since $\tilde{X}_1$ is a *zero-row-ending* matrix, we have

$$x_{1j} = 0 \quad \text{and} \quad x_{nj} = 0 \quad \text{for } 1 \le j \le n-1. \tag{5.2.29}$$

We can simplify (5.2.28) to

$$B_m \begin{bmatrix} 0 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_{n-1} \\ 0 \end{bmatrix}_{n\times 1} + \begin{bmatrix} 0 \\ \sum_{k=1}^{n-1} x_{2k} \\ \vdots \\ \sum_{k=1}^{n-1} x_{n-1,k} \\ 0 \end{bmatrix}_{n\times 1} = 0. \tag{5.2.30}$$

Since the Shoenberg-Whitney matrix B_m is invertible, we can find the solution for $\{\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n\}$ from (5.2.30) as follows,

$$\begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix} = -B_m^{-1} \begin{bmatrix} 0 \\ \sum_{k=1}^{n-1} x_{2k} \\ \vdots \\ \sum_{k=1}^{n-1} x_{n-1,k} \\ 0 \end{bmatrix}. \tag{5.2.31}$$

Now we claim that $\tilde{b}_1 = 0$ and $\tilde{b}_n = 0$. Indeed, from (5.2.31), we have

$$B_m \begin{bmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_n \end{bmatrix} = - \begin{bmatrix} 0 \\ \sum_{k=1}^{n-1} x_{2k} \\ \vdots \\ \sum_{k=1}^{n-1} x_{n-1,k} \\ 0 \end{bmatrix}. \tag{5.2.32}$$

Notice that the first row of B_m is $[1, 0, \ldots, 0]$, and the last row of B_m is $[0, \ldots, 0, 1]$. By comparing the two sides of (5.2.32), we get $\tilde{b}_1 = 0$ and $\tilde{b}_n = 0$. Thus, we find $\tilde{b}_2, \ldots, \tilde{b}_{n-1}$ that satisfy (5.2.30).

To make the final conclusion, we observe that $B_m\tilde{B}_2 + \tilde{X}_1$ is an $n \times (n-1)$ *zero-row-ending* matrix and it satisfies (5.2.28). By 5.2.3, there exists an $n \times (n-2)$ *zero-row-ending* matrix $\tilde{X}_2$ that satisfies (5.2.25), and we complete the proof. □

Then we combine the above two results, and get the following factorization result.

Proposition 5.2.6. ., *Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a, b]$ with the condition: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions on the interval $[a, b]$ defined on the knots $\vec{t} := \{t_i\}_{-m+1}^{n}$. Assume that $\vec{t}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{t}$ and $\vec{y}$. There exists an $n \times n$ band matrix $\tilde{B}_0$ with bandwidth 3, such that*

$$B_m\tilde{B}_0 - I = \tilde{X}E_2E_1, \quad (m \geq 2) \tag{5.2.33}$$

for some $n \times (n-2)$ matrix $\tilde{X}$.

Proof. : With the given conditions, we apply (5.2.4) first on B_m, and get

$$B_m - I = \tilde{X}_1E_1 \tag{5.2.34}$$

for some $n \times (n-1)$ *zero-row-ending* matrix $\tilde{X}_1$. With this $\tilde{X}_1$, we apply on B_m, and get an $n \times (n-1)$ band matrix $\tilde{B}_2$ with bandwidth 2, such that

$$B_m\tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2E_2 \tag{5.2.35}$$

for some $n \times (n-2)$ *zero-row-ending* matrix $\tilde{X}_2$. It follows from (5.2.34)- (5.2.35) that

$$B_m(\tilde{B}_2E_1 + I) - I = \tilde{X}_2E_2E_1. \tag{5.2.36}$$

Now we take

$$B_0 = \tilde{B}_2 E_1 + I. \tag{5.2.37}$$

It is easy to see that $\tilde{B}_0$ is an $n \times n$ band matrix of bandwidth 3 that satisfies (5.2.33), and we complete the proof.

□

Proposition 5.2.7. *With the same condition as above for $m \geq 2$, there exists an $n \times n$ banded matrix $\tilde{B}_0$ with bandwidth 3, such that*

$$\tilde{B}_0 \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \rho^1_{0,m} \\ \vdots & \vdots \\ 1 & \rho^1_{n-1,m} \end{bmatrix}. \tag{5.2.38}$$

Proof. By Proposition (5.2.6), we can find a banded matrix $\tilde{B}_0$ of bandwidth 3, such that

$$B_m \tilde{B}_0 - I = \tilde{X} E_2 E_1 \tag{5.2.39}$$

for some $n \times (n-2)$ matrix $\tilde{X}$. Then by Proposition (5.1.1), we have

$$(B_m \tilde{B}_0 - I) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \tilde{X} E_2 E_1 \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = 0,$$

which implies that

$$B_m \tilde{B}_0 \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}.$$

Hence, we obtain

$$\tilde{B}_0 \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = B_m^{-1} \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}. \tag{5.2.40}$$

By identity (4.2.9), the right-hand-side of (5.2.40) equals the right-hand-side of (5.2.38), and we complete the proof. □

Remark 5.2.8. *In the proof of **Proposition 5.2.5**, we use the matrix inverse B_m^{-1} to get the band matrix $\tilde{B}_2$. But in the real world applications, we should avoid taking any matrix inverse, because it is too expensive (even if for $m = 2$). Thus, we need to find the explicit formula for $\tilde{B}_2$ when $m = 2$.*

In the next section, we will use a special case for $\mathcal{B}_2$ with $n = 8$ to find the explicit formula for $\tilde{B}_0$ in (5.2.33).

5.3 Determine the Band Matrix of Approximate Inverse for a Special Case

In this section, we will use the following matrix factorization steps to find the explicit formula for $\tilde{B}_0$ in (5.2.33). The first time, we work on a special case, and we will work on the general case in the next section.

In the special case, we assume that

$$a = t_0 = (y_0) \rightharpoonup (y_1) \rightharpoonup t_1 \rightharpoonup (y_2) \rightharpoonup t_2 \rightharpoonup (y_3) \rightharpoonup t_3$$

$$\rightharpoonup t_4 \rightharpoonup (y_4) \rightharpoonup t_5 \rightharpoonup (y_5) \rightharpoonup t_6 \rightharpoonup (y_6) \rightharpoonup b = t_7 = (y_7)$$

for the knots-samples relationship to start our investigation. We do the matrix factorization in the following steps.

The structure of the Shronberg-Whitney matrix can be represented as the following format:

$$\mathcal{B}_2 := [b_{ji}]_{8\times 8},$$

where $b_{ij} := B_{i,2}(y_j)$, that is,

$$\mathcal{B}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{01} & b_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & b_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & b_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{55} & b_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{66} & b_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Notice that there is no data sample between the knots t_3 and t_4, the $\mathcal{B}_2$ matrix is a block-diagonal matrix with two diagonal blocks.

1. Find $\tilde{B}_1$ and $\tilde{X}_1$, such that

$$\mathcal{B}_2\tilde{B}_1 - I = \tilde{X}_1 E_1, \tag{5.3.1}$$

where

$$E_1 = \begin{bmatrix} \frac{1}{y_1-y_0} & & & \\ & \frac{1}{y_2-y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_7-y_6} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{7\times 8}. \tag{5.3.2}$$

We take $\tilde{B}_1 = I$. Denote $\mathbf{1}_8$ as the column matrix with all entries 1, that is,

$$\mathbf{1}_8 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{1\times 8}. \tag{5.3.3}$$

The *partition of unit* property of the B-spline functions implies that

$$(\mathcal{B}_2\tilde{B}_1 - I)\mathbf{1}_8 = 0, \tag{5.3.4}$$

which allows us to do the following factorization:

$$\mathcal{B}_2\tilde{B}_1 - I = X_1 D_1, \tag{5.3.5}$$

where

$$D_1 := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{7\times 8}. \tag{5.3.6}$$

To find X_1 in (5.3.5), we need the pseudo-inverse of D_1, that is,

$$D_1^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{8\times 7}. \tag{5.3.7}$$

Thus, we have

$$X_1 = (\mathcal{B}_2\tilde{B}_1 - I)D_1^+ = \tag{5.3.8}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{66} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we have

$$\tilde{X}_1 = X_1 \begin{bmatrix} y_1 - y_0 & & & \\ & y_2 - y_1 & & \\ & & \ddots & \\ & & & y_7 - y_6 \end{bmatrix}. \tag{5.3.9}$$

Denote

$$d_j^k := y_j - y_{j-k}. \tag{5.3.10}$$

Now we can write $\tilde{X}_1$ as follows

$$\tilde{X}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1^1 b_{01} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2^1 b_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3^1 b_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_4^1 b_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_5^1 b_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_6^1 b_{66} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.3.11}$$

2. Find $\tilde{B}_2$ and $\tilde{X}_2$, such that

$$\mathcal{B}_2 \tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2 E_2, \tag{5.3.12}$$

where

$$E_2 = \begin{bmatrix} \frac{1}{y_2 - y_0} & & & \\ & \frac{1}{y_3 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_7 - y_5} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{6\times 7}. \tag{5.3.13}$$

