Construct Linear Quasi-Interpolants on Infinite Intervals

Using Linear B-Splines

Johara Farah Albaliwi

M.S. in Mathematics, University of Missouri-St.Louis, 2018
M.S. in Mathematics with a concentration in Teaching, University of Incarnate Word, 2015
B.S. in Mathematics, University of Tabuk,2011

A dissertation submitted to the Graduate School of the
University of Missouri-St. Louis
in partial fulfillment of the requirements for the degree
Doctor of Philosophy in Mathematical and Computational Sciences
with an emphasis in Mathematics.

May 2021

Advisory Committee:
Wenjie He, Ph.D.
(Chairperson)
Qingtang Jiang, Ph.D.
Adrian Clingher, Ph.D.
Haiyan Cai, Ph.D.

Abstract

In solving the *data interpolation problem*, which is fundamental in data analysis, we typically deal with the data samples spread in a finite interval $[a, b]$, which results in the operations involving finite-dimensional matrices. There are many interesting results developed under this framework. However, when the data samples are given from an infinite interval $[a, \infty)$ (for certain special types of real-world applications), many existing results would not work anymore due to the special properties of the *infinite data samples*. A new framework should be established to support the infinite data samples.

In this dissertation, we develop a special tool called *local linear quasi-interpolant* for an infinite interval with the following properties: 1) Each linear functional of the quasi-interpolant is determined by at most three data samples, so that the spline coefficients can be calculated in real-time; 2) The quasi-interpolant preserves all the linear polynomials; 3) Our framework does not impose any restriction on the relationship between the sample locations and the spline knots, which provides us the necessary flexibility in the real-world applications.

Our construction is based on a matrix factorization method with respect to infinite-dimensional matrices. In order to ensure that the infinite version of the Shoenberg-Whitney matrices are invertible, we take the constructive approach that results in both the left-inverses and the right-inverses. Furthermore, since the *associative law* of the matrix multiplication does not work for the infinite matrices, we verify all the formulas derived from the infinite matrix operations. Finally, our local method allows us to calculate the spline interpolating coefficients in real-time on the fly for the infinite data samples.

Key Words:B-spline/ Reproduction and Marsden's Identity/ Shoenberg-Whitney/ Quasi interpolation/ Coefficients of the Marsden's Identities.

Acknowledgments

Throughout the writing of this dissertation I have received a great deal of support and assistance.

I would like to acknowledge my indebtedness and render my warmest thanks to my advisor, Professor Winjie He, who expertise was invaluable in formulating the research questions and methodology. Thank you for helping me broaden my knowledge of the area of specialization and research. His friendly guidance and expert advice have been invaluable throughout all stages of the work. Without his guidance and constant feedback this PhD would not have been achievable.

Then I express my sincere thanks and praise for my advisory committee, Professors : Qingtang Jiang , Adrian Clingher, Haiyan Cai, , for their time and their attendance at the dissertation discussion.

I would also wish to express my gratitude to my government for their support in completing my studies and their cost of all tuition and medical insurance.

Special thanks are due to my dear father and my loving mother, for their continuous support and constant encouragement. They spared no effort to help and encourage me in my scientific and research journey, and assumed many responsibilities for the sake of achieving my ambition. I am forever indebted to my parents for giving me the opportunities and experiences that have made me who I am. They selflessly encouraged me to explore new directions in life and seek my own destiny. This journey would not have been possible if not for them, and I dedicate this milestone to them.

Finally, I thank my sisters and brothers who offered invaluable support over the years, I am grateful to them for always being there for me as a friends. I thank them for their support and constant question. I am proud of them and their presence in my life.

Thanks again to everyone who made this dissertation and me possible.

Sincerely,

Johara Albaliwi

Contents

1 *Introduction* **4**

2 *Preliminaries* **6**

2.1 *B-splines* . . . 6

2.2 *Data Interpolation* . . . 7

2.3 *Local Quasi-Interpolation Operator* . . . 8

2.4 *Three-Point Quasi-Interpolant* . . . 9

2.5 *General Expression of* $\mathcal{B}_2$ . . . 10

2.6 *Marsden's Identity* . . . 12

3 *Construction of Local Quasi-Interpolation Operator* **14**

3.1 *Matrix Criterion for Polynomial Preservation* . . . 14

3.2 *Matrix Factorization Procedure:* . . . 17

3.3 *General 3-Point Solutions* . . . 27

3.4 *3-Point Solutions for Digital Filters* . . . 27

4 *Shoenberg-Whitney Matrix:* **31**

4.1 *Properties of infinite matrices* . . . 31

4.2 General representation for linear Shoenberg-Whitney matrix . . . 33

5 *Inverses of linear Shoenberg-Whitney matrices:* **38**

5.1 *Inverse Shoenberg-Whitney Matrix Case I:* . . . 38

5.2 *Inverse Shoenberg-Whitney Matrix Case II:* . . . 45

5.3 *Inverse Shoenberg-Whitney Matrix Case III:* . . . 64

5.4 *Inverse Shoenberg-Whitney Matrix Case IV:* . . . 66

5.5 *Inverse Shoenberg-Whitney Matrix Case V:* . . . 70

6 *Conclusion and Future Research* **72**

Chapter 1

Introduction

In data analysis, *interpolation* is a fundamental technique to represent given data using functions with "good" properties. Interpolation is a type of estimation, a method of constructing new data points within the range of a discrete set of known data points. It is the process of deriving a simple function from a set of discrete data points so that the function passes through all the given data points. The spline interpolation is the most commonly used method due to the excellent properties of the B-splines. The *quasi-interpolation* is a very useful concept in that it emphasizes *polynomial preservation* aspect of data approximation. Many applications rely on *quasi-interpolation* to represent data samples. After that, we can apply many powerful mathematical transforms to process data.

In this dissertation, we will construct *local quasi-interpolants* for the *linear B-splines* on *unbounded intervals*. The current research results in this field are basically established on the special settings for the B-spline knots and data sampling locations. We will consder the more general setting for which only the *Shoenberg-Whitney condition* is needed, so that it can provide the *flexibility* for many real-world applications. The linear operators associated the quasi-interpolants are required to be *local*, which means that each coefficient of the spline representation only needs $O(1)$ computation, so that it supports the *real-time* data processing.

We focus our constructions on the interval $[0, \infty)$, which could be useful for some special types of applications. To this end, we need to deal with operations of the *infinite matrices* which have different properties from their finite conterpart. More specifically, we need to use the inverses of the *Shoenberg-Whitney matrices* for the linear B-splines on $[0, \infty)$. We need to overcome many technical difficulties to find those inverse matrices. To calculate the infinite inverses of the *Shoenberg-Whitney matrices*, we divide them into 5 categories based on their structures.

Another important part of this dissertation is developing a special matrix factorization technique that is used to explore the structures of the inverse of the *Shoenberg-Whitney matrices*. Since the inverse matrices of the *Shoenberg-Whitney matrices* could be very complicated in general, we have to use an indirect way to avoid working on those inverse matrices. The *polynomial preservation* property for the quasi-interpolants can be represented in some matrix conditions based on the coefficients of the *Marsden's identity*. This technique could be generalized to more general B-splines.

Finally, we have a good chance to extend our method to other basis functions, for example, many *refinable functions*, because they have their versions of the *"Marsden's identitties"*, and our method is based on the matrix factorization on the matrices related to the Marsden's coefficients. Another interesting generalization is on 2-d basis functions, such as *box-splines*.

Chapter 2

Preliminaries

2.1 *B-splines*

Setting for the interval $[0, \infty)$

We consider a set of B-spline basis functions of order m with $m > 0$ as $\{B_{i,m}(x)\}_{i=0}^{\infty}$ on the interval $[0, \infty)$ that are defined on the knot sequence $\{t_j\}_{j=-m+1}^{\infty}$. The knot sequence satisfy the following conditions:

$$0 = t_{-m+1} = \cdots = t_0 < t_1 \leq t_2 \leq \cdots \leq t_n < \infty, \tag{2.1.1}$$

and

$$t_{j-m} < t_j, \qquad \text{for} \quad j = 1, \cdots, n, \cdots . \tag{2.1.2}$$

Definition 2.1.1. *the B-splines is given by the following recurrence relation:*

$$B_{i,m}(x) = \frac{x - t_{i-m+1}}{t_i - t_{i-m+1}} B_{i-1,m-1}(x) + \frac{t_{i+1} - x}{t_{i+1} - t_{i-m+2}} B_{i,m-1}(x) \tag{2.1.3}$$

for $x \in [0, \infty)$*, with*

$$B_{i,1}(x) = \begin{cases} 1, & \text{if } t_i \leq x < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases}$$

One can see that $B_{i,m}(x)$ relies on the $m+1$ knots: $\{t_{i-m+1}, \ldots, t_i, t_{i+1}\}$, that is, the subscript i of the basis function $B_{i,m}(x)$ corresponds to the *rightmost inner* knot t_i, not the rightmost knot t_{i+1}. In this way, we can make the subscript i running through all nonnegative integers.

To do data interpolation, we are given a sequence of n sample locations $\{x_i\}_{i=0}^{\infty}$ that satisfy the following condition:

$$0 = x_0 < x_1 < x_2 < \cdots < x_n < \infty. \tag{2.1.4}$$

Eventually, the data interpolation will occur on these sample points: $\{(x_i, y_i)\}_{i=0}^{\infty}$. We observe some obvious properties of the sample locations:

- The left boundary sample location is x_0 that matches the left endpoint of the interval 0;
- The inner sample locations are $\{x_1, x_2, \ldots, x_n, \ldots, \}$, and they are distinct.

Similarly, we can get the basic properties for the knot squence $\{t_j\}_{j=-m+1}^{\infty}$ as follows:

- The left boundry knot is t_0 that matches the left endpoint of the interval 0;
- The left boundary knot is repeated m times, that is,

$$0 = t_{-m+1} = \cdots = t_0;$$

- The inner knots $\{t_1, t_2, \ldots, t_n, \ldots\}$ are not necessarily distinct,

$$t_1 \leq t_2 \leq \cdots \leq t_n < \infty,$$

with

$$t_{j-m} < t_j, \qquad \text{for} \quad j = 1, \cdots, n, \cdots .$$

2.2 *Data Interpolation*

We consider the spline space $S_{m,t}$ defined as follows

$$S_{m,t} = \left\{ \sum_{j=0}^{\infty} c_j B_{j,m}(x) \,|\, c_j \in \mathbb{R} \text{ for } 0 \leq j < \infty \right\}. \tag{2.2.1}$$

The space $S_{m,t}$ is a linear space, and we will use it to approximate the continuous functions in $C[0, \infty)$. To find a spline function $f(x)$ in $S_{m,t}$ with the form of

$$f(x) = \sum_{j=0}^{\infty} c_j B_{j,m}(x),$$

that satisfies the interpolation condition:

$$f(x_i) = y_i, \qquad \text{for} \quad i = 0, 1, \ldots, n, \ldots, \tag{2.2.2}$$

we need to find the coefficients $c_0, c_1, \ldots, c_n, \ldots$, such that

$$\begin{bmatrix} B_{0,m}(x_0) & \cdots & B_{n,m}(x_0) & \cdots \\ \vdots & \ddots & \vdots & \\ B_{0,m}(x_n) & \cdots & B_{n,m}(x_n) & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ \vdots \end{bmatrix}. \tag{2.2.3}$$

Denote

$$\mathcal{B}_m := \begin{bmatrix} B_{0,m}(x_0) & \cdots & B_{n,m}(x_0) & \cdots \\ \vdots & \ddots & \vdots & \\ B_{0,m}(x_n) & \cdots & B_{n,m}(x_n) & \cdots \\ \vdots & & \vdots & \ddots \end{bmatrix}. \tag{2.2.4}$$

Based on the finite interval Shoenberg-Whitney theorem, we have the following condition between knots and samples

$$t_{i-m+1} < x_i < t_{i+1}, \qquad \text{for all} \quad i = 1, 2, \ldots, n, \ldots. \tag{2.2.5}$$

For the linear case, i.e. $m = 2$, it becomes

$$t_{i-1} < x_i < t_{i+1}, \qquad \text{for all} \quad i = 1, 2, \ldots, n, \ldots. \tag{2.2.6}$$

Equivalently, we can also write as

$$x_{i-1} < t_i < x_{i+1}, \qquad \text{for all} \quad i = 1, 2, \ldots, n, \ldots. \tag{2.2.7}$$

In our discussion below, we assume that $\mathcal{B}_2$ is invertible, which will be shown in Chapter 5. Thus, we have

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \\ \vdots \end{bmatrix} = \mathcal{B}_2^{-1} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \\ \vdots \end{bmatrix}. \tag{2.2.8}$$

2.3 *Local Quasi-Interpolation Operator*

In order to define a quasi-interpolant on $C[0, \infty)$, we need to construct a sequence of linear functionals $\{\lambda_k\}_{k=0}^{\infty}$ from $C[0, \infty)$ to $\mathbb{R}$ as follows, for any $f(x) \in C[0, \infty)$, we define

$$\lambda_k f = \sum_{i=0}^{\infty} q_{ki} f(x_i), \qquad \text{for} \quad 0 \le k < \infty, \tag{2.3.1}$$

with certain real coefficients $q_{k0}, q_{k1}, \ldots, q_{k,n}, \ldots$. With these linear functionals, we can define a quasi-interpolation operator from $C[0,\infty)$ to $S_{m,t}$ as follows

$$(Qf)(x) = \sum_{j=0}^{\infty} (\lambda_j f) B_{j,m}(x). \tag{2.3.2}$$

In particular, a quasi-interpolant should have the *polynomial preservation* property. That is, for any polynomial $p(x)$ of degree up to $m-1$, denoted by $p(x) \in \pi_{m-1}$, we have

$$(Qp)(x) = p(x). \tag{2.3.3}$$

We are more interested in the *local* quasi-interpolants, that is, for each linear functional λ_k with $0 \le k < \infty$, the nonzero coefficients can only appear in the sequence $\{q_{k,k-a}, \ldots, q_{kk}, \ldots, q_{k,k+b}\}$ for some fixed non-negative integers a and b. Of course, we need to exclude those coefficients whose subscripts are out of bounds of $\{0, 1, \ldots, \}$.

2.4 *Three-Point Quasi-Interpolant*

Our goal for constructing a local quasi-interpolant is: Find a tridiagonal matrix $\mathcal{Q}_2$ in the form of

$$\mathcal{Q}_2 := \begin{bmatrix} q_{00} & q_{01} & 0 & \cdots & \cdots & \cdots \\ q_{10} & q_{11} & q_{12} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & q_{n,n-1} & q_{n,n} & q_{n,n+1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{2.4.1}$$

which defines the linear functionals $\{\lambda_k\}_{k=0}^{\infty}$ in two parts:

- For the internal linear functionals with $1 \le k < \infty$, we have

$$\lambda_k f = q_{k,k-1} f(x_{k-1}) + q_{k,k} f(x_k) + q_{k,k+1} f(x_{k+1}). \tag{2.4.2}$$

- For the boundary linear functionals, we have

$$\lambda_0 f = q_{00} f(x_0) + q_{01} f(x_1). \tag{2.4.3}$$

Then the quasi-interpolation operator Qf as in (2.3.2) for $m = 2$ given by

$$(Qf)(x) = \sum_{j=0}^{\infty} (\lambda_j f) B_{j,2}(x) \tag{2.4.4}$$

preserves the linear polynomials as follows

$$(Qp)(x) = p(x), \qquad \text{for all} \quad p(x) \in \pi_1. \tag{2.4.5}$$

In other words, we will find the formulas for the entries of $\mathcal{Q}_2$: q_{kj} in terms of the knots $\{t_i\}_{i=-1}^{\infty}$ and the samples $\{x_i\}_{i=0}^{\infty}$. The definition of matrix $\mathcal{Q}_2$ can be easily generalized to that of $\mathcal{Q}_m$ for m-th order B-splines.

2.5 *General Expression of* $\mathcal{B}_2$

A straightforward application of the recurrence relation (2.1.3) gives us the explicit formula for $B_{i,2}(x)$ as follows

$$\begin{aligned} B_{i,2}(x) &= \frac{x - t_{i-1}}{t_i - t_{i-1}} B_{i-1,1}(x) + \frac{t_{i+1} - x}{t_{i+1} - t_i} B_{i,1}(x) \\ &= \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}}, & \text{if} \quad t_{i-1} \le x < t_i, \\ \dfrac{t_{i+1} - x}{t_{i+1} - t_i}, & \text{if} \quad t_i \le x < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5.1}$$

To write the explicit formula for $\mathcal{B}_2$, we consider the $(k+1)$-th row of the infinite matrix $\mathcal{B}_2$ for $0 \le k < \infty$, which has the form of

$$\begin{bmatrix} B_{0,2}(x_k) & B_{1,2}(x_k) & \cdots & B_{k,2}(x_k) & \cdots \end{bmatrix}. \tag{2.5.2}$$

We still need to resolve the *uncertainty* about the sample locations $\{x_k\}_{k=0}^{\infty}$ with respect to the knots $\{t_k\}_{k=-1}^{\infty}$. Specifically, for each internal x_k with $k = 1, 2, \ldots,$, we have that either $x_k \in (t_{k-1}, t_k]$ or $x_k \in (t_k, t_{k+1})$ by condition (2.2.6). To examine what entries in the above row are nonzero, we consider the following two cases:

- When $x_k \in (t_{k-1}, t_k]$, notice that the support of $B_{i,2}(x)$ is $[t_{i-1}, t_{i+1}]$. We can see that only the supports of $B_{k-1,2}(x)$ and $B_{k,2}(x)$ have an overlap with $(t_{k-1}, t_k]$. Thus, we can write the above row as

$$\Big[\underbrace{0 \ \cdots \ 0}_{k-1} \quad B_{k-1,2}(x_k) \quad B_{k,2}(x_k) \quad 0 \quad \cdots \Big].$$

- When $x_k \in (t_k, t_{k+1})$, with similar analysis, we can write (4.2.2) as

$$\Big[\underbrace{0 \ \cdots \ 0}_{k} \quad B_{k,2}(x_k) \quad B_{k+1,2}(x_k) \quad 0 \quad \cdots \Big].$$

If we combine the above two cases, we can say that the only possible nonzero entries of the $(k+1)$-th row are: $B_{k-1,2}(x_k), B_{k,2}(x_k)$, and $B_{k+1,2}(x_k)$, and we can write the general form of the row as

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{k-1} & B_{k-1,2}(x_k) & B_{k,2}(x_k) & B_{k+1,2}(x_k) & 0 & \cdots \end{bmatrix}.$$

In order to use a more compact way to write the matrix $\mathcal{B}_2$, we use the notation $b_{ij} := B_{i,2}(x_j)$. Thus, we have

$$\mathcal{B}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} & b_{n+1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.5.3}$$

We need to address the uncertainty that which of the intervals $(t_{k-1}, t_k]$ and (t_k, t_{k+1}) contains x_k. To this end, we introduce a set of indicator variables $\{\sigma_i\}_{i=1}^{\infty}$ as follows,

$$\sigma_i = \begin{cases} 1 & \text{if} \quad x_i \in (t_{i-1}, t_i) \\ 0 & \text{if} \quad x_i \in [t_i, t_{i+1}). \end{cases} \tag{2.5.4}$$

Now we can write explicit expressions for $B_{i-1,2}(x_i), B_{i,2}(x_i), B_{i+1,2}(x_i)$ as follows,

$$\begin{cases} B_{i-1,2}(x_i) = \sigma_i \dfrac{t_i - x_i}{t_i - t_{i-1}}, \\ B_{i,2}(x_i) = \sigma_i \dfrac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i) \dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(x_i) = (1-\sigma_i) \dfrac{x_i - t_i}{t_{i+1} - t_i} \end{cases} \tag{2.5.5}$$

for $i = 1, 2, \ldots$. With this new notation, we can verify the *partition of unity* property of the linear B-splines easily as follows,

$$B_{i-1,2}(x_i) + B_{i,2}(x_i) + B_{i+1,2}(x_i) = \sigma_i \frac{t_i - x_i}{t_i - t_{i-1}} + \left(\sigma_i \frac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i) \frac{t_{i+1} - x_i}{t_{i+1} - t_i} \right)$$

$$+(1-\sigma_i) \frac{x_i - t_i}{t_{i+1} - t_i} = \sigma_i \left(\frac{t_i - x_i}{t_i - t_{i-1}} + \frac{x_i - t_{i-1}}{t_i - t_{i-1}} \right) + (1-\sigma_i) \left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i} + \frac{x_i - t_i}{t_{i+1} - t_i} \right) = 1$$

for $i = 1, 2, \ldots$. The *partition of unity* property of the linear B-splines can also be written as a matrix form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} & b_{n+1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix}, \tag{2.5.6}$$

which will be a critical property in our matrix factorization technique.

2.6 *Marsden's Identity*

In 1970, Marsden expressed in [1] $(\cdot - y)^\alpha$ in terms of a linear combination of B-splines, which is called Marsden's identity. This identity plays an important role in change of basis procedures and B-spline curve approximation. Moreover this identity is deeply studied and extended in various setting by many researchers. Denote $\{\rho^r_{k,m}, r = 0, 1, ..., m-1\}$ as the coefficients of the Marsden's Identities given by

$$\rho^0_{k,m} = 1,$$

$$\rho^r_{k,m} = \frac{1}{\binom{m-1}{r}} \sum_{k+1 \le j_1 < j_2 < ... < j_r \le k+m-1} t_{j_1} t_{j_2} \dots t_{j_r}, \quad 1 \le r \le m-1$$

for $k \in \mathbb{Z}$. We use the Marsden's identity to develop a matrix version criterion for the polynomial preservation property.