Assume that

$$\tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{b}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{b}_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{8\times 7}. \tag{5.3.14}$$

We will determine $\tilde{b}_1, \ldots, \tilde{b}_6$ such that

$$(\mathcal{B}_2\tilde{B}_2 + \tilde{X}_1)\mathbf{1}_7 = 0, \tag{5.3.15}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_{01} & b_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{12} & b_{22} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{23} & b_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{44} & b_{45} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_{55} & b_{56} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_{66} & b_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \\ \tilde{b}_5 \\ \tilde{b}_6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -d_1^1\, b_{01} \\ -d_2^1\, b_{12} \\ -d_3^1\, b_{23} \\ -d_4^1\, b_{44} \\ -d_5^1\, b_{55} \\ -d_6^1\, b_{66} \\ 0 \end{bmatrix}. \tag{5.3.16}$$

For this particular example, we find that

$$\begin{aligned} &\tilde{b}_1 = y_1 - t_1, \quad \tilde{b}_2 = y_2 - t_2, \quad \tilde{b}_3 = y_3 - t_3, \\ &\tilde{b}_4 = y_4 - t_4, \quad \tilde{b}_5 = y_5 - t_5, \quad \tilde{b}_6 = y_6 - t_6. \end{aligned} \tag{5.3.17}$$

Now we can calculate $\tilde{B}_0$ using the formula (5.2.37).

$$\tilde{B}_0 = \tilde{B}_2 X_1 + I =$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{y_1-t_1}{y_1-y_0} & \frac{t_1-y_0}{y_1-y_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{y_2-t_2}{y_2-y_1} & \frac{t_2-y_1}{y_2-y_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{y_3-t_3}{y_3-y_2} & \frac{t_3-y_2}{y_3-y_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{y_4-t_4}{y_4-y_3} & \frac{t_4-y_3}{y_4-y_3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{y_5-t_5}{y_5-y_4} & \frac{t_5-y_4}{y_5-y_4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{y_6-t_6}{y_6-y_5} & \frac{t_6-y_5}{y_6-y_5} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{5.3.18}
$$

5.4 Determine the Band Matrix of Approximate Inverse for the General Case

This time we will use the general expression of $\mathcal{B}_2$ with the indicator variables $\{\sigma_i\}_{i=1}^{n-2}$ as in *Chapter 3*. Specifically, we take equation (3.1.6) here,

$$
\mathcal{B}_2 := \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
B_{0,2}(y_1) & B_{1,2}(y_1) & B_{2,2}(y_1) & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & B_{n-3,2}(y_{n-2}) & B_{n-2,2}(y_{n-2}) & B_{n-1,2}(y_{n-2}) \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}_{n\times n},
$$

where $B_{i-1,2}(y_i), B_{i,2}(y_i), B_{i+1,2}(y_i)$ can be represented as

$$
\begin{cases}
B_{i-1,2}(y_i) = \sigma_i \dfrac{t_i - y_i}{t_i - t_{i-1}}, \\
B_{i,2}(y_i) = \sigma_i \dfrac{y_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i)\dfrac{t_{i+1} - y_i}{t_{i+1} - t_i}, \\
B_{i+1,2}(y_i) = (1-\sigma_i)\dfrac{y_i - t_i}{t_{i+1} - t_i}.
\end{cases} \tag{5.4.1}
$$

with

$$
\sigma_i = \begin{cases} 1 & \text{if} \quad y_i \in (t_{i-1}, t_i) \\ 0 & \text{if} \quad y_i \in [t_i, t_{i+1}). \end{cases} \tag{5.4.2}
$$

We will use the same method as in the previous section under the new formulation.

1. Find $\tilde{B}_1$ and $\tilde{X}_1$, such that

$$\mathcal{B}_2\tilde{B}_1 - I = \tilde{X}_1 E_1, \tag{5.4.3}$$

where

$$E_1 = \begin{bmatrix} \frac{1}{y_1 - y_0} & & & \\ & \frac{1}{y_2 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_{n-1} - y_{n-2}} \end{bmatrix}_{(n-1)\times(n-1)} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{(n-1)\times n}. \tag{5.4.4}$$

We take $\tilde{B}_1 = I_n$. Denote $\mathbf{1}_n$ as the column matrix with all entries 1, that is,

$$\mathbf{1}_n = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{1\times n}. \tag{5.4.5}$$

The *partition of unit* property of the B-spline functions implies that

$$(\mathcal{B}_2\tilde{B}_1 - I_n)\mathbf{1}_n = 0, \tag{5.4.6}$$

which allows us to do the following factorization:

$$\mathcal{B}_2\tilde{B}_1 - I = X_1 D_1, \tag{5.4.7}$$

where

$$D_1 := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{(n-1)\times n}. \tag{5.4.8}$$

To find X_1 in (5.4.7), we need the pseudo-inverse of D_1, that is,

$$D_1^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times(n-1)}. \tag{5.4.9}$$

Thus, we have

$$X_1 = (\mathcal{B}_2\tilde{B}_1 - I)D_1^+ = \tag{5.4.10}$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
B_{0,2}(y_1) & -B_{2,2}(y_1) & 0 & 0 & 0 & 0 \\
0 & B_{1,2}(y_2) & -B_{3,2}(y_2) & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & B_{i-1,2}(y_i) & -B_{i+1,2}(y_i) & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ddots & B_{n-3,2}(y_{n-2}) & -B_{n-1,2}(y_{n-2}) \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{n\times(n-1)}.
$$

With the notations in (5.4.1), we have

$$
X_1 =
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_1\frac{t_1-y_1}{t_1-t_0} & (\sigma_1-1)\frac{y_1-t_1}{t_2-t_1} & 0 & 0 & 0 & 0 \\
0 & \sigma_2\frac{t_2-y_2}{t_2-t_1} & (\sigma_2-1)\frac{y_2-t_2}{t_3-t_2} & 0 & 0 & 0 \\
0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & \sigma_i\frac{t_i-y_i}{t_i-t_{i-1}} & (\sigma_i-1)\frac{y_i-t_i}{t_{i+1}-t_i} & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 \\
0 & 0 & 0 & \ddots & \sigma_{n-2}\frac{t_{n-2}-y_{n-2}}{t_{n-2}-t_{n-3}} & (\sigma_{n-2}-1)\frac{y_{n-2}-t_{n-2}}{t_{n-1}-t_{n-2}} \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{n\times(n-1)}. \tag{5.4.11}
$$

Then we use

$$
\tilde{X}_1 = X_1 \begin{bmatrix}
y_1-y_0 & & & \\
 & y_2-y_1 & & \\
 & & \ddots & \\
 & & & y_{n-1}-y_{n-2}
\end{bmatrix}_{(n-1)\times(n-1)}. \tag{5.4.12}
$$

Denote

$$
d_j^k := y_j - y_{j-k}. \tag{5.4.13}
$$

We can write $\tilde{X}_1$ as

$$
\tilde{X}_1 = X_1 \begin{bmatrix}
d_1^1 & & & \\
 & d_2^1 & & \\
 & & \ddots & \\
 & & & d_{n-1}^1
\end{bmatrix}_{(n-1)\times(n-1)} =
$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \sigma_1 d_1^1 \frac{t_1 - y_1}{t_1 - t_0} & (\sigma_1 - 1) d_2^1 \frac{y_1 - t_1}{t_2 - t_1} & 0 & 0 & 0 \\ 0 & \sigma_2 d_2^1 \frac{t_2 - y_2}{t_2 - t_1} & (\sigma_2 - 1) d_3^1 \frac{y_2 - t_2}{t_3 - t_2} & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \sigma_i d_i^1 \frac{t_i - y_i}{t_i - t_{i-1}} & (\sigma_i - 1) d_{i+1}^1 \frac{y_i - t_i}{t_{i+1} - t_i} & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & \sigma_{n-2} d_{n-2}^1 \frac{t_{n-2} - y_{n-2}}{t_{n-2} - t_{n-3}} & (\sigma_{n-2} - 1) d_{n-1}^1 \frac{y_{n-2} - t_{n-2}}{t_{n-1} - t_{n-2}} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

2. Find $\tilde{B}_2$ and $\tilde{X}_2$, such that

$$\mathcal{B}_2 \tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2 E_2, \tag{5.4.14}$$

where

$$E_2 = \begin{bmatrix} \frac{1}{y_2 - y_0} & & & \\ & \frac{1}{y_3 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_{n-1} - y_{n-3}} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{(n-2)\times(n-1)}. \tag{5.4.15}$$

To make $\tilde{B}_2$ as simple as possible, we take

$$\tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{b}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n\times(n-1)}. \tag{5.4.16}$$