When we use the B-splines to represent the polynomials, we need the Marsden's identity, which rely on the following dual polynomial corresponding to $B_{j,m}(x)$ using its internal knots $\{t_{j-m+2}, t_{j-m+3}, \dots, t_j\}$ as follows

$$\begin{cases} \rho_{j,1}(y) = 1 \\ \rho_{j,m}(y) = (y - t_{j-m+2})(y - t_{j-m+3}) \dots (y - t_j), \quad m \ge 2. \end{cases} \tag{2.6.1}$$

Then the Marsden's identity tells that

$$(y - x)^{m-1} = \sum_{j=0}^{\infty} \rho_{j,m}(y) B_{j,m}(x), \quad \text{for } x \in [0, \infty). \tag{2.6.2}$$

When we view the above identity, we fix x and treat y as a variable. Then we can expand both sides of (2.6.2) as polynomials of y and compare their coefficients, which results in the following sequence of identities: For $r = 0, 1, 2, ...m-1$, we have

$$x^r = \sum_{j=0}^{\infty} \rho^r_{j,m} B_{j,m}(x), \qquad \text{for } x \in [0, \infty), \tag{2.6.3}$$

where $\rho^r_{j,m}$ are the symmetric polynomials given by:

$$\rho^r_{j,m} = \frac{1}{\binom{m-1}{r}} \sum t_{j_1} t_{j_2} \dots t_{j_r} \tag{2.6.4}$$

and the sum is over all integers $j_1, j_2, \dots, j_r$ such that

$$j - m + 2 \le j_1 < \dots < j_r \le j$$

and the total number of terms is $\binom{m-1}{r}$. When $r = 0, 1, 2$, we can write (2.6.3) in the following explicit and convenient forms:

$$
\begin{aligned}
1 &= \sum_{j=0}^{\infty} B_{j,m}(x), \quad \text{for} \quad m \geq 1, \\
x &= \sum_{j=0}^{\infty} t^{*}_{j,m} B_{j,m}(x), \quad \text{for} \quad m \geq 2, \\
x^2 &= \sum_{j=0}^{\infty} t^{**}_{j,m} B_{j,m}(x), \quad \text{for} \quad m \geq 3
\end{aligned} \tag{2.6.5}
$$

on the interval $[0, \infty)$ where

$$
\begin{aligned}
t^{*}_{j,m} &= \frac{1}{m-1}(t_{j-m+2} + \ldots + t_j), \\
t^{**}_{j,m} &= \frac{1}{\binom{m-1}{2}} \sum_{i=j-m+2}^{j-1} \sum_{k=i+1}^{j} t_i t_k.
\end{aligned}
$$

Chapter 3

Construction of Local Quasi-Interpolation Operator

3.1 *Matrix Criterion for Polynomial Preservation*

When we apply (2.6.3) on all the data samples $\{x_0, x_1, \ldots, x_n, \ldots\}$, we get

$$x_j^r = \sum_{j=0}^{\infty} \rho_{i,m}^r B_{i,m}(x_j), \qquad \text{for } j = 0, 1, \ldots, n, \ldots, \tag{3.1.1}$$

which can be written in a matrix form of

$$\begin{bmatrix} x_0^r \\ x_1^r \\ \vdots \\ x_n^r \\ \vdots \end{bmatrix} = [B_{i,m}(x_j)] \begin{bmatrix} \rho_{0,m}^r \\ \rho_{1,m}^r \\ \vdots \\ \rho_{n,m}^r \\ \vdots \end{bmatrix}. \tag{3.1.2}$$

If we combine m equations of (3.1.2) for $r = 0, 1, \ldots, m-1$, we get the following matrix equation

$$\begin{bmatrix} 1 & x_0 & \cdots & x_0^{m-1} \\ 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \mathcal{B}_m \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^{m-1} \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \rho_{n,m}^1 & \cdots & \rho_{n,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}. \tag{3.1.3}$$

Assume that $\mathcal{B}_m$ is invertible (*use the generalization of the Shoenberg-Whitney theorem to be proved*), we can write (3.1.3) to the following equivalent form

$$\mathcal{B}_m^{-1} \begin{bmatrix} 1 & x_0 & \cdots & x_0^{m-1} \\ 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^{m-1} \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \rho_{n,m}^1 & \cdots & \rho_{n,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}. \tag{3.1.4}$$

To ensure that our quasi-interpolant has the polynomial preservation property, (here we can consider the general m case, not necessarily restricted to $m = 2$), we need our quasi-interpolantion operator Qf defined as in (2.4.4) to have the following property:

$$(Qp)(x) = p(x), \qquad \text{for all} \quad p(x) \in \pi_{m-1}. \tag{3.1.5}$$

Since $\{1, x, x^2, \ldots, x^{m-1}\}$ form a basis for the linear space π_{m-1}, we can use an equivalent form of (3.1.5) as follows

$$(Q\eta^r)(x) = \eta^r(x), \qquad \text{for} \quad r = 0, 1, \ldots, m-1, \tag{3.1.6}$$

where $\eta^r(x)$ is the monomial x^r. (The reason that we use a new symbol η for the monomials is that we want to avoid the confusion when we apply the quasi-interpolation operator Q on the monomials.) We apply (2.4.4) on the monomials $\{\eta^r(x)\}_{r=0}^{m-1}$, and get

$$(Q\eta^r)(x) = \textstyle\sum_{j=0}^{\infty}(\lambda_j\eta^r)B_{j,m}(x), \quad \text{for } r = 0, 1, \ldots, m-1. \tag{3.1.7}$$

On the other hand, the Marsden's identity (2.6.3) can be represented as

$$\eta^r(x) = \sum_{j=0}^{\infty} \rho_{j,m}^r B_{j,m}(x), \qquad \text{for } x \in [0, \infty). \tag{3.1.8}$$

From (3.1.6), (3.1.7), and (3.1.8), we get

$$\lambda_j\eta^r = \rho_{j,m}^r, \quad r = 0, 1, \ldots, m-1, \ j = 0, 1, \ldots, n, \ldots. \tag{3.1.9}$$

Let us look at the expressions of $(\lambda_j\eta^r)$'s with respect to the samples $\{x_i\}_{i=0}^{\infty}$. By the definition of the linear functionals $\{\lambda_k\}_{k=0}^{\infty}$ in (2.3.1), we have

$$\lambda_k\eta^r = \sum_{i=0}^{\infty} q_{ki}\, \eta^r(x_i), \qquad \text{for} \quad 0 \le k < \infty. \tag{3.1.10}$$

Since $\eta^r(x_i) = x_i^r$, we can write (3.1.10) in the form of

$$\lambda_k \eta^r = \begin{bmatrix} q_{k0} & q_{k1} & \cdots & q_{k,n} & \cdots \end{bmatrix} \begin{bmatrix} x_0^r \\ x_1^r \\ \vdots \\ x_n^r \\ \vdots \end{bmatrix}. \tag{3.1.11}$$

Replacing $\lambda_k \eta^r$ by $\rho_{k,m}^r$ in (3.1.11) based on (3.1.9), we simply get

$$\begin{bmatrix} q_{k0} & q_{k1} & \cdots & q_{k,n} & \cdots \end{bmatrix} \begin{bmatrix} x_0^r \\ x_1^r \\ \vdots \\ x_n^r \\ \vdots \end{bmatrix} = \rho_{k,m}^r. \tag{3.1.12}$$

Combining all the equations of (3.1.12) for $k = 0, 1, \ldots, n, \ldots$, we obtain

$$\mathcal{Q}_m \begin{bmatrix} x_0^r \\ x_1^r \\ \vdots \\ x_n^r \\ \vdots \end{bmatrix} = \begin{bmatrix} \rho_{0,m}^r \\ \rho_{1,m}^r \\ \vdots \\ \rho_{n,m}^r \\ \vdots \end{bmatrix}, \tag{3.1.13}$$

where

$$\mathcal{Q}_m := [q_{kj}]_{k=0,j=0}^{\infty,\infty}.$$

With the above investigation, we summarize our findings as the following theorem.

Lemma: *Given a set of data samples* $\{x_i\}_{i=0}^{\infty}$ *that satisfy the conditions* (2.1.4) *and* (2.2.6) *with respect to the B-spline space* $S_{m,t}$ *as in* (2.2.1), *let* $\mathcal{Q}_m$ *be an infinite matrix, that defines a linear operator* Qf *from* $C[0,\infty)$ *to* $S_{m,t}$ *as in* (2.4.4). *If* $\mathcal{Q}_m$ *satisfies the equation:*

$$\mathcal{Q}_m \begin{bmatrix} 1 & x_0 & \cdots & x_0^{m-1} \\ 1 & x_1 & \cdots & x_1^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_n & \cdots & x_n^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & \rho_{0,m}^1 & \cdots & \rho_{0,m}^{m-1} \\ 1 & \rho_{1,m}^1 & \cdots & \rho_{1,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \rho_{n,m}^1 & \cdots & \rho_{n,m}^{m-1} \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}, \tag{3.1.14}$$

where $\rho_{0,m}^r, \ldots, \rho_{n,m}^r, \ldots$ *are the Marsden's coefficients defined as in* (2.6.4), *then the* Q *operator preserves the polynomials in* π_{m-1}, *that is,*

$$(Qp)(x) = p(x), \quad \textit{for all} \quad p \in \pi_{m-1}([0,\infty)).$$

With this criterion, we can convert our construction of local linear quasi-interpolant problem to a linear algebra problem: Find an infinite tridiagonal matrix $\mathcal{Q}_2$ in the form of (2.4.1), such that

$$\mathcal{Q}_2 \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & \rho^1_{0,2} \\ 1 & \rho^1_{1,2} \\ \vdots & \vdots \\ 1 & \rho^1_{n,2} \\ \vdots & \vdots \end{bmatrix}, \tag{3.1.15}$$

for the data samples $\{x_i\}_{i=0}^{\infty}$ and the linear B-spline space $S_{2,t}$. In this way, we can apply certain linear algebra techniques to solve this problem.

3.2 *Matrix Factorization Procedure:*

To find an infinite tridiagonal matrix $\mathcal{Q}_2$ that satisfies (3.1.15), we would like to convert it to a matrix factorization problem. First we rewrite equation (3.1.4) for $m = 2$ case:

$$\mathcal{B}_2^{-1} \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 1 & \rho^1_{0,2} \\ 1 & \rho^1_{1,2} \\ \vdots & \vdots \\ 1 & \rho^1_{n,2} \\ \vdots & \vdots \end{bmatrix}. \tag{3.2.1}$$

Combining (3.1.15) and (3.2.1), we get

$$\mathcal{Q}_2 \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \\ \vdots & \vdots \end{bmatrix} = \mathcal{B}_2^{-1} \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \\ \vdots & \vdots \end{bmatrix},$$

which is equivalent to

$$(\mathcal{B}_2\mathcal{Q}_2 - I) \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \\ \vdots & \vdots \end{bmatrix} = 0. \tag{3.2.2}$$

An obvious advantage of the form of (3.2.2) comparing with the one in (3.1.15) is that we do not need to deal with the Marsden's coefficients $\rho^1_{j,2}$'s directly, which reduces the complexity of the problem significantly.

To solve equation (3.2.2), we split it into two equations, so that we can apply the *divide-and-conquer* approach to solve it:

$$(\mathcal{B}_2\mathcal{Q}_2 - I)\begin{bmatrix}1\\1\\\vdots\\1\\\vdots\end{bmatrix} = 0, \tag{3.2.3}$$

and

$$(\mathcal{B}_2\mathcal{Q}_2 - I)\begin{bmatrix}x_0\\x_1\\\vdots\\x_n\\\vdots\end{bmatrix} = 0. \tag{3.2.4}$$

By the *partition of unity* for linear B-splines as in (4.2.6), we can write it as the following form

$$(\mathcal{B}_2 - I)\begin{bmatrix}1\\1\\\vdots\\1\\\vdots\end{bmatrix} = 0. \tag{3.2.5}$$

Comparing the structures of equations (3.2.3) and (3.2.5), we write $\mathcal{Q}_2$ in the forllowing form

$$\mathcal{Q}_2 = I + T_0, \tag{3.2.6}$$

where T_0 is also a tridiagonal matrix to be determined. Thus, (3.2.3) is equivalent to

$$\mathcal{B}_2 T_0\begin{bmatrix}1\\1\\\vdots\\1\\\vdots\end{bmatrix} = 0. \tag{3.2.7}$$

Since $\mathcal{B}_2$ is invertible, equation (3.2.7) can be simplified to

$$T_0\begin{bmatrix}1\\1\\\vdots\\1\\\vdots\end{bmatrix} = 0. \tag{3.2.8}$$

Observe that

$$\begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ & & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 0. \tag{3.2.9}$$

We take T_0 in the form of

$$T_0 = T_1 D_0, \tag{3.2.10}$$

where D_0 is the difference matrix

$$D_0 := \begin{bmatrix} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & -1 & \\ & & & & \ddots & \ddots \end{bmatrix}, \tag{3.2.11}$$

and T_1 is an infinite matrix to be determined.

In other words, if we take $\mathcal{Q}_2$ in the following form

$$\mathcal{Q}_2 = I + T_1 D_0, \tag{3.2.12}$$

for any infinite matrix T_1, then equation (3.2.3) is automatically satisfied. Next, we need to determine T_1, such that (3.2.4) is satisfied. To this end, we write (3.2.4) as follows

$$(\mathcal{B}_2 - I) \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} + \mathcal{B}_2 T_1 D_0 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = 0. \tag{3.2.13}$$

Next we will simplify each term of the left-hand side in (3.2.13), so that we can determine the structure of T_1.

- *Simplify the first term of* (3.2.13):

  Equation (3.2.5) implies that there exists an infinite matrix X_1, such that

  $$\mathcal{B}_2 - I = X_1 D_0 \tag{3.2.14}$$

  using the similar discussion from (3.2.8) to (3.2.11). Furthermore, we can find the explicit expression for X_1. To solve the equation (3.2.14) for X_1, we multiply both sides by the pseudo-inverse of D_0, which is given by

$$D_0^+ = \begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots \\ 0 & 1 & \ddots & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & 1 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.2.15}$$

Since

$$D_0 D_0^+ = I, \tag{3.2.16}$$

we get

$$X_1 = (\mathcal{B}_2 - I)D_0^+.$$

From (4.2.3), we can write $\mathcal{B}_2 - I$ as

$$\mathcal{B}_2 - I = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} - 1 & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} - 1 & b_{n+1,n} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{3.2.17}$$

which has the property that the sum of all the entries in each row is zero by the *partition of unity* for the linear B-splines. Thus,

$$X_1 = (\mathcal{B}_2 - I)D_0^+ = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & -b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & b_{n-3,n-2} & -b_{n-1,n-2} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.2.18}$$

Now we do the following simplification

$$(\mathcal{B}_2 - I)\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = X_1 D_0 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = X_1 \begin{bmatrix} x_0 - x_1 \\ x_1 - x_2 \\ \vdots \\ x_{n-1} - x_n \\ \vdots \end{bmatrix}.$$

Denote, for $0 < k \leq m$

$$d_j^k := x_j - x_{j-k}, \quad \text{for } k \leq j < \infty. \tag{3.2.19}$$

We can simplify the above expression to

$$(\mathcal{B}_2 - I)\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = -X_1 \begin{bmatrix} d_1^1 \\ d_2^1 \\ \vdots \\ d_n^1 \\ \vdots \end{bmatrix}.$$

Condition (2.1.4) implies that $d_j^k \neq 0$. To further simplify the above expression, we let

$$\tilde{X}_1 = X_1 \begin{bmatrix} d_1^1 & & & & \\ & d_2^1 & & & \\ & & \ddots & & \\ & & & d_n^1 & \\ & & & & \ddots \end{bmatrix}. \tag{3.2.20}$$

Thus, we have

$$(\mathcal{B}_2 - I)\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = -X_1 \begin{bmatrix} d_1^1 \\ d_2^1 \\ \vdots \\ d_n^1 \\ \vdots \end{bmatrix} = -\tilde{X}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}. \tag{3.2.21}$$

It follows from (3.2.18) and (3.2.20) that

$$\tilde{X}_1 = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots \\ d_1^1 b_{01} & -d_2^1 b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & d_{n-1}^1 b_{n-2,n-1} & -d_n^1 b_{n,n-1} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{3.2.22}$$

In our formulations above, we notice that

$$X_1 D_0 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = X_1 \begin{bmatrix} d_1^1 & & & \\ & \ddots & & \\ & & d_n^1 & \\ & & & \ddots \end{bmatrix} \times$$

$$\begin{bmatrix} \frac{1}{d_1^1} & & & \\ & \ddots & & \\ & & \frac{1}{d_n^1} & \\ & & & \ddots \end{bmatrix} D_0 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = -\tilde{X}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}. \tag{3.2.23}$$

We would like to introduce a new notation E_1 as

$$E_1 = \begin{bmatrix} \frac{1}{d_1^1} & & \\ & \ddots & \\ & & \frac{1}{d_{n-1}^1} \end{bmatrix} D_0. \tag{3.2.24}$$

Thus, we have

$$X_1 D_0 = \tilde{X}_1 E_1, \tag{3.2.25}$$

and

$$E_1 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}. \tag{3.2.26}$$

- *Simplify the second term of* (3.2.13):

In view of (3.2.23) through (3.2.26), we would like to write $T_1 D_0$ as

$$T_1 D_0 = \tilde{T}_1 E_1, \tag{3.2.27}$$

where

$$\tilde{T}_1 = T_1 \begin{bmatrix} d_1^1 & & & \\ & \ddots & & \\ & & d_n^1 & \\ & & & \ddots \end{bmatrix}. \tag{3.2.28}$$

Thus, the second term of (3.2.13) can be written as

$$\mathcal{B}_2 T_1 D_0 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = \mathcal{B}_2 \tilde{T}_1 E_1 \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \\ \vdots \end{bmatrix} = -\mathcal{B}_2 \tilde{T}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}. \tag{3.2.29}$$

It follows from (3.2.13), (3.2.21) and (3.2.29) that

$$(\mathcal{B}_2 \tilde{T}_1 + \tilde{X}_1) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = 0. \tag{3.2.30}$$

Now our problem becomes (3.2.30). Specifically, find an appropriate infinite matrix $\tilde{T}_1$ for (3.2.30). To this end, we rewrite it in the following new form

$$\mathcal{B}_2\tilde{T}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = -\tilde{X}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ -d_1^1 b_{01} + d_2^1 b_{21} \\ \vdots \\ -d_{n-1}^1 b_{n-2,n-1} + d_n^1 b_{n,n-1} \\ \vdots \end{bmatrix}. \tag{3.2.31}$$

The easiest form we can see for $\tilde{T}$ is as follows

$$\tilde{T}_1 = \begin{bmatrix} 0 & & & & & & & \\ \tilde{t}_1 & 0 & & & & & & \\ & \tilde{t}_2 & 0 & & & & & \\ & & \ddots & 0 & & & & \\ & & & \ddots & 0 & & & \\ & & & & \ddots & 0 & & \\ & & & & & \tilde{t}_n & \ddots & \\ & & & & & & \ddots & \end{bmatrix}, \tag{3.2.32}$$

and with this form, we can write

$$\tilde{T}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \tilde{t}_1 \\ \tilde{t}_2 \\ \vdots \\ \tilde{t}_n \\ \vdots \end{bmatrix}. \tag{3.2.33}$$

So we further reduce our problem (3.2.30) to a new form: Determine $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n$, such that

$$\mathcal{B}_2 \begin{bmatrix} 0 \\ \tilde{t}_1 \\ \tilde{t}_2 \\ \vdots \\ \tilde{t}_n \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ -d_1^1 b_{01} + d_2^1 b_{21} \\ \vdots \\ -d_n^1 b_{n-1,n} + d_{n+1}^1 b_{n+1,n} \\ \vdots \end{bmatrix}. \tag{3.2.34}$$

We can represent the $(i+1)$-th row of the above matrix equation as

$$b_{i-1,i}\tilde{t}_{i-1} + b_{i,i}\tilde{t}_i + b_{i+1,i}\tilde{t}_{i+1} = -d_i^1 b_{i-1,i} + d_{i+1}^1 b_{i+1,i} \tag{3.2.35}$$

for $i = 1, 2, \ldots, n, \ldots$, and we assume that $\tilde{t}_0 = 0$. After we do the symbolic computation using the **Maple**, we get the following result:

$$\tilde{t}_i = x_i - t_i, \qquad \text{for } i = 1, 2, \ldots, n, \ldots. \tag{3.2.36}$$

Using (3.2.19) and (3.2.36) in (3.2.35), we get the following expression for (3.2.35)

$$\begin{aligned}(x_{i-1}-t_{i-1})\, b_{i-1,i} + (x_i - t_i)\, b_{i,i} + (x_{i+1} - t_{i+1})\, b_{i+1,i} \\ = -\,(x_i - x_{i-1})\, b_{i-1,i} + (x_{i+1} - x_i)\, b_{i+1,i}.\end{aligned}$$

Notice that $b_{i,j} = B_{i,2}(x_j)$, we apply (4.2.5) for $b_{i,j}$ in the above equality and get an equivalent expression for (3.2.35) as follows

$$\begin{aligned}&(x_{i-1} - t_{i-1})\, \sigma_i \frac{t_i - x_i}{t_i - t_{i-1}} + (x_i - t_i) \left(\sigma_i \frac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1 - \sigma_i) \frac{t_{i+1} - x_i}{t_{i+1} - t_i} \right) \\ &+ (x_{i+1} - t_{i+1})\, (1 - \sigma_i) \frac{x_i - t_i}{t_{i+1} - t_i} \\ =& -\,(x_i - x_{i-1})\, \sigma_i \frac{t_i - x_i}{t_i - t_{i-1}} + (x_{i+1} - x_i)\, (1 - \sigma_i) \frac{x_i - t_i}{t_{i+1} - t_i}.\end{aligned} \tag{3.2.37}$$

Next we will verify that the identity (3.2.37) is true.