We will determine $\tilde{b}_1, \ldots, \tilde{b}_{n-2}$ such that

$$(\mathcal{B}_2 \tilde{B}_2 + \tilde{X}_1)\mathbf{1}_{n-1} = 0, \tag{5.4.17}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ B_{0,2}(y_1) & B_{1,2}(y_1) & B_{2,2}(y_1) & 0 & 0 & 0 \\ 0 & B_{1,2}(y_2) & B_{2,2}(y_2) & B_{3,2}(y_2) & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & B_{n-3,2}(y_{n-2}) & B_{n-2,2}(y_{n-2}) & B_{n-1,2}(y_{n-2}) \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \vdots \\ \vdots \\ \tilde{b}_{n-2} \\ 0 \end{bmatrix}$$

$$= - \begin{bmatrix} 0 \\ \sigma_1 d_1^1 \frac{t_1 - y_1}{t_1 - t_0} + (\sigma_1 - 1) d_2^1 \frac{y_1 - t_1}{t_2 - t_1} \\ \sigma_2 d_2^1 \frac{t_2 - y_2}{t_2 - t_1} + (\sigma_2 - 1) d_3^1 \frac{y_2 - t_2}{t_3 - t_2} \\ \vdots \\ \vdots \\ \sigma_{n-2} d_{n-2}^1 \frac{t_{n-2} - y_{n-2}}{t_{n-2} - t_{n-3}} + (\sigma_{n-2} - 1) d_{n-1}^1 \frac{y_{n-2} - t_{n-2}}{t_{n-1} - t_{n-2}} \\ 0 \end{bmatrix}.$$

Based on our example in the previous section, we make the following guess,

$$\tilde{b}_1 = y_1 - t_1, \quad \tilde{b}_2 = y_2 - t_2, \quad \cdots, \quad \tilde{b}_{n-2} = y_{n-2} - t_{n-2}. \tag{5.4.18}$$

Next we need to prove the above conjecture. Let us do the calculation as follows,

$$\begin{aligned} LHS \quad &= \tilde{b}_{i-1}\, B_{i-1,2}(y_i) + \tilde{b}_i\, B_{i,2}(y_i) + \tilde{b}_{i+1}\, B_{i+1,2}(y_i) \\ &= (y_{i-1} - t_{i-1})\, \sigma_i \frac{t_i - y_i}{t_i - t_{i-1}} + (y_i - t_i)\, \sigma_i \frac{y_i - t_{i-1}}{t_i - t_{i-1}} + \\ &\quad (y_i - t_i)\,(1 - \sigma_i) \frac{t_{i+1} - y_i}{t_{i+1} - t_i} + (y_{i+1} - t_{i+1})\,(1 - \sigma_i) \frac{y_i - t_i}{t_{i+1} - t_i} \\ &= \sigma_i \left((y_{i-1} - t_{i-1})\, \frac{t_i - y_i}{t_i - t_{i-1}} + (y_i - t_i)\, \frac{y_i - t_{i-1}}{t_i - t_{i-1}} \right) \\ &\quad + (1 - \sigma_i) \left((y_i - t_i)\, \frac{t_{i+1} - y_i}{t_{i+1} - t_i} + (y_{i+1} - t_{i+1})\, \frac{y_i - t_i}{t_{i+1} - t_i} \right). \end{aligned}$$

and

$$\begin{aligned} RHS \quad &= -\sigma_i\, d_i^1\, \frac{t_i - y_i}{t_i - t_{i-1}} + (1 - \sigma_i)\, d_{i+1}^1\, \frac{y_i - t_i}{t_{i+1} - t_i} \\ &= -\sigma_i\, (y_i - y_{i-1})\, \frac{t_i - y_i}{t_i - t_{i-1}} + (1 - \sigma_i)\, (y_{i+1} - y_i)\, \frac{y_i - t_i}{t_{i+1} - t_i}. \end{aligned}$$

Comparing the two sides above, we need to verify the following identities:

$$(y_{i-1} - t_{i-1})(t_i - y_i) + (y_i - t_i)(y_i - t_{i-1}) = (y_{i-1} - y_i)(t_i - y_i)$$

and

$$(y_i - t_i)(t_{i+1} - y_i) + (y_{i+1} - t_{i+1})(y_i - t_i) = (y_{i+1} - y_i)(y_i - t_i).$$

We can that the above two identities are true by factorizing the common factors $(t_i - y_i)$ and $(y_i - t_i)$, respectively. Thus, our conjecture is true. Based on this property, we can get the following theorem.

Theorem 5.4.1. *Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a, b]$ with the endpoints interpolating property: $y_0 = a$ and $y_{n-1} = b$, we consider the linear B-splines with n basis functions $\{B_{i,2}(x)\}_{i=0}^{n-1}$ on the interval $[a, b]$ defined on the knots $\{t_i\}_{-1}^{n}$ with*

$$a = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-2} < t_{n-1} = t_n = b.$$

Assume that

$$t_{i-1} < y_i < t_{i+1}, \qquad 1 \leq i \leq n-2,$$

and we denote $\vec{t} := \{t_i\}_0^{n-1}$. Then the Shoenberg-Whitney matrix defined as follows

$$\mathcal{B}_2(\vec{t}, \vec{y}) := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ B_{0,2}(y_1) & B_{1,2}(y_1) & B_{2,2}(y_1) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & B_{n-3,2}(y_{n-2}) & B_{n-2,2}(y_{n-2}) & B_{n-1,2}(y_{n-2}) \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}_{n \times n} \tag{5.4.19}$$

has an approximate inverse *of the form $\mathcal{B}_2(\vec{y}, \vec{t})$, i.e.*

$$\mathcal{B}_2(\vec{t}, \vec{y}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \mathcal{B}_2(\vec{y}, \vec{t}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}. \tag{5.4.20}$$

Proof. By the matrix factorization steps above, we can find $\tilde{X}_1$ of the form (5.4.12), such that

$$\mathcal{B}_2 - I_n = \tilde{X}_1 E_1, \tag{5.4.21}$$

where E_1 is defined in (5.4.4). Since $\mathcal{B}_2$ contains the indicator variables $\{\sigma_i\}_{i=1}^{n-2}$, we choose our $\tilde{B}_2$ factoring in the σ-structure as follows

$$\tilde{B}_2 =$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\sigma_1 (y_1 - t_1) & (1-\sigma_1)(y_1 - t_1) & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_2 (y_2 - t_2) & (1-\sigma_2)(y_2 - t_2) & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_{n-2} (y_{n-2} - t_{n-2}) & (1-\sigma_{n-2})(y_{n-2} - t_{n-2}) \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{n\times(n-1)}, \tag{5.4.22}
$$

and we can get

$$
(\mathcal{B}_2\tilde{B}_2 + \tilde{X}_1)\,\mathbf{1}_{n-1} = 0 \tag{5.4.23}
$$

by the following identity

$$
\begin{aligned}
&(y_{i-1} - t_{i-1})\,B_{i-1,2}(y_i) + (y_i - t_i)\,B_{i,2}(y_i) + (y_{i+1} - t_{i+1})\,B_{i+1,2}(y_i) \\
&= -\sigma_i\,(y_i - y_{i-1})\,\frac{t_i - y_i}{t_i - t_{i-1}} + (1-\sigma_i)\,(y_{i+1} - y_i)\,\frac{y_i - t_i}{t_{i+1} - t_i},
\end{aligned}
$$

where the general expressions of $B_{i-1,2}(y_i), B_{i,2}(y_i)$, and $B_{i+1,2}(y_i)$ are given by (5.4.1). The identity (5.4.23) implies that there exists an $n \times (n-2)$ matrix $\tilde{X}_2$, such that

$$
\mathcal{B}_2\tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2 E_2,
$$

with E_2 given by (5.4.15). Combine (5.4.21) and (5.4.23), we get

$$
\mathcal{B}_2\,(I_n + \tilde{B}_2 E_1) - I_n = \tilde{X}_2 E_2 E_1. \tag{5.4.24}
$$

Now we define

$$
\tilde{B}_0 := I_n + \tilde{B}_2 E_1. \tag{5.4.25}
$$

We can write (5.4.24) as

$$
\mathcal{B}_2\,\tilde{B}_0 - I_n = \tilde{X}_2 E_2 E_1. \tag{5.4.26}
$$

By *Proposition* (5.1.1), we have that

$$
(\mathcal{B}_2\,\tilde{B}_0 - I_n)\begin{bmatrix} 1 & y_0 \\ 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}_{n\times 2} = \tilde{X}_2 E_2 E_1 \begin{bmatrix} 1 & y_0 \\ 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}_{n\times 2} = 0,
$$

which implies that

$$
\mathcal{B}_2^{-1}\begin{bmatrix} 1 & y_0 \\ 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}_{n\times 2} = \tilde{B}_0 \begin{bmatrix} 1 & y_0 \\ 1 & y_1 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}_{n\times 2}.
$$

By the definition of the *approximate inverse*, we can see that $\tilde{B}_0$ is an *approximate inverse* of $\mathcal{B}_2$. Next we will find the expression of $\tilde{B}_0$ by calculating $I_n + \tilde{B}_2 E_1$ in the following steps.