We simplify the expression of the left-hand-side of (3.2.37) as follows

$$\begin{aligned}LHS =& \sigma_i \left((x_{i-1} - t_{i-1})\, \frac{t_i - x_i}{t_i - t_{i-1}} + (x_i - t_i)\, \frac{x_i - t_{i-1}}{t_i - t_{i-1}} \right) + \\ &(1 - \sigma_i) \left((x_i - t_i)\, \frac{t_{i+1} - x_i}{t_{i+1} - t_i} + (x_{i+1} - t_{i+1})\, \frac{x_i - t_i}{t_{i+1} - t_i} \right) \\ =&(t_i - x_i)\, \sigma_i \left(\frac{x_{i-1} - t_{i-1}}{t_i - t_{i-1}} - \frac{x_i - t_{i-1}}{t_i - t_{i-1}} \right) \\ &+(x_i - t_i)\, (1 - \sigma_i) \left(\frac{t_{i+1} - x_i}{t_{i+1} - t_i} + \frac{x_{i+1} - t_{i+1}}{t_{i+1} - t_i} \right) \\ =& -\,(x_i - x_{i-1})\, \sigma_i \frac{t_i - x_i}{t_i - t_{i-1}} + (x_{i+1} - x_i)\, \frac{x_i - t_i}{t_{i+1} - t_i},\end{aligned}$$

which is exactly the same as the right-hand-side of (3.2.37). Thus, we verified the identity (3.2.35) for $\tilde{t}_i$ given by (3.2.36).

To summarize our analysis above, we get this result: If we take $\mathcal{Q}_2$ in the following form

$$\mathcal{Q}_2 = I_n + \tilde{T}_1 E_1, \tag{3.2.38}$$

where $\tilde{T}_1$ is given by (3.2.32) and (3.2.36), and E_1 is given by (3.2.24), then we have (3.2.2), and hence (3.1.15). Thus, the matrix defined in (3.2.38) already gives us a *local quasi-interpolant* for the linear B-splines. Since the solution for a local quasi-interpolant is not unique, we would like to see if there is any other interesting solution.

We go back to examine $\tilde{T}_1$ as in (3.2.32), and notice that there is some flexibility in choosing the form of $\tilde{T}_1$. If we take $\tilde{T}_1$ in the following form

$$\tilde{T}_1 = \begin{bmatrix} 0 & & & & & \\ \sigma_1\tilde{t}_1 & (1-\sigma_1)\tilde{t}_1 & & & & \\ & \sigma_2\tilde{t}_2 & (1-\sigma_2)\tilde{t}_2 & & & \\ & & \ddots & \ddots & & \\ & & & \sigma_n\tilde{t}_n & (1-\sigma_n)\tilde{t}_n & \\ & & & & \ddots & \ddots \end{bmatrix}, \tag{3.2.39}$$

where σ_i's are the indicator variables defined by (4.2.5), it still satisfies (3.2.33). In other words, if we choose $\tilde{T}_1$ defined by (3.2.39) and (3.2.36), then $\mathcal{Q}_2$ given by (3.2.38) also gives us another local quasi-interpolant for the linear B-splines. Moreover, we would like to view the explicit form of the second solution. To this end, we calculate $\mathcal{Q}_2$ in the following steps.

First, we calculate $\tilde{T}_1 E_1$ using the following expression:

$$\tilde{T}_1 E_1 = \tilde{T}_1 \operatorname{diag}\left(\frac{1}{d_1^1}, \ldots, \frac{1}{d_n^1}, \ldots\right) D_0$$

based on (3.2.24), and get

$$\tilde{T}_1 \operatorname{diag}\left(\frac{1}{d_1^1}, \ldots, \frac{1}{d_n^1}, \ldots\right) =$$

$$\begin{bmatrix} 0 & & & & & \\ \sigma_1\frac{\tilde{t}_1}{d_1^1} & (1-\sigma_1)\frac{\tilde{t}_1}{d_2^1} & & & & \\ & \sigma_2\frac{\tilde{t}_2}{d_2^1} & (1-\sigma_2)\frac{\tilde{t}_2}{d_3^1} & & & \\ & & \ddots & \ddots & & \\ & & & \sigma_{n-1}\frac{\tilde{t}_{n-1}}{d_{n-1}^1} & (1-\sigma_{n-1})\frac{\tilde{t}_{n-1}}{d_n^1} & \\ & & & & \ddots & \ddots \end{bmatrix}.$$

To represent the matrix $\tilde{T}_1 E_1$, in order to handle its large size, we just write its $(i+1)$-th row general expression for $1 \le i < \infty$, because its first row and last row are all-zero rows. By (3.2.36), (3.2.19), and (3.2.11), we get the $(i+1)$-th row of $\tilde{T}_1 E_1$ whose nonzero entries are given below

$$\begin{cases} (i\text{-th}) & := \sigma_i \dfrac{x_i - t_i}{x_i - x_{i-1}}, \\ ((i+1)\text{-th}) & := -\sigma_i \dfrac{x_i - t_i}{x_i - x_{i-1}} + (1-\sigma_i)\dfrac{x_i - t_i}{x_{i+1} - x_i}, \\ ((i+2)\text{-th}) & := (1-\sigma_i)\dfrac{t_i - x_i}{x_{i+1} - x_i} \end{cases} \tag{3.2.40}$$

for $1 \leq i < \infty$.

To write the expression for $\mathcal{Q}_2$, which is $I_n + \tilde{T}_1 E_1$, we only need to add 1 to each of the diagonal entries of $\tilde{T}_1 E_1$. Thus, we simplify the following expression, which corresponds to the $(i+1)$-th diagonal entry,

$$\begin{aligned}
&1 - \sigma_i \frac{x_i - t_i}{x_i - x_{i-1}} + (1-\sigma_i)\frac{x_i - t_i}{x_{i+1} - x_i} \\
=&\sigma_i + (1-\sigma_i) - \sigma_i \frac{x_i - t_i}{x_i - x_{i-1}} + (1-\sigma_i)\frac{x_i - t_i}{x_{i+1} - x_i} \\
=&\sigma_i \frac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1-\sigma_i)\frac{x_{i+1} - t_i}{x_{i+1} - x_i}.
\end{aligned}$$

Finally, we can write the $(i+1)$-th row of $\mathcal{Q}_2$ in its nonzero entries as follows,

$$\begin{cases}
(i\text{-th}) & := \sigma_i \dfrac{x_i - t_i}{x_i - x_{i-1}}, \\
((i+1)\text{-th}) & := \sigma_i \dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1-\sigma_i)\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\
((i+2)\text{-th}) & := (1-\sigma_i)\dfrac{t_i - x_i}{x_{i+1} - x_i}
\end{cases} \tag{3.2.41}$$

for $1 \leq i < \infty$.

In order to compare $\mathcal{B}_2$ and $\mathcal{Q}_2$, we use two vector variables: the first one is the knot vector defined as $\vec{t} := [t_0, t_1, \ldots, t_n, \ldots]$; and the second one is the sample vector defined as $\vec{x} := [x_0, x_1, \ldots, x_n, \ldots]$. Now we can view both infinite matrices $\mathcal{B}_2$ and $\mathcal{Q}_2$ as matrix-valued functions with variables $\vec{t}$ and $\vec{x}$. Specifically, we write them as $\mathcal{B}_2(\vec{t}, \vec{x})$ and $\mathcal{Q}_2(\vec{t}, \vec{x})$, respectively.

First, $\mathcal{B}_2$ and $\mathcal{Q}_2$ have the same top row: $[1, 0, \ldots, 0, \ldots]$. Next, we compare their general $(i+1)$-th row for $1 \leq i < \infty$ at their corresponding nonzero entries, i.e. i-th, $(i+1)$-th, and $(i+2)$-th positions, as follows

$$\begin{cases}
\sigma_i \dfrac{t_i - x_i}{t_i - t_{i-1}} & \leftrightarrow \sigma_i \dfrac{x_i - t_i}{x_i - x_{i-1}}, \\[2ex]
\sigma_i \dfrac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1-\sigma_i)\dfrac{t_{i+1} - x_i}{t_{i+1} - t_i} & \leftrightarrow \sigma_i \dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1-\sigma_i)\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\[2ex]
(1-\sigma_i)\dfrac{x_i - t_i}{t_{i+1} - t_i} & \leftrightarrow (1-\sigma_i)\dfrac{t_i - x_i}{x_{i+1} - x_i}.
\end{cases} \tag{3.2.42}$$

We can see that there is a *duality* property between $\mathcal{B}_2(\vec{t}, \vec{x})$ and $\mathcal{Q}_2(\vec{t}, \vec{x})$ in the sense that when we switch $\vec{t}$ and $\vec{x}$, one becomes the other, i.e.

$$\mathcal{Q}_2(\vec{x}, \vec{t}) = \mathcal{B}_2(\vec{t}, \vec{x}). \tag{3.2.43}$$

3.3 *General 3-Point Solutions*

To get a general expression for the 3-point solutions, we would like to make $\tilde{T}_1$ more general than the forms in (3.2.32) and in (3.2.39). If we take $\tilde{T}_1$ in the following form

$$\tilde{T}_1 = \begin{bmatrix} 0 & & & & & & \\ \omega_1\tilde{t}_1 & (1-\omega_1)\tilde{t}_1 & & & & & \\ & \omega_2\tilde{t}_2 & (1-\omega_2)\tilde{t}_2 & & & & \\ & & \ddots & \ddots & & & \\ & & & \omega_n\tilde{t}_n & (1-\omega_n)\tilde{t}_n & & \\ & & & & \ddots & \ddots \end{bmatrix}, \tag{3.3.1}$$

where $\omega_1, \ldots, \omega_n, \ldots$ are free parameters to be determined for specific properties in applications. It is easy to see that $\tilde{T}_1$ in (3.3.1) satisfies (3.2.33), and $\mathcal{Q}_2$ given by (3.2.38) gives us general 3-point quasi-interpolants for the linear B-splines. With this $\tilde{T}_1$, and by the similar calculation above, we can get that the nonzero entries of the $(i+1)$-th row of $\mathcal{Q}_2$ are given by

$$\begin{cases} (i\text{-th}) & := \quad \omega_i \dfrac{x_i - t_i}{x_i - x_{i-1}}, \\ ((i+1)\text{-th}) & := \quad \omega_i \dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1-\omega_i)\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\ ((i+2)\text{-th}) & := \quad (1-\omega_i)\dfrac{t_i - x_i}{x_{i+1} - x_i} \end{cases} \tag{3.3.2}$$

for $1 \leq i < \infty$.