1. Calculate $\tilde{B}_2 \operatorname{diag}\left(\dfrac{1}{y_1 - y_0}, \ldots, \dfrac{1}{y_{n-1} - y_{n-2}}\right)$

   By (5.4.22), we have

$$\tilde{B}_2 \operatorname{diag}\left(\frac{1}{y_1 - y_0}, \ldots, \frac{1}{y_{n-1} - y_{n-2}}\right) =$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sigma_1 \frac{y_1 - t_1}{y_1 - y_0} & (1-\sigma_1) \frac{y_1 - t_1}{y_2 - y_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 \frac{y_2 - t_2}{y_2 - y_1} & (1-\sigma_2) \frac{y_2 - t_2}{y_3 - y_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{n-2} \frac{y_{n-2} - t_{n-2}}{y_{n-2} - y_{n-3}} & (1-\sigma_{n-2}) \frac{y_{n-2} - t_{n-2}}{y_{n-1} - y_{n-2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5.4.27}$$

2. Calculate $\tilde{B}_2 \, E_1$

   By (5.4.4), we have

$$\tilde{B}_2 \, E_1 = \tilde{B}_2 \operatorname{diag}\left(\frac{1}{y_1 - y_0}, \ldots, \frac{1}{y_{n-1} - y_{n-2}}\right) D_1,$$

   where D_1 is the difference matrix defined in (5.4.8). Notice that the $(i+1)$-th row of $\tilde{B}_2 \operatorname{diag}\left(\dfrac{1}{y_1 - y_0}, \ldots, \dfrac{1}{y_{n-1} - y_{n-2}}\right)$ is

$$\begin{bmatrix} 0 & \cdots & 0 & \sigma_i \frac{y_i - t_i}{y_i - y_{i-1}}, & (1-\sigma_i) \frac{y_i - t_i}{y_{i+1} - y_i}, & 0 & \cdots & 0 \end{bmatrix},$$

   where the first nonzero entry is at the i-th location in the row. Then we can get the $(i+1)$-th row of $\tilde{B}_2 \, E_1$ as

$$\begin{bmatrix} 0 & \cdots & 0 & \sigma_i \frac{y_i - t_i}{y_i - y_{i-1}}, & -\sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} + (1-\sigma_i) \frac{y_i - t_i}{y_{i+1} - y_i}, & -(1-\sigma_i) \frac{y_i - t_i}{y_{i+1} - y_i}, & 0 & \cdots & 0 \end{bmatrix},$$

   where the first nonzero entry is also at the i-th location of the row.

3. Calculate $I_n + \tilde{B}_2 E_1$

   It is easy to see that the $(i+1)$-th row of $I_n + \tilde{B}_2 E_1$ is

$$\begin{bmatrix} 0 & \cdots & 0 & \sigma_i \dfrac{y_i - t_i}{y_i - y_{i-1}}, & 1 - \sigma_i \dfrac{y_i - t_i}{y_i - y_{i-1}} + (1-\sigma_i) \dfrac{y_i - t_i}{y_{i+1} - y_i}, & (1-\sigma_i) \dfrac{t_i - y_i}{y_{i+1} - y_i}, & 0 & \cdots & 0 \end{bmatrix},$$

   which can be simplified as

$$\begin{bmatrix} 0 & \cdots & 0 & \sigma_i \dfrac{y_i - t_i}{y_i - y_{i-1}}, & \sigma_i \dfrac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1-\sigma_i) \dfrac{y_{i+1} - t_i}{y_{i+1} - y_i}, & (1-\sigma_i) \dfrac{t_i - y_i}{y_{i+1} - y_i}, & 0 & \cdots & 0 \end{bmatrix}, \tag{5.4.28}$$

   where the first nonzero entry is also at the i-th location of the row.

In order to compare $\mathcal{B}_2$ with $\tilde{B}_0$, we also list the $(i+1)$-th row of $\mathcal{B}_2$ here, i.e.

$$\begin{bmatrix} 0 & \cdots & 0 & B_{i-1,2}(y_i) & B_{i,2}(y_i) & B_{i+1,2}(y_i) & 0 & \cdots & 0 \end{bmatrix},$$

which is

$$\begin{bmatrix} 0 & \cdots & 0 & \sigma_i \dfrac{t_i - y_i}{t_i - t_{i-1}}, & \sigma_i \dfrac{y_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i) \dfrac{t_{i+1} - y_i}{t_{i+1} - t_i}, & (1-\sigma_i) \dfrac{y_i - t_i}{t_{i+1} - t_i}, & 0 & \cdots & 0 \end{bmatrix}. \tag{5.4.29}$$

By examining the expressions of (5.4.28) and (5.4.29), we observe the following relationship between them: If we switch the positions of t and y in the first expression, we will get the second expression. In other words, these two vectors have the *duality property* with respect to $\vec{t}$ and $\vec{y}$. By extending this property to the whole matrix, we get

$$\mathcal{B}_2(\vec{t}, \vec{y}) = \tilde{B}_0(\vec{y}, \vec{t}),$$

which is the same as (5.4.20), and we complete the proof.

□

From this theorem, we can easily get the following corollary.

Corollary 5.4.2. *With the same conditions and notations as in Theorem 5.4.1, we have the following identity:*

$$\left(\mathcal{B}_2\left(\vec{t}, \vec{y}\right) \mathcal{B}_2(\vec{y}, \vec{t}) - I_n\right) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = 0. \tag{5.4.30}$$

Proof. By Theorem 5.4.1, we have

$$\mathcal{B}_2(\vec{t}, \vec{y}) = \tilde{B}_0(\vec{y}, \vec{t}),$$

or equivalently,

$$\tilde{B}_0(\vec{t}, \vec{y}) = \mathcal{B}_2(\vec{y}, \vec{t}). \tag{5.4.31}$$

By the definition of *approximate inverse*, we have that

$$\mathcal{B}_2^{-1}(\vec{t}, \vec{y}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \tilde{B}_0(\vec{t}, \vec{y}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix}. \tag{5.4.32}$$

It follows from (5.4.31) and (5.4.32) that

$$\mathcal{B}_2^{-1}(\vec{t}, \vec{y}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix} = \mathcal{B}_2(\vec{y}, \vec{t}) \begin{bmatrix} 1 & y_0 \\ \vdots & \vdots \\ 1 & y_{n-1} \end{bmatrix},$$

which is the same as (5.4.30), and we complete the proof. □

Next we will use the band matrix $\tilde{B}_0$ to construct a *local quasi-interpolating* operator.

5.5 Local Quasi-Interpolating Operator for Linear B-Splines

(*Compare our new result with the following identity:*)

$$\sum_{k=-\infty}^{\infty} p(k) N_m(x-k) = \sum_{k=0}^{m-1} N_m(k) p(x-k).$$

Looking from this angle: Switching the knots $\vec{t}$ and the sampling points $\vec{y}$. For the linear case ($m = 2$), there is a *duality* property between $\vec{t}$ and $\vec{y}$. But for the general case, this property may be lost.

With the explicit formula for the approximate inverse of $\mathcal{B}_2$ as in (5.4.20), we can write the data-induced (DI) linear operator for $\tilde{B}_0$ as follows.

With n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a, b]$ satisfying the end-points interpolating property: $y_0 = a$ and $y_{n-1} = b$, we use the n linear B-splines $\{B_{i,2}(x)\}_{i=0}^{n-1}$ on the interval $[a, b]$ defined on the knots $\{t_i\}_{-1}^{n}$ to define the following linear operator:

$\tilde{B}_y^0 : C[a, b] \to S_{m,t}$ with

$$(\tilde{B}_y^0 f)(x) = \sum_{k=0}^{n-1} (\tilde{B}_0 \vec{f}_y)_k B_{k,m}(x), \tag{5.5.1}$$

where

$$\vec{f}_y = [f(y_0), \cdots, f(y_{n-1})]^T. \tag{5.5.2}$$

In order to see this operator clearly, we would like to write its expression explicitly.

First, let us write the matrix $\tilde{B}_0$ explicitly. We use $\tilde{b}_0^i, i = 0, 1, \ldots, n-1$ to denote the n rows of $\tilde{B}_0$. Thus the matrix-vector product $\tilde{B}_0 \vec{f}_y$ can be represented as

$$\tilde{B}_0 \vec{f}_y = \begin{bmatrix} \langle \tilde{b}_0^0, \vec{f}_y \rangle \\ \langle \tilde{b}_0^1, \vec{f}_y \rangle \\ \vdots \\ \langle \tilde{b}_0^{n-1}, \vec{f}_y \rangle \end{bmatrix},$$

where the notation $\langle \cdot, \cdot \rangle$ is the regular inner-product of two vectors. Hence, (5.5.1) can be written as

$$(\tilde{B}_y^0 f)(x) = \sum_{i=0}^{n-1} \langle \tilde{b}_0^i, \vec{f}_y \rangle \, B_{i,m}(x).$$

Furthermore, let us write the expression of $\langle \tilde{b}_0^i, \vec{f}_y \rangle$ explicitly. By (5.4.28), we have

$$\langle \tilde{b}_0^i, \vec{f}_y \rangle = \sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} f(y_{i-1}) + \left(\sigma_i \frac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1 - \sigma_i) \frac{y_{i+1} - t_i}{y_{i+1} - y_i} \right) f(y_i) + (1 - \sigma_i) \frac{t_i - y_i}{y_{i+1} - y_i} f(y_{i+1}).$$