3.4 *3-Point Solutions for Digital Filters*

Since the sum of each row is 1 (*we can give a short proof*), the linear functional $\lambda_k f$ in the following form

$$\lambda_k f = q_{k,k-1} f_{k-1} + q_{k,k} f_k + q_{k,k+1} f_{k+1},$$

can be treated as a weighted average of f_{k-1}, f_k, and f_{k+1}. In some of the applications, such as *Computer Graphics* and *Image Processing*, we would like to have these weights $q_{k,k-1}, q_{k,k}$, and $q_{k,k+1}$ connect to digital filters. In other words, we want to make $\{q_{k,k-1}, q_{k,k}, q_{k,k+1}\}$ correspond to *low-pass* or *high-pass* filters. To this end, we want to enforce the following condition:

$$q_{k,k-1}\, q_{k,k+1} > 0 \quad \text{and} \quad q_{k,k} > 0. \tag{3.4.1}$$

Under this condition, $q_{k,k-1}$ and $q_{k,k+1}$ must have the same sign: either both positive that corresponds to the low-pass filter case, or both negative that corresponds to the high-pass filter case. By (3.3.2), we have

$$\left(\omega_i \frac{x_i - t_i}{x_i - x_{i-1}}\right)\left((1-\omega_i)\frac{t_i - x_i}{x_{i+1} - x_i}\right) > 0,$$

which can be written as

$$\omega_i\,(\omega_i - 1)\,\frac{(x_i - t_i)^2}{(x_i - x_{i-1})(x_{i+1} - x_i)} > 0.$$

Notice that the fraction part must be positive. Thus, we get the condition for ω_i as follows

$$\omega_i < 0 \quad \text{or} \quad \omega_i > 1. \tag{3.4.2}$$

In order to make our discussion focus on a positive variable, we introduce a new variable $\tilde{\omega}_i$, which is derived from ω_i as follows

$$\begin{cases} \tilde{\omega}_i = -\omega_i & \text{when} \quad \omega_i < 0; \\ \tilde{\omega}_i = \omega_i - 1 & \text{when} \quad \omega_i > 1. \end{cases} \tag{3.4.3}$$

This condition takes care of the case that $q_{k,k-1}\,q_{k,k+1} > 0$. Next, we will work on the condition for the case that $q_{k,k} > 0$.

- $\sigma_i = 1$, which implies that $x_i \in (t_{i-1}, t_i]$;

  First, we consider the low-pass case: $q_{k,k-1} > 0$ and $q_{k,k+1} > 0$, and we get the following expressions:

$$\begin{cases} q_{k,k-1} &=& -\tilde{\omega}_i\,\dfrac{x_i - t_i}{x_i - x_{i-1}}, \\ \\ q_{k,k} &=& -\tilde{\omega}_i\,\dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1 + \tilde{\omega}_i)\,\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\ \\ q_{k,k+1} &=& (1 + \tilde{\omega}_i)\,\dfrac{t_i - x_i}{x_{i+1} - x_i} \end{cases} \tag{3.4.4}$$

  with $\tilde{\omega}_i > 0$. To make $q_{k,k} > 0$, we do the following calculation:

$$-\tilde{\omega}_i\,\frac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1 + \tilde{\omega}_i)\,\frac{x_{i+1} - t_i}{x_{i+1} - x_i} = \frac{x_{i+1} - t_i}{x_{i+1} - x_i} + \tilde{\omega}_i\,\left(\frac{x_{i+1} - t_i}{x_{i+1} - x_i} - \frac{t_i - x_{i-1}}{x_i - x_{i-1}}\right),$$

  and we want to solve

$$\frac{x_{i+1} - t_i}{x_{i+1} - x_i} > \tilde{\omega}_i\,\left(\frac{t_i - x_{i-1}}{x_i - x_{i-1}} - \frac{x_{i+1} - t_i}{x_{i+1} - x_i}\right),$$

  which is

$$\frac{x_{i+1} - t_i}{x_{i+1} - x_i} > \tilde{\omega}_i\,\frac{(t_i - x_i)(x_{i+1} - x_{i-1})}{(x_i - x_{i-1})(x_{i+1} - x_i)}.$$

  Thus, we get

$$0 < \tilde{\omega}_i < \frac{(x_{i+1} - t_i)(x_i - x_{i-1})}{(t_i - x_i)(x_{i+1} - x_{i-1})}. \tag{3.4.5}$$

In order to make the structure as simple as possible, we introduce another variable ξ_i associated with $\tilde{\omega}_i$ as follows,

$$\tilde{\omega}_i = \xi_i \frac{(x_{i+1} - t_i)(x_i - x_{i-1})}{(t_i - x_i)(x_{i+1} - x_{i-1})}. \tag{3.4.6}$$

In this way, we can get an equivalent inequality for (3.4.5), which is as simple as

$$0 < \xi_i < 1. \tag{3.4.7}$$

Here ξ_i is an independent parameter in $(0, 1)$, which does not rely on $\{x_i\}$ and $\{t_i\}$. We can use it to control $\tilde{\omega}_i$. Next we will write (3.4.4) in terms of ξ_i.

By (3.4.6), (3.4.4) becomes

$$\begin{cases} q_{k,k-1} &= \xi_i \dfrac{x_{i+1} - t_i}{x_{i+1} - x_{i-1}}, \\ q_{k,k} &= (1 - \xi_i) \dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\ q_{k,k+1} &= \dfrac{t_i - x_i}{x_{i+1} - x_i} + \xi_i \dfrac{(x_{i+1} - t_i)(x_i - x_{i-1})}{(x_{i+1} - x_i)(x_{i+1} - x_{i-1})}. \end{cases} \tag{3.4.8}$$

Next, we consider the high-pass case: $q_{k,k-1} < 0$ and $q_{k,k+1} < 0$, and we get the following expressions:

$$\begin{cases} q_{k,k-1} &= \omega_i \dfrac{x_i - t_i}{x_i - x_{i-1}}, \\ q_{k,k} &= \omega_i \dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} + (1 - \omega_i) \dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\ q_{k,k+1} &= (1 - \omega_i) \dfrac{t_i - x_i}{x_{i+1} - x_i}, \end{cases} \tag{3.4.9}$$

where

$$0 < \omega_i < 1.$$

Obviously, we have $q_{k,k} > 0$.

- $\sigma_i = 0$, which implies that $x_i \in (t_i, t_{i+1})$.

For the high-pass case: $q_{k,k-1} > 0$ and $q_{k,k+1} > 0$, we have the following expressions:

$$\begin{cases} q_{k,k-1} &= (1+\tilde{\omega}_i)\,\dfrac{x_i - t_i}{x_i - x_{i-1}}, \\[2ex] q_{k,k} &= (1+\tilde{\omega}_i)\,\dfrac{t_i - x_{i-1}}{x_i - x_{i-1}} - \tilde{\omega}_i\,\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \\[2ex] q_{k,k+1} &= -\tilde{\omega}_i\,\dfrac{t_i - x_i}{x_{i+1} - x_i} \end{cases} \tag{3.4.10}$$

with $\tilde{\omega}_i > 0$. To make $q_{k,k} > 0$, we do the following calculation:

$$(1+\tilde{\omega}_i)\,\frac{t_i - x_{i-1}}{x_i - x_{i-1}} - \tilde{\omega}_i\,\frac{x_{i+1} - t_i}{x_{i+1} - x_i} = \frac{t_i - x_{i-1}}{x_i - x_{i-1}} + \tilde{\omega}_i\left(\frac{t_i - x_{i-1}}{x_i - x_{i-1}} - \frac{x_{i+1} - t_i}{x_{i+1} - x_i}\right),$$

and we want to solve

$$\frac{t_i - x_{i-1}}{x_i - x_{i-1}} > \tilde{\omega}_i\left(\frac{x_{i+1} - t_i}{x_{i+1} - x_i} - \frac{t_i - x_{i-1}}{x_i - x_{i-1}}\right),$$

which is

$$\frac{t_i - x_{i-1}}{x_i - x_{i-1}} > \tilde{\omega}_i\,\frac{(x_i - t_i)(x_{i+1} - x_{i-1})}{(x_i - x_{i-1})(x_{i+1} - x_i)}.$$

Thus, we get

$$0 < \tilde{\omega}_i < \frac{(x_{i+1} - x_i)(t_i - x_{i-1})}{(x_i - t_i)(x_{i+1} - x_{i-1})}. \tag{3.4.11}$$

In order to make the structure as simple as possible, we introduce another variable ξ_i associated with $\tilde{\omega}_i$ as follows,

$$\tilde{\omega}_i = \xi_i\,\frac{(x_{i+1} - x_i)(t_i - x_{i-1})}{(x_i - t_i)(x_{i+1} - x_{i-1})}. \tag{3.4.12}$$

In this way, we can get an equivalent inequality for (3.4.11), which is as simple as

$$0 < \xi_i < 1.$$

Here ξ_i is an independent parameter in $(0,1)$, which does not rely on $\{x_i\}$ and $\{t_i\}$. We can use it to control $\tilde{\omega}_i$. Next we will write (3.4.10) in terms of ξ_i.

By (3.4.12), (3.4.10) becomes

$$\begin{cases} q_{k,k-1} &= \xi_i\,\dfrac{x_{i+1} - t_i}{x_{i+1} - x_{i-1}}, \quad \text{(to do)} \\[2ex] q_{k,k} &= (1-\xi_i)\,\dfrac{x_{i+1} - t_i}{x_{i+1} - x_i}, \quad \text{(to do)} \\[2ex] q_{k,k+1} &= \xi_i\,\dfrac{t_i - x_{i-1}}{x_{i+1} - x_{i-1}}. \end{cases} \tag{3.4.13}$$

Chapter 4

Shoenberg-Whitney Matrix:

4.1 *Properties of infinite matrices*

Since the behaviors of the infinite matrices are quite different from those of the regular finite matrices, we would like to study some basic properties of the infinite matrices in this section.

To represent a general infinite matrix, we use the following notation:

$$A = [a_{ij}]_{i=0,j=0}^{\infty}, \quad a_{ij} \in \mathbb{R}. \tag{4.1.1}$$

In this case, we write it as $A \in \mathbb{M}(\infty \times \infty)$.

We need to consider a special type of infinite matrices that have the property: For each row, only finitely many entries are nonzero. We denote this kind of matrices as $\mathbb{M}^0(\infty \times \infty)$.

- (*Two infinite matrices are equal.*)

  For two matrices $A, B \in \mathbb{M}(\infty \times \infty)$, if $A = B$, where $A = [a_{ij}]_{i=0,j=0}^{\infty}$ and $B = [b_{ij}]_{i=0,j=0}^{\infty}$, if and only if

  $$a_{ij} = b_{ij}, \quad \text{for all} \quad 0 \le i, j < \infty.$$

- (*An infinite matrix is zero.*)

  For a matrix $A \in \mathbb{M}(\infty \times \infty)$, if $A = 0$, where $A = [a_{ij}]_{i=0,j=0}^{\infty}$, if and only if

  $$a_{ij} = 0, \quad \text{for all} \quad 0 \le i, j < \infty.$$

- (*Transpose operation*)

  For two matrices $A, C \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$ and $C = [c_{ij}]_{i=0,j=0}^{\infty}$, if $C = A^T$, then we have

$$c_{ij} = a_{ji}, \quad \text{for all} \quad 0 \leq i, j < \infty.$$

- (*Addition operation*)

  For three matrices $A, B, C \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$, $B = [b_{ij}]_{i=0,j=0}^{\infty}$, and $C = [c_{ij}]_{i=0,j=0}^{\infty}$, if $C = A + B$, then we have

$$c_{ij} = a_{ji} + b_{ij}, \quad \text{for all} \quad 0 \leq i, j < \infty.$$

- (*Scalar-matrix multiplication operation*)

  For two matrices $A, C \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$ and $C = [c_{ij}]_{i=0,j=0}^{\infty}$, if $C = \alpha\, A$ for some $\alpha \in \mathbb{R}$, then we have

$$c_{ij} = \alpha\, a_{ji}, \quad \text{for all} \quad 0 \leq i, j < \infty.$$

- (*Multiplication operation*)

  For three matrices $A, B, C \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$, $B = [b_{ij}]_{i=0,j=0}^{\infty}$, and $C = [c_{ij}]_{i=0,j=0}^{\infty}$, if $A \in \mathbb{M}^0(\infty \times \infty)$ or $B \in \mathbb{M}^0(\infty \times \infty)$, then $C = AB$ is well-defined as follows

$$c_{ij} = \sum a_{ik} b_{kj}, \quad \text{for all} \quad 0 \leq i, k, j < \infty.$$

- (*No associative law for multiplications*)

  For three matrices $A, B, C \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$, $B = [b_{ij}]_{i=0,j=0}^{\infty}$, and $C = [c_{ij}]_{i=0,j=0}^{\infty}$, the two products $(AB)C$ and $A(BC)$ could be different.