In order to see a complete picture of (5.5.1), we write it as follows,

$$(\tilde{B}_y^0 f)(x) = \sum_{i=0}^{n-1} \left[\sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} f(y_{i-1}) + \left(\sigma_i \frac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1 - \sigma_i) \frac{y_{i+1} - t_i}{y_{i+1} - y_i} \right) f(y_i) + (1 - \sigma_i) \frac{t_i - y_i}{y_{i+1} - y_i} f(y_{i+1}) \right] B_{i,m}(x). \tag{5.5.3}$$

In order to show that the linear operator in (5.5.1) is a quasi-interpolating operator, we need to show that

$$(\tilde{B}_y^0 1)(x) = 1 \quad \text{and} \quad (\tilde{B}_y^0 x)(x) = x.$$

The first identity can be verified easily by the *partition of unity* for B-splines as follows,

$$(\tilde{B}_y^0 1)(x) = \sum_{i=0}^{n-1} \left[\sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} \cdot 1 + \left(\sigma_i \frac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1-\sigma_i) \frac{y_{i+1} - t_i}{y_{i+1} - y_i} \right) \cdot 1 + (1-\sigma_i) \frac{t_i - y_i}{y_{i+1} - y_i} \cdot 1 \right] B_{i,m}(x)$$

$$= \sum_{i=0}^{n-1} [\sigma_i \cdot 1 + (1-\sigma_i) \cdot 1] B_{i,m}(x) = \sum_{i=0}^{n-1} B_{i,m}(x) = 1.$$

To verify the second identity, we need to calculate the following expression:

$$(\tilde{B}_y^0 x)(x) = \sum_{i=0}^{n-1} \left[\sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} \cdot y_{i-1} + \left(\sigma_i \frac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1-\sigma_i) \frac{y_{i+1} - t_i}{y_{i+1} - y_i} \right) \cdot y_i + (1-\sigma_i) \frac{t_i - y_i}{y_{i+1} - y_i} \cdot y_{i+1} \right] B_{i,m}(x).$$

Let us simplify the coefficient for each $B_{i,m}(x)$ first.

$$\sigma_i \frac{y_i - t_i}{y_i - y_{i-1}} \cdot y_{i-1} + \left(\sigma_i \frac{t_i - y_{i-1}}{y_i - y_{i-1}} + (1-\sigma_i) \frac{y_{i+1} - t_i}{y_{i+1} - y_i} \right) \cdot y_i + (1-\sigma_i) \frac{t_i - y_i}{y_{i+1} - y_i} \cdot y_{i+1}$$

$$= \sigma_i \left(\frac{y_i - t_i}{y_i - y_{i-1}} \cdot y_{i-1} + \frac{t_i - y_{i-1}}{y_i - y_{i-1}} \cdot y_i \right) + (1-\sigma_i) \left(\frac{y_{i+1} - t_i}{y_{i+1} - y_i} \cdot y_i + \frac{t_i - y_i}{y_{i+1} - y_i} \cdot y_{i+1} \right)$$

$$= \sigma_i t_i + (1-\sigma_i) t_i = t_i.$$

With this simplification, we have

$$(\tilde{B}_y^0 x)(x) = \sum_{i=0}^{n-1} t_i B_{i,m}(x) = x,$$

where the last equality is based on the Marsden's identity for the linear case.

Next Question: How to connect the following two identities?

$$\sum_{k=-\infty}^{\infty} p(k) N_m(x-k) = \sum_{k=0}^{m-1} N_m(k) p(x-k)$$

and

$$\mathcal{B}_2(\vec{t}, \vec{y}) = \tilde{B}_0(\vec{y}, \vec{t}).$$

To see this, we need to go back to the definition of the *approximate inverse* and the definition of the DI-operator.

First we simplify the first identity by using a monomial to replace the polynomial, i.e., replacing $p(x)$ by x^r for some $0 \le r \le m-1$. We have

$$\sum_{k=-\infty}^{\infty} k^r N_m(x-k) = \sum_{k=0}^{m-1} N_m(k)(x-k)^r.$$

Here our focus is on: How the knot vector and the data sample vector switch.

In the Cardinal B-spline case, both the knot vector and the data sample vector are using integers. It is hard to separate them. Let us list the following two properties.

Polynomial preservation property:

$$L \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,m}^r \\ \rho_{1,m}^r \\ \vdots \\ \rho_{n-1,m}^r \end{bmatrix}, \quad \text{for} \quad r = 0, 1, \cdots, m-1,$$

then

$$(L_y p)(x) = p(x), \quad \text{for all} \quad p \in \pi_{m-1},$$

where $\rho_{0,m}^r, \ldots, \rho_{n-1,m}^r$ are the Marsden's coefficients.

The B-spline interpolating operator preserves the polynomials.

$$B_m^{-1} \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,m}^r \\ \rho_{1,m}^r \\ \vdots \\ \rho_{n-1,m}^r \end{bmatrix}.$$

For the linear case, we have the following results:

$$\mathcal{B}_2^{-1}(\vec{t}, \vec{y}) \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,2}^r \\ \rho_{1,2}^r \\ \vdots \\ \rho_{n-1,2}^r \end{bmatrix},$$

which is equivalently as

$$\mathcal{B}_2(\vec{y}, \vec{t}) \begin{bmatrix} y_0^r \\ y_1^r \\ \vdots \\ y_{n-1}^r \end{bmatrix} = \begin{bmatrix} \rho_{0,2}^r \\ \rho_{1,2}^r \\ \vdots \\ \rho_{n-1,2}^r \end{bmatrix},$$

For the Cardinal B-splines, we can view $N_m(x-k)$ as $N_{k,m}(x)$ for any integer k. Now we can write

$$\sum_{k=-\infty}^{\infty} p(k) N_m(x-k) = \begin{bmatrix} \cdots & N_{k-1,m}(x) & N_{k,m}(x) & N_{k+1,m}(x) & \cdots \end{bmatrix} \begin{bmatrix} \vdots \\ (k-1)^r \\ k^r \\ (k+1)^r \\ \vdots \end{bmatrix}$$

Chapter 6

Conclusion and Future Research

In this work, we constructed a local quasi-interpolating operator for the linear B-splines which can be used to do data interpolation without using any matrix inverse. The construction is based on the matrix factorization technique. The success of this method relies on a matrix criterion for the polynomial reproduction using the coefficients of the Marsden's identities.

We believe that this method can be extended to more general situation. For example, we can consider the spaces generated by refinable functions. We can also consider the basis functions on the real axis. In these cases, how to define the *approximate inverse* concept, and what are the properties for the approximate inverses in these general cases?

We also plan to apply our quasi-interpolation operators on some real-world application problems. In many data processing problems, real-time response is very important. Since our method avoids matrix inverse computation that is needed in most existing data interpolating methods, we have a very good change to achieve the linear ($O(n)$) performance in those problems.

We will develop computer programs in Matlab, R, and Python with user friendly interfaces and make them available in the community, so that our methods can be applied to many data processing problems in the community.

In order to get the approximate inverse formulas for $m \geq 3$, we do the following calculations for some special cases.

6.1 Quadratic B-spline - General case

Next, we will do factorization on $\mathcal{B}_3$. Assume that we are given n sample points: $\{y_k\}_{k=0}^{n-1}$ spread on the interval $[a, b]$ with the following condition:

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b.$$

To do the quadratic B-spline interpolation on these sample points, we construct a set of quadratic B-splines $\{B_{i,3}(x)\}_{i=0}^{n-1}$ using the knots $\{t_k\}_{k=-2}^{n}$ with the following form:

$$a = t_{-2} = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-3} < t_{n-2} = t_{n-1} = t_n = b,$$

where the basis function $B_{i,3}(x)$ is constructed from the knots: $\{t_{i-2}, t_{i-1}, t_i, t_{i+1}\}$ for $i = 0, \ldots, n-1$. Furthermore, the Shoenberg-Whitney condition must be satisfied, that is,

$$t_{i-2} < y_i < t_{i+1}, \quad \text{for } 1 \le i \le n-2,$$

which implies that

$$y_{i-1} < t_i < y_{i+2}, \quad \text{for } 1 \le i \le n-3.$$

1. Find $\tilde{B}_1$ and $\tilde{X}_1$, such that

$$\mathcal{B}_3\tilde{B}_1 - I = \tilde{X}_1 E_1, \tag{6.1.1}$$

where

$$E_1 = \begin{bmatrix} \frac{1}{y_1-y_0} & & & \\ & \frac{1}{y_2-y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_8-y_7} \end{bmatrix}_{9\times 9} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{8\times 9}. \tag{6.1.2}$$

We take $\tilde{B}_1 = I$. Denote $\mathbf{1}_9$ as the column matrix with all entries 1, that is,

$$\mathbf{1}_9 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{1\times 9}. \tag{6.1.3}$$

The *partition of unit* property of the B-spline functions implies that

$$(\mathcal{B}_3\tilde{B}_1 - I)\mathbf{1}_9 = 0, \tag{6.1.4}$$

which allows us to do the following factorization:

$$\mathcal{B}_3\tilde{B}_1 - I = X_1 D_1, \tag{6.1.5}$$

where

$$D_1 := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{8\times 9}. \tag{6.1.6}$$

To find X_1 in (6.1.5), we need the pseudo-inverse of D_1, that is,

$$D_1^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{9\times 8}. \tag{6.1.7}$$

Thus, we have

$$X_1 = (\mathcal{B}_3\tilde{B}_1 - I)D_1^+ = \tag{6.1.8}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & -B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & -B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & -B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & -B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{45} & -B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & -B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{67} & -B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 8},$$

where B_{ij} means $B_{i,3}(y_j)$. Then we have

$$\tilde{X}_1 = X_1 \begin{bmatrix} y_1 - y_0 & & & \\ & y_2 - y_1 & & \\ & & \ddots & \\ & & & y_8 - y_7 \end{bmatrix}_{8\times 8}. \tag{6.1.9}$$

Denote

$$d_j^k := y_j - y_{j-k}. \tag{6.1.10}$$

Now we can write $\tilde{X}_1$ as follows

$$\tilde{X}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ d_1^1 B_{01} & -d_2^1 B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d_2^1 B_{12} & -d_3^1 B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & d_3^1 B_{23} & -d_4^1 B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4^1 B_{34} & -d_5^1 B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d_5^1 B_{45} & -d_6^1 B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d_6^1 B_{56} & -d_7^1 B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_7^1 B_{67} & -d_8^1 B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 8}. \tag{6.1.11}$$

2. Find $\tilde{B}_2$ and $\tilde{X}_2$, such that

$$\mathcal{B}_3 \tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2 E_2, \tag{6.1.12}$$

where

$$E_2 = \begin{bmatrix} \frac{1}{y_2-y_0} & & & \\ & \frac{1}{y_3-y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_8-y_6} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{7\times 8}. \tag{6.1.13}$$

Assume that

$$\tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{b}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{b}_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{b}_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 8}. \tag{6.1.14}$$

We will determine $\tilde{b}_1, \ldots, \tilde{b}_7$ such that

$$(\mathcal{B}_3 \tilde{B}_2 + \tilde{X}_1)\mathbf{1}_8 = 0, \tag{6.1.15}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & B_{11} & B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & B_{22} & B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & B_{33} & B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & B_{44} & B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{45} & B_{55} & B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & B_{66} & B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{67} & B_{77} & B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \\ \tilde{b}_4 \\ \tilde{b}_5 \\ \tilde{b}_6 \\ \tilde{b}_7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -d_1^1 B_{01} + d_2^1 B_{21} \\ -d_2^1 B_{23} + d_3^1 B_{32} \\ -d_3^1 B_{34} + d_4^1 B_{43} \\ -d_4^1 B_{45} + d_5^1 B_{54} \\ -d_5^1 B_{56} + d_6^1 B_{65} \\ -d_6^1 B_{67} + d_7^1 B_{76} \\ -d_7^1 B_{78} + d_8^1 B_{87} \\ 0 \end{bmatrix}. \tag{6.1.16}$$

Based on our experiment, we guess that

$$\begin{gathered} \tilde{b}_1 = -\frac{t_0 + t_1}{2} + y_1, \quad \tilde{b}_2 = -\frac{t_1 + t_2}{2} + y_2, \quad \tilde{b}_3 = -\frac{t_2 + t_3}{2} + y_3, \\ \tilde{b}_4 = -\frac{t_3 + t_4}{2} + y_4, \quad \tilde{b}_5 = -\frac{t_4 + t_5}{2} + y_5, \\ \tilde{b}_6 = -\frac{t_5 + t_6}{2} + y_6, \quad \tilde{b}_7 = -\frac{t_6 + t_7}{2} + y_7. \end{gathered} \tag{6.1.17}$$

To verify the correctness of each row in (6.1.16), we need to show that

$$\tilde{b}_{i-1} B_{i-1,i} + \tilde{b}_i B_{i,i} + \tilde{b}_{i+1} B_{i+1,i} = -d_i^1 B_{i-1,i} + d_{i+1}^1 B_{i+1,i}. \tag{6.1.18}$$

Let us verify it directly. We calculate the two sides of (6.1.18) separately.

$$\begin{aligned} LHS &= \tilde{b}_{i-1} B_{i-1,i} + \tilde{b}_i B_{i,i} + \tilde{b}_{i+1} B_{i+1,i} \\ &= \left(-\tfrac{t_{i-2}+t_{i-1}}{2} + y_{i-1}\right) B_{i-1,i} + \left(-\tfrac{t_{i-1}+t_i}{2} + y_i\right) B_{i,i} + \left(-\tfrac{t_i+t_{i+1}}{2} + y_{i+1}\right) B_{i+1,i} \\ &= y_{i-1} B_{i-1,i} + y_i B_{i,i} + y_{i+1} B_{i+1,i} - \textstyle\sum_{k=0}^2 \left(\tfrac{t_{i+k-2}+t_{i+k-1}}{2}\right) B_{i+k-1,i}. \end{aligned}$$

The last summation term can be simplified by the Marsden's identity as follows

$$\sum_{k=0}^{2} \left(\frac{t_{j+k-2} + t_{j+k-1}}{2}\right) B_{j+k-1,3}(y_j) = y_j. \tag{6.1.19}$$

Thus we have (using the *partition of unity*)

$$\begin{aligned} LHS &= y_{i-1} B_{i-1,i} + y_i B_{i,i} + y_{i+1} B_{i+1,i} - y_i \\ &= y_{i-1} B_{i-1,i} + y_i B_{i,i} + y_{i+1} B_{i+1,i} - y_i (B_{i-1,i} + B_{i,i} + B_{i+1,i}) \\ &= (y_{i-1} - y_i) B_{i-1,i} + (y_{i+1} - y_i) B_{i+1,i} = -d_i^1\, B_{i-1,i} + d_{i+1}^1\, B_{i+1,i}, \end{aligned}$$

which is exactly the right-hand-side of (6.1.18).

To get the factorization, we need to compute $\mathcal{B}_3\tilde{B}_2 + \tilde{X}_1$, that is,

$$\mathcal{B}_3\tilde{B}_2 + \tilde{X}_1 = \tag{6.1.20}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{b}_1 B_{11} - d_1^1 B_{01} & (\tilde{b}_2 + d_2^1)B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{b}_1 B_{12} & \tilde{b}_2 B_{22} - d_2^1 B_{12} & (\tilde{b}_3 + d_3^1)B_{32} & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{b}_2 B_{23} & \tilde{b}_3 B_{33} - d_3^1 B_{23} & (\tilde{b}_4 + d_4^1)B_{43} & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{b}_3 B_{34} & \tilde{b}_4 B_{44} - d_4^1 B_{34} & (\tilde{b}_5 + d_5^1)B_{54} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{b}_4 B_{45} & \tilde{b}_5 B_{55} - d_5^1 B_{45} & (\tilde{b}_6 + d_6^1)B_{65} & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{b}_5 B_{56} & \tilde{b}_6 B_{66} - d_6^1 B_{56} & (\tilde{b}_7 + d_7^1)B_{76} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{b}_6 B_{67} & \tilde{b}_7 B_{77} - d_7^1 B_{67} & d_8^1 B_{87} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} \tag{6.1.21}$$

To find X_2, such that

$$\mathcal{B}_3\tilde{B}_2 + \tilde{X}_1 = X_2 D_2, \tag{6.1.22}$$

where

$$D_2 := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{7\times 8}, \tag{6.1.23}$$

we need to use the pseudo-inverse of D_2, that is,

$$D_2^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{8\times 7}. \tag{6.1.24}$$

Thus,

$$X_2 = (\mathcal{B}_3\tilde{B}_2 + \tilde{X}_1)D_2^+ = \tag{6.1.25}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-(\tilde{b}_2 + d_2^1)B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\
\tilde{b}_1 B_{12} & -(\tilde{b}_3 + d_3^1)B_{32} & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{b}_2 B_{23} & -(\tilde{b}_4 + d_4^1)B_{43} & 0 & 0 & 0 & 0 \\
0 & 0 & \tilde{b}_3 B_{34} & -(\tilde{b}_5 + d_5^1)B_{54} & 0 & 0 & 0 \\
0 & 0 & 0 & \tilde{b}_4 B_{45} & -(\tilde{b}_6 + d_6^1)B_{65} & 0 & 0 \\
0 & 0 & 0 & 0 & \tilde{b}_5 B_{56} & -(\tilde{b}_7 + d_7^1)B_{76} & 0 \\
0 & 0 & 0 & 0 & 0 & \tilde{b}_6 B_{67} & -d_8^1 B_{87} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{9\times 7}.$$