- (*Identity matrix*)

  We call a matrix $A \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$ an identity matrix, if $a_{ii} = 1$ for all $0 \leq i < \infty$, and the remaining entries of A are all zero. We denote it as I_{∞}.

- (*Inverse matrix: Left-inverse and right-inverse*)

  Given a matrix $A \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$, if there exists another matrix $B \in \mathbb{M}(\infty \times \infty)$ with $B = [b_{ij}]_{i=0,j=0}^{\infty}$, such that $AB = I_\infty$, then we say that B is a right-inverse matrix of A.

  Given a matrix $A \in \mathbb{M}(\infty \times \infty)$ with $A = [a_{ij}]_{i=0,j=0}^{\infty}$, if there exists another matrix $C \in \mathbb{M}(\infty \times \infty)$ with $C = [c_{ij}]_{i=0,j=0}^{\infty}$, such that $CA = I_\infty$, then we say that C is a left-inverse matrix of A.

  The left-inverse and the right-inverse of the same matrix usually are different.

4.2 General representation for linear Shoenberg-Whitney matrix

Explicit formula for $B_{i,2}(x)$:

$$\begin{aligned} B_{i,2}(x) &= \frac{x - t_{i-1}}{t_i - t_{i-1}} B_{i-1,1}(x) + \frac{t_{i+1} - x}{t_{i+1} - t_i} B_{i,1}(x) \\ &= \begin{cases} \dfrac{x - t_{i-1}}{t_i - t_{i-1}}, & \text{if} \quad t_{i-1} \le x < t_i, \\ \dfrac{t_{i+1} - x}{t_{i+1} - t_i}, & \text{if} \quad t_i \le x < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2.1}$$

Consider the $(k+1)$-th row of the infinite matrix $\mathcal{B}_2$ for $0 \le k < \infty$,

$$\begin{bmatrix} B_{0,2}(x_k) & B_{1,2}(x_k) & \cdots & B_{k,2}(x_k) & \cdots \end{bmatrix}. \tag{4.2.2}$$

When $x_k \in (t_{k-1}, t_k]$, we can write the above row as

$$\begin{bmatrix} \underbrace{0 \cdots 0}_{k-1} & B_{k-1,2}(x_k) & B_{k,2}(x_k) & 0 & \cdots \end{bmatrix}.$$

- When $x_k \in (t_k, t_{k+1})$, we can write (4.2.2) as

$$\begin{bmatrix} \underbrace{0 \cdots 0}_{k} & B_{k,2}(x_k) & B_{k+1,2}(x_k) & 0 & \cdots \end{bmatrix}.$$

Combine the above two cases,

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{k-1} & B_{k-1,2}(x_k) & B_{k,2}(x_k) & B_{k+1,2}(x_k) & 0 & \cdots \end{bmatrix}.$$

In order to use a more compact way to write the matrix $\mathcal{B}_2$, we use the notation $b_{ij} := B_{i,2}(x_j)$. Thus, we have

$$\mathcal{B}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} & b_{n+1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.2.3}$$

Address the uncertainty that which of the intervals $(t_{k-1}, t_k]$ and (t_k, t_{k+1}) contains x_k.

To this end, we introduce a set of indicator variables $\{\sigma_i\}_{i=1}^{\infty}$ as follows,

$$\sigma_i = \begin{cases} 1 & \text{if} \quad x_i \in (t_{i-1}, t_i) \\ 0 & \text{if} \quad x_i \in [t_i, t_{i+1}). \end{cases} \tag{4.2.4}$$

Write explicit expressions for $B_{i-1,2}(x_i), B_{i,2}(x_i), B_{i+1,2}(x_i)$:

$$\begin{cases} B_{i-1,2}(x_i) = \sigma_i \dfrac{t_i - x_i}{t_i - t_{i-1}}, \\ B_{i,2}(x_i) = \sigma_i \dfrac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1 - \sigma_i) \dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(x_i) = (1 - \sigma_i) \dfrac{x_i - t_i}{t_{i+1} - t_i} \end{cases} \tag{4.2.5}$$

for $i = 1, 2, \ldots$.

The *partition of unity* property of the linear B-splines can be written as a matrix form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} & b_{n+1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ \vdots \end{bmatrix}, \tag{4.2.6}$$

which will be a critical property in our matrix factorization technique.

Although our matrix factorization technique does not require the explicit forms of the inverse matrices of the Shoenberg-Whitney matrices, we require the *existence* of those inverse matrices. To show the existence for those infinite inverse matrices, it is a very challenging task. So we take the *constructuve approach* to solve this problem.

In general, finding the inverse matrices of the Shoenberg-Whitney matrices directly is very complicated. Here we propose a feasible approach to solve this problem. We get into the concrete structure of the Shoenberg-Whitney matrices, and characterize them into the following 5 special forms.

We start with the general expression of the Shoenberg-Whitney matrices for the linear B-splines as follows,

$$\mathcal{B}_2 := \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots \\ b_{01} & b_{11} & b_{21} & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & b_{n-1,n} & b_{n,n} & b_{n+1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \sigma_i = \begin{cases} 1 & \text{if} \quad x_i \in (t_{i-1}, t_i) \\ 0 & \text{if} \quad x_i \in [t_i, t_{i+1}). \end{cases}$$

where

$$\begin{cases} b_{i-1,i} & = \sigma_i \dfrac{t_i - x_i}{t_i - t_{i-1}}, \\ b_{i,i} & = \sigma_i \dfrac{x_i - t_{i-1}}{t_i - t_{i-1}} + (1 - \sigma_i) \dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}, \\ b_{i-1,i} & = (1 - \sigma_i) \dfrac{x_i - t_i}{t_{i+1} - t_i} \end{cases} \tag{4.2.7}$$

for $i = 1, 2, \ldots$. We will examine all the possibilities for the indicator variables $\{\sigma_i\}_{i=1}^{\infty}$.

Case I: Infinite bi-diagonal upper triangular matrix that corresponds to $\sigma_i = 0$ for $i = 1, \ldots, n, \ldots$:

$$B_2^I = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \dfrac{t_2 - x_1}{t_2 - t_1} & \dfrac{x_1 - t_1}{t_2 - t_1} & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & \dfrac{t_3 - x_2}{t_3 - t_2} & \dfrac{x_2 - t_2}{t_3 - t_2} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \dfrac{t_n - x_{n-1}}{t_n - t_{n-1}} & \dfrac{x_{n-1} - t_{n-1}}{t_n - t_{n-1}} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.2.8}$$

Case II: Infinite bi-diagonal lower triangular matrix that corresponds to $\sigma_i = 1$ for $i = 1, \dots, n, \dots$:

$$B_2^{II} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots \\ \frac{t_1 - x_1}{t_1 - t_0} & \frac{x_1 - t_0}{t_1 - t_0} & 0 & \ddots & \ddots & \ddots \\ 0 & \frac{t_2 - x_2}{t_2 - t_1} & \frac{x_2 - t_1}{t_2 - t_1} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \frac{t_n - x_{n-1}}{t_n - t_{n-1}} & \frac{x_n - t_{n-1}}{t_n - t_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.2.9}$$

Case III: Infinite block-diagonal matrix that corresponds to $\sigma_1 = \sigma_3 = \cdots = \sigma_{2k-1} = \cdots = 0$ and $\sigma_2 = \sigma_4 = \cdots = \sigma_{2k} = \cdots = 1$:

$$B_2^{III} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2 - x_1}{t_2 - t_1} & \frac{x_1 - t_1}{t_2 - t_1} & \ddots & \ddots & \ddots & \ddots \\ 0 & \frac{t_2 - x_2}{t_2 - t_1} & \frac{x_2 - t_1}{t_2 - t_1} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & \ddots & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \frac{t_{2k} - x_{2k-1}}{t_{2k} - t_{2k-1}} & \frac{x_{2k-1} - t_{2k-1}}{t_{2k} - t_{2k-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{t_{2k} - x_{2k}}{t_{2k} - t_{2k-1}} & \frac{x_{2k} - t_{2k-1}}{t_{2k} - t_{2k-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.2.10}$$

The first three cases are very simple, and we can find their inverses in the straightforward way. Then we will move to the more general cases.

Case IV: Assume that $\sigma_2 = \sigma_3 = \cdots = \sigma_k = 0$ and $\sigma_{k+1} = \cdots = \sigma_n = \cdots = 1$ for $2 \leq k < \infty$.

Then $\mathcal{B}_2$ has the form of

$$B_2^{IV} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2 - x_2}{t_2 - t_1} & \frac{x_2 - t_1}{t_2 - t_1} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \frac{t_{k-1} - x_{k-1}}{t_{k-1} - t_{k-2}} & \frac{x_{k-1} - t_{k-2}}{t_{k-1} - t_{k-2}} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{t_k - x_k}{t_k - t_{k-1}} & \frac{x_k - t_{k-1}}{t_k - t_{k-1}} & \ddots \\ \vdots & \ddots & \ddots & 0 & \frac{t_k - x_{k+1}}{t_k - t_{k-1}} & \frac{x_{k+1} - t_{k-1}}{t_k - t_{k-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \end{bmatrix} \tag{4.2.11}$$

We will use this case as the building block for the lase case.

Case V: Assume that $\sigma_2 = \sigma_3 = \cdots = \sigma_{k_1} = 0, \sigma_{k_1+1} = \cdots = \sigma_{k_2} = 1, \cdots, \sigma_{k_{2s-2}+1} = \cdots = \sigma_{k_{2s-1}} = 0$ and $\sigma_{k_{2s-1}+1} = \cdots = \sigma_{k_{2s}} = 1, \cdots$.

Then $\mathcal{B}_2$ has the form of

$$B_2^V := \begin{bmatrix} \mathcal{B}_2^1 & 0 & \cdots & \cdots & \cdots \\ \vdots & \mathcal{B}_2^2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathcal{B}_2^k & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.2.12}$$

where each block matrix $\mathcal{B}_2^k$ has the similar structure as in *Case IV*.

Chapter 5

Inverses of linear Shoenberg-Whitney matrices:

5.1 *Inverse Shoenberg-Whitney Matrix Case I:*

Let us modify the existing results for the finite case.