Now we calculate

$$\tilde{X}_2 = X_2 \begin{bmatrix} y_2 - y_0 & & & \\ & y_3 - y_1 & & \\ & & \ddots & \\ & & & y_8 - y_6 \end{bmatrix}_{7\times 7} = \tag{6.1.26}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -(\tilde{b}_2 + d_2^1)d_2^2 B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{b}_1 d_2^2 B_{12} & -(\tilde{b}_3 + d_3^1)d_3^2 B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 d_3^2 B_{23} & -(\tilde{b}_4 + d_4^1)d_4^2 B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_3 d_4^2 B_{34} & -(\tilde{b}_5 + d_5^1)d_5^2 B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_4 d_5^2 B_{45} & -(\tilde{b}_6 + d_6^1)d_6^2 B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{b}_5 d_6^2 B_{56} & -(\tilde{b}_7 + d_7^1)d_7^2 B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_6 d_7^2 B_{67} & -d_8^1 d_8^2 B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3. Find $\tilde{B}_3$ and $\tilde{X}_3$, such that

$$\mathcal{B}_3 \tilde{B}_3 + \tilde{X}_2 = \tilde{X}_3 E_3, \tag{6.1.27}$$

where

$$E_3 = \begin{bmatrix} \frac{1}{y_3 - y_0} & & & \\ & \frac{1}{y_4 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_8 - y_5} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{6\times 7}. \tag{6.1.28}$$

Assume that

$$\tilde{B}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{c}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{c}_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{c}_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{c}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{c}_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{c}_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{c}_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{9\times 7}. \tag{6.1.29}$$

We will determine $\tilde{c}_1, \ldots, \tilde{c}_7$ such that

$$(\mathcal{B}_3 \tilde{B}_3 + \tilde{X}_2)\mathbf{1}_7 = 0, \tag{6.1.30}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & B_{11} & B_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & B_{22} & B_{32} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & B_{33} & B_{43} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & B_{44} & B_{54} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{45} & B_{55} & B_{65} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{56} & B_{66} & B_{76} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{67} & B_{77} & B_{87} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \tilde{c}_3 \\ \tilde{c}_4 \\ \tilde{c}_5 \\ \tilde{c}_6 \\ \tilde{c}_7 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ (\tilde{b}_2 + d_2^1)d_2^2 B_{21} \\ -\tilde{b}_1 d_2^2 B_{12} + (\tilde{b}_3 + d_3^1)d_3^2 B_{32} \\ -\tilde{b}_2 d_3^2 B_{23} + (\tilde{b}_4 + d_4^1)d_4^2 B_{43} \\ -\tilde{b}_3 d_4^2 B_{34} + (\tilde{b}_5 + d_5^1)d_5^2 B_{54} \\ -\tilde{b}_4 d_5^2 B_{45} + (\tilde{b}_6 + d_6^1)d_6^2 B_{65} \\ -\tilde{b}_5 d_6^2 B_{56} + (\tilde{b}_7 + d_7^1)d_7^2 B_{76} \\ -\tilde{b}_6 d_7^2 B_{67} + (\tilde{b}_8 + d_8^1)d_8^2 B_{87} \\ 0 \end{bmatrix}. \tag{6.1.31}$$

The expressions of $\tilde{c}_1, \dots, \tilde{c}_7$ should be related to the quadratic Marsden's identity.

Research Problem 1

Given n sample points: $\{y_k\}_{k=0}^{n-1}$ on the interval $[a, b]$ with

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b,$$

consider the quadratic B-splines $\{B_{i,3}(x)\}_{i=0}^{n-1}$ defined on the knots $\{t_k\}_{k=-2}^{n}$ with

$$a = t_{-2} = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-3} < t_{n-2} = t_{n-1} = t_n = b.$$

Let $\mathcal{B}_3$ be the Shoenberg-Whitney matrix. How to find a band matrix $\tilde{B}_0$, such that

$$\mathcal{B}_3^{-1} \begin{bmatrix} 1 & y_0 & y_0^2 \\ \vdots & \vdots & \vdots \\ 1 & y_{n-1} & y_{n-1}^2 \end{bmatrix} = \tilde{B}_0 \begin{bmatrix} 1 & y_0 & y_0^2 \\ \vdots & \vdots & \vdots \\ 1 & y_{n-1} & y_{n-1}^2 \end{bmatrix}.$$

The reason to find this matrix $\tilde{B}_0$ is that it can help us define a *local quasi-interpolating* operator.

(*For this problem, if we can find the explicit formulas for* $\tilde{c}_1, \dots, \tilde{c}_{n-2}$ in above discussion, we can find this $\tilde{B}_0$.)

6.2 Cubic B-Spline - General Case

Next, we will do factorization on $\mathcal{B}_4$. Assume that we are given n sample points: $\{y_k\}_{k=0}^{n-1}$ spread on the interval $[a, b]$ with the following condition:

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b.$$

To do the cubic B-spline interpolation on these sample points, we construct a set of cubic B-splines $\{B_{i,4}(x)\}_{i=0}^{n-1}$ using the knots $\{t_k\}_{k=-3}^{n}$ with the following form:

$$a = t_{-3} = t_{-2} = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-4} < t_{n-3} = t_{n-2} = t_{n-1} = t_n = b,$$

where the basis function $B_{i,4}(x)$ is constructed from the knots: $\{t_{i-3}, t_{i-2}, t_{i-1}, t_i, t_{i+1}\}$ for $i = 0, \dots, n-1$. Furthermore, the Shoenberg-Whitney condition must be satisfied, that is,

$$t_{i-3} < y_i < t_{i+1}, \quad \text{for } 1 \leq i \leq n-2,$$

which implies that

$$y_{i-1} < t_i < y_{i+3}, \quad \text{for } 1 \le i \le n-4.$$

1. Find $\tilde{B}_1$ and $\tilde{X}_1$, such that

$$\mathcal{B}_4 \tilde{B}_1 - I = \tilde{X}_1 E_1, \tag{6.2.1}$$

where

$$E_1 = \begin{bmatrix} \frac{1}{y_1 - y_0} & & & \\ & \frac{1}{y_2 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_9 - y_8} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{9\times 10}. \tag{6.2.2}$$

We take $\tilde{B}_1 = I$. Denote $\mathbf{1}_{10}$ as the column matrix with all entries 1, that is,

$$\mathbf{1}_{10} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{1\times 10}. \tag{6.2.3}$$

The *partition of unit* property of the B-spline functions implies that

$$(\mathcal{B}_4 \tilde{B}_1 - I)\mathbf{1}_{10} = 0, \tag{6.2.4}$$

which allows us to do the following factorization:

$$\mathcal{B}_4 \tilde{B}_1 - I = X_1 D_1, \tag{6.2.5}$$

where

$$D_1 := \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{9\times 10}. \tag{6.2.6}$$

To find X_1 in (6.2.5), we need the pseudo-inverse of D_1, that is,

$$D_1^+ = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \ddots & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{10\times 9}. \tag{6.2.7}$$

Thus, we have

$$X_1 = (\mathcal{B}_4 \tilde{B}_1 - I) D_1^+ = \tag{6.2.8}$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_{01} & -B_{21} - B_{31} & -B_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & B_{12} & -B_{32} - B_{42} & -B_{42} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_{23} & -B_{43} - B_{53} & -B_{53} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & B_{34} & -B_{54} - B_{64} & -B_{64} & 0 & 0 & 0 \\
0 & 0 & 0 & B_{35} & B_{35} + B_{45} & -B_{65} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_{46} & B_{46} + B_{56} & -B_{76} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & B_{57} & B_{57} + B_{67} & -B_{87} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B_{68} & B_{68} + B_{78} & -B_{98} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Then we have

$$\tilde{X}_1 = X_1 \begin{bmatrix} y_1 - y_0 & & & \\ & y_2 - y_1 & & \\ & & \ddots & \\ & & & y_9 - y_8 \end{bmatrix}. \tag{6.2.9}$$

Denote

$$d_j^k := y_j - y_{j-k}. \tag{6.2.10}$$

Now we can write $\tilde{X}_1$ as follows

$$\tilde{X}_1 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_1^1 B_{01} & -d_2^1(B_{21} + B_{31}) & -d_3^1 B_{31} & 0 & 0 & 0 & 0 & 0 \\
0 & d_2^1 B_{12} & -d_3^1(B_{32} + B_{42}) & -d_4^1 B_{42} & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & d_6^1 B_{57} & d_7^1(B_{57} + B_{67}) & -d_8^1 B_{87} & 0 \\
0 & 0 & 0 & 0 & 0 & d_7^1 B_{68} & d_8^1(B_{68} + B_{78}) & -d_9^1 B_{98} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{6.2.11}$$

2. Find $\tilde{B}_2$ and $\tilde{X}_2$, such that

$$\mathcal{B}_4 \tilde{B}_2 + \tilde{X}_1 = \tilde{X}_2 E_2, \tag{6.2.12}$$

where

$$E_2 = \begin{bmatrix} \frac{1}{y_2 - y_0} & & & \\ & \frac{1}{y_3 - y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_9 - y_7} \end{bmatrix} \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{bmatrix}_{8 \times 9}. \tag{6.2.13}$$

Assume that

$$\tilde{B}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \tilde{b}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{b}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{b}_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{b}_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{b}_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{b}_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{b}_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tilde{b}_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{10\times 9}. \tag{6.2.14}$$

We will determine $\tilde{b}_1, \ldots, \tilde{b}_8$ such that

$$(\mathcal{B}_4 \tilde{B}_2 + \tilde{X}_1)\mathbf{1}_9 = 0, \tag{6.2.15}$$

that is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{01} & B_{11} & B_{21} & B_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{12} & B_{22} & B_{32} & B_{42} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{23} & B_{33} & B_{43} & B_{53} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{34} & B_{44} & B_{54} & B_{64} & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{35} & B_{45} & B_{55} & B_{65} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{46} & B_{56} & B_{66} & B_{76} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_{57} & B_{67} & B_{77} & B_{87} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & B_{68} & B_{78} & B_{88} & B_{98} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{b}_1 \\ \vdots \\ \tilde{b}_8 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -d_1^1 B_{01} + d_2^1(B_{21} + B_{31}) + d_3^1 B_{31} \\ -d_2^1 B_{12} + d_3^1(B_{32} + B_{42}) + d_4^1 B_{42} \\ \vdots \\ -d_6^1 B_{57} - d_7^1(B_{57} + B_{67}) + d_8^1 B_{87} \\ -d_7^1 B_{68} - d_8^1(B_{68} + B_{78}) + d_9^1 B_{98} \\ 0 \end{bmatrix}. \tag{6.2.16}$$

Based on our experiment, we guess that

$$\tilde{b}_1 = \frac{2a + t_1}{3} - y_1, \quad \tilde{b}_2 = \frac{a + t_1 + t_2}{3} - y_2, \quad \tilde{b}_3 = \frac{t_1 + t_2 + t_3}{3} - y_3,$$
$$\tilde{b}_4 = \frac{t_2 + t_3 + t_4}{3} - y_4, \quad \tilde{b}_5 = \frac{t_3 + t_4 + t_5}{3} - y_5, \quad \tilde{b}_6 = \frac{t_4 + t_5 + t_6}{3} - y_6, \tag{6.2.17}$$
$$\tilde{b}_7 = \frac{t_5 + t_6 + b}{3} - y_7, \quad \tilde{b}_8 = \frac{t_6 + 2b}{3} - y_8.$$

In particular, we have

$$\tilde{b}_0 = \frac{a + a + a}{3} - y_0 = 0, \quad \text{and} \quad \tilde{b}_9 = \frac{b + b + b}{3} - y_9 = 0.$$

To verify (6.2.16), we consider the following two cases:

- For $2 \leq i \leq 5$, we need to verify that

$$\tilde{b}_{i-1}B_{i-1,i}+\tilde{b}_iB_{i,i}+\tilde{b}_{i+1}B_{i+1,i}+\tilde{b}_{i+2}B_{i+2,i} = d_i^1B_{i-1,i}-d_{i+1}^1(B_{i+1,i}+B_{i+2,i})-d_{i+2}^1B_{i+2,i}, \tag{6.2.18}$$

which is equivalent to

$$(\tilde{b}_{i-1}-d_i^1)B_{i-1,i}+\tilde{b}_iB_{i,i}+(\tilde{b}_{i+1}+d_{i+1}^1)B_{i+1,i}+(\tilde{b}_{i+2}+d_{i+2}^1+d_{i+1}^1)B_{i+2,i} = 0,$$

that is,

$$\left(\frac{x_{\sigma(j)-3}+x_{\sigma(j)-2}+x_{\sigma(j)-1}}{3}-y_i\right)B_{i-1,i}+\left(\frac{x_{\sigma(j)-2}+x_{\sigma(j)-1}+x_{\sigma(j)}}{3}-y_i\right)B_{i,i}+$$

$$\left(\frac{x_{\sigma(j)-1}+x_{\sigma(j)}+x_{\sigma(j)+1}}{3}-y_i\right)B_{i+1,i}+\left(\frac{x_{\sigma(j)}+x_{\sigma(j)+1}+x_{\sigma(j)+2}}{3}-y_i\right)B_{i+2,i} = 0.$$

This formula is just the linear case of the Marsden's identity:

$$y_j = \sum_{k=0}^{3}\left(\frac{x_{\sigma(j)-3+k}+x_{\sigma(j)-2+k}+x_{\sigma(j)-1+k}}{3}-y_j\right)B_{i-1+k,4}(y_j). \tag{6.2.19}$$

Research Problem 2

Given n sample points: $\{y_k\}_{k=0}^{n-1}$ on the interval $[a,b]$ with

$$a = y_0 < y_1 < \cdots < y_{n-2} < y_{n-1} = b,$$

consider the cubic B-splines $\{B_{i,4}(x)\}_{i=0}^{n-1}$ defined on the knots $\{t_k\}_{k=-3}^{n}$ with

$$a = t_{-3} = t_{-2} = t_{-1} = t_0 < t_1 < t_2 < \cdots < t_{n-4} < t_{n-3} = t_{n-2} = t_{n-1} = t_n = b.$$

Let $\mathcal{B}_4$ be the Shoenberg-Whitney matrix. How to find a band matrix $\tilde{B}_0$, such that

$$\mathcal{B}_4^{-1}\begin{bmatrix}1 & y_0 & y_0^2 & y_0^3\\ \vdots & \vdots & \vdots & \vdots\\ 1 & y_{n-1} & y_{n-1}^2 & y_{n-1}^3\end{bmatrix} = \tilde{B}_0\begin{bmatrix}1 & y_0 & y_0^2 & y_0^3\\ \vdots & \vdots & \vdots & \vdots\\ 1 & y_{n-1} & y_{n-1}^2 & y_{n-1}^3\end{bmatrix}.$$

The reason to find this matrix $\tilde{B}_0$ is that it can help us define a *local quasi-interpolating* operator.

For the general factorization, we would like to have the following results.

Proposition 6.2.1. *Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a,b]$ with the condition: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions $\{B_{i,m}(x)\}_{i=0}^{n-1}$ on the interval $[a,b]$ defined on the knots $\vec{t} := \{t_i\}_{-m+1}^{n}$. Assume that $\vec{t}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{t}$ and $\vec{y}$. Show that for any $n \times (n-r)$* zero-row-ending *matrix $\tilde{X}_r$ with $0 \leq r \leq m-1$, there exists an $n \times (n-r)$ band matrix $\tilde{B}_{r+1}$ of bandwidth up to $\max(2, r-1)$, such that*

$$B_m \tilde{B}_{r+1} + \tilde{X}_r = \tilde{X}_{r+1} E_{r+1} \tag{6.2.20}$$

for some $n \times (n-r-1)$ zero-row-ending *matrix $\tilde{X}_{r+1}$.*

Proposition 6.2.2. *., Given n data samples $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a,b]$ with the condition: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions on the interval $[a,b]$ defined on the knots $\vec{x} := \{t_i\}_{-m+1}^{n}$. Assume that $\vec{x}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{x}$ and $\vec{y}$. Show that there exists an $n \times n$ band matrix $\tilde{B}_0$ with bandwidth 5, such that*

$$B_m \tilde{B}_0 - I = \tilde{X} E_4 E_3 E_2 E_1, \quad (m \geq 4) \tag{6.2.21}$$

for some $n \times (n-4)$ matrix $\tilde{X}$.

Proposition 6.2.3. *Given n data points $\vec{y} := \{y_i\}_0^{n-1}$ on the interval $[a,b]$ with the condition: $y_0 = a$ and $y_{n-1} = b$, we consider the m-th order B-splines with n basis functions $\{B_{i,m}(x)\}_{i=0}^{n-1}$ on the interval $[a,b]$ defined on the knots $\vec{t} := \{t_i\}_{-m+1}^{n}$ in the form of*

$$\{\underbrace{a, \cdots, a}_{m}, t_1, t_2, \cdots, t_{n-m}, \underbrace{b, \cdots, b}_{m}\}$$

satisfying

$$a < t_1 < t_2 < \cdots < t_{n-m} < b.$$

Here we take

$$t_{-m+1} = \cdots = t_0 = a \quad \text{and} \quad t_{n-m+1} = \cdots = t_n = b.$$

Assume that $\vec{t}$ and $\vec{y}$ satisfy the Shoenberg-Whitney condition. Denote B_m as the Shoenberg-Whitney matrix with respect to $\vec{t}$ and $\vec{y}$. Show that there exists an $n \times n$ band matrix $\tilde{B}_0$ with bandwidth at most $\max(4, 2r-2)$ for $1 \leq r \leq m$, such that

$$B_m \tilde{B}_0 - I = \tilde{X} E_r \cdots E_1, \quad (1 \leq r \leq m) \tag{6.2.22}$$

for some $n \times (n-r)$ matrix $\tilde{X}$. Furthermore, the computation complexity for $\tilde{B}_0$ is in $O(n)$.

Proposition 6.2.4. *With the same condition as above for $m \geq 4$, show that there exists an $n \times n$ banded matrix $\tilde{B}_0$ with bandwidth 5, such that*

$$\tilde{B}_0 \begin{bmatrix} 1 & y_0 & y_0^2 & y_0^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & y_{n-1} & y_{n-1}^2 & y_{n-1}^3 \end{bmatrix} = \begin{bmatrix} 1 & \rho_{0,m}^1 & \rho_{0,m}^2 & \rho_{0,m}^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \rho_{n-1,m}^1 & \rho_{n-1,m}^2 & \rho_{n-1,m}^3 \end{bmatrix}. \tag{6.2.23}$$

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