- *Case I*: $\sigma_i = 0$ for $i = 1, \ldots, n, \ldots$

  In this case, we have that $x_i \in [t_i, t_{i+1})$ for all $i = 1, \ldots, n, \ldots$, and we have

$$\begin{cases} B_{i-1,2}(x_i) = 0, \\ B_{i,2}(x_i) = \dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}, \\ B_{i+1,2}(x_i) = \dfrac{x_i - t_i}{t_{i+1} - t_i}. \end{cases} \tag{5.1.1}$$

  $\mathcal{B}_2$ becomes an upper triangular matrix in the form of

$$B_2^I = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2 - x_1}{t_2 - t_1} & \frac{x_1 - t_1}{t_2 - t_1} & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \frac{t_3 - x_2}{t_3 - t_2} & \frac{x_2 - t_2}{t_3 - t_2} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 & \frac{t_{n-1} - x_{n-2}}{t_{n-1} - t_{n-2}} & \frac{x_{n-2} - t_{n-2}}{t_{n-1} - t_{n-2}} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.1.2}$$

In order to make sure that the inverse of B_2^I exists, we require that all the diagonal entries are nonzero, i.e. $x_i \neq t_{i+1}$ for all $i = 1, 2, \ldots$.

We introduce another notation to make the expression of B_2^I a little simpler,

$$\eta_{i,j} = \frac{t_i - x_j}{t_i - t_{i-1}}. \tag{5.1.3}$$

Then B_2^I becomes

$$B_2^I = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \eta_{2,1} & 1-\eta_{2,1} & 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \eta_{3,2} & 1-\eta_{3,2} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 & \eta_{n,n-1} & 1-\eta_{n,n-1} & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.1.4}$$

In order to represent $\left(B_2^I\right)^{-1}$ in a concise way, we need another notation,

$$\xi_{i,j} = \left(\frac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}\right)\left(\frac{x_i - t_i}{t_{i+1} - x_i}\right)\cdots\left(\frac{x_{j-1} - t_{j-1}}{t_j - x_{j-1}}\right), \tag{5.1.5}$$

for $i > j$ and $\xi_{i,i} = 1$.

Then the result from the finite case, we can write

$$\left(B_2^I\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \frac{t_2 - t_1}{x_1 - t_1}\xi_{2,2} & -\frac{t_3 - t_2}{x_2 - t_2}\xi_{2,3} & \frac{t_4 - t_3}{x_3 - t_3}\xi_{2,4} & \cdots & (-1)^{n+1}\frac{t_{n-1} - t_{n-2}}{x_{n-2} - t_{n-2}}\xi_{2,n-1} & \vdots \\ \vdots & \ddots & \frac{t_3 - t_2}{x_2 - t_2}\xi_{3,3} & -\frac{t_4 - t_3}{x_3 - t_3}\xi_{3,4} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \frac{t_4 - t_3}{x_3 - t_3}\xi_{4,4} & \ddots & \frac{t_{n-1} - t_{n-2}}{x_{n-2} - t_{n-2}}\xi_{n-3,n-1} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\frac{t_{n-1} - t_{n-2}}{x_{n-2} - t_{n-2}}\xi_{n-2,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \frac{t_{n-1} - t_{n-2}}{x_{n-2} - t_{n-2}}\xi_{n-1,n-1} & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix} \tag{5.1.6}$$

which can be further simplified as

$$\left(B_2^I\right)^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \xi_{2,2} & -\xi_{2,3} & \xi_{2,4} & \cdots & (-1)^{n+1}\xi_{2,n-1} & \ddots \\ \vdots & \ddots & \xi_{3,3} & -\xi_{3,4} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \xi_{4,4} & \ddots & \xi_{n-3,n-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\xi_{n-2,n-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \xi_{n-1,n-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \Lambda_2,$$

where

$$\Lambda_2 = \begin{bmatrix} 1 & & & & & \\ & \frac{1}{\eta_{2,1}} & & & & \\ & & \frac{1}{\eta_{3,2}} & & & \\ & & & \ddots & & \\ & & & & \frac{1}{\eta_{n,n-1}} & \\ & & & & & \ddots \end{bmatrix}.$$

Define $C_2^I \in \mathbb{M}_{\infty\times\infty}$ as follows,

$$C_2^I = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{\xi_{2,2}}{\eta_{2,1}} & -\frac{\xi_{2,3}}{\eta_{3,2}} & \frac{\xi_{2,4}}{\eta_{4,3}} & \cdots & (-1)^{n-2}\frac{\xi_{2,n}}{\eta_{n,n-1}} & \ddots \\ \vdots & 0 & \frac{\xi_{3,3}}{\eta_{3,2}} & -\frac{\xi_{3,4}}{\eta_{4,3}} & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & \frac{\xi_{4,4}}{\eta_{4,3}} & \ddots & \frac{\xi_{n-2,n}}{\eta_{n,n-1}} & \ddots \\ \vdots & \ddots & \ddots & 0 & \ddots & -\frac{\xi_{n-1,n}}{\eta_{n,n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \frac{\xi_{n,n}}{\eta_{n,n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.1.7}$$

For convenience, we can define $\xi_{i,j} = 0$ for $i < j$.

Goal: Verify that

$$B_2^I C_2^I = I_\infty. \tag{5.1.8}$$

In order to show that $B_2^I C_2^I = I_\infty$, we would like to find the representation of the (i, j) entry of the matrix $B_2^I C_2^I$ for $1 \leq i, j < \infty$.

The i-th row of B_2^I:

$$\left[\underbrace{0 \;\cdots\; 0}_{i-1} \quad \eta_{i,i-1} \quad 1-\eta_{i,i-1} \quad 0 \quad \cdots \right].$$

The j-th column of C_2^I with $j > 1$:

$$\left[0 \quad (-1)^{j-2}\frac{\xi_{2,j}}{\eta_{j,j-1}} \quad (-1)^{j-3}\frac{\xi_{3,j}}{\eta_{j,j-1}} \quad \cdots \quad (-1)^{0}\frac{\xi_{j,j}}{\eta_{j,j-1}} \quad 0 \quad \cdots \right]^T.$$

For $j \geq i$, (the upper triangular entries, including the diagonal entries), the (i,j) entry is:

$$\eta_{i,i-1} \cdot (-1)^{j-i}\frac{\xi_{i,j}}{\eta_{j,j-1}} + (1-\eta_{i,i-1}) \cdot (-1)^{j-i-1}\frac{\xi_{i+1,j}}{\eta_{j,j-1}}$$

To verify that $B_2^I C_2^I = I_\infty$, we denote $A_2^I := B_2^I C_2^I$ with its entries a_{ij} at (i,j).

Main diagonal case: For $i = j$,

$$a_{ii} = \eta_{i,i-1} \cdot \frac{\xi_{i,i}}{\eta_{i,i-1}} - (1-\eta_{i,i-1}) \cdot \frac{\xi_{i+1,i}}{\eta_{i,i-1}} = \frac{1}{\eta_{i,i-1}}[\eta_{i,i-1} \cdot 1 - (1-\eta_{i,i-1}) \cdot 0] = 1.$$

The lower triangular case: For $i > j$,

$$a_{ij} = \eta_{i,i-1} \cdot (-1)^{j-i}\frac{\xi_{i,j}}{\eta_{j,j-1}} + (1-\eta_{i,i-1}) \cdot (-1)^{j-i-1}\frac{\xi_{i+1,j}}{\eta_{j,j-1}} = 0.$$

The upper triangular case: For $i < j$,

$$\begin{aligned} a_{ij} &= \eta_{i,i-1} \cdot (-1)^{j-i}\frac{\xi_{i,j}}{\eta_{j,j-1}} + (1-\eta_{i,i-1}) \cdot (-1)^{j-i-1}\frac{\xi_{i+1,j}}{\eta_{j,j-1}} \\ &= (-1)^{j-i}\frac{1}{\eta_{j,j-1}}[\eta_{i,i-1} \cdot \xi_{i,j} - (1-\eta_{i,i-1}) \cdot \xi_{i+1,j}]. \end{aligned}$$

In order to show that $a_{ij} = 0$ in this case, we just need to show that

$$\eta_{i,i-1} \cdot \xi_{i,j} - (1-\eta_{i,i-1}) \cdot \xi_{i+1,j} = 0,$$

which is equivalent to

$$\frac{\eta_{i,i-1}}{1-\eta_{i,i-1}} = \frac{\xi_{i+1,j}}{\xi_{i,j}}.$$

Indeed,

$$\frac{\eta_{i,i-1}}{1-\eta_{i,i-1}} = \frac{\dfrac{t_i - x_{i-1}}{t_i - t_{i-1}}}{1-\dfrac{t_i - x_{i-1}}{t_i - t_{i-1}}} = \frac{t_i - x_{i-1}}{x_{i-1} - t_{i-1}}.$$

On the other hand, we have

$$\frac{\xi_{i+1,j}}{\xi_{i,j}} = \frac{\left(\dfrac{x_i - t_i}{t_{i+1} - x_i}\right)\left(\dfrac{x_{i+1} - t_{i+1}}{t_{i+2} - x_{i+1}}\right)\cdots\left(\dfrac{x_{j-1} - t_{j-1}}{t_j - x_{j-1}}\right)}{\left(\dfrac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}\right)\left(\dfrac{x_i - t_i}{t_{i+1} - x_i}\right)\cdots\left(\dfrac{x_{j-1} - t_{j-1}}{t_j - x_{j-1}}\right)} = \frac{t_i - x_{i-1}}{x_{i-1} - t_{i-1}}. \quad (5.1.9)$$

Thus, this case is verified. So we can say that C_2^I is the right-inverse of B_2^I.

Now we would like to find $G_2^I \in \mathbb{M}(\infty \times \infty)$, such that $G_2^I B_2^I = I_\infty$. To this end, we take G_2^I as an upper triangular matrix, and determine its entries sub-diagonal by sub-diagonal.

Define $G_2^I \in \mathbb{M}_{\infty\times\infty}$ as follows,

$$G_2^I = \begin{bmatrix} g_{11} & g_{12} & g_{13} & \cdots & \cdots & g_{1,n} & \cdots \\ 0 & g_{22} & g_{23} & g_{24} & \cdots & g_{2,n} & \ddots \\ \vdots & 0 & g_{33} & g_{34} & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & g_{44} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \ddots & g_{n-1,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & g_{n,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \quad (5.1.10)$$

we would like to find the representation of the (i,j) entry of the matrix $G_2^I B_2^I$ for $1 \le i,j < \infty$.

The i-th row of G_2^I with $i \ge 1$:

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{i-1} & g_{i,i} & g_{i,i+1} & \cdots & g_{i,n} & \cdots \end{bmatrix}.$$

The j-th column of B_2^I with $j > 2$:

$$\begin{bmatrix} \underbrace{0 \ \cdots \ 0}_{j-2} & 1-\eta_{j-1,j-2} & \eta_{j,j-1} & 0 & \cdots \end{bmatrix}^T.$$

To ensure that $G_2^I B_2^I = I_\infty$, we denote $A_2^I := G_2^I B_2^I$ with its entries a_{ij} at (i,j).

Main diagonal case: For $i = j$ and $i > 2$,

$$a_{ii} = 0 \cdot (1 - \eta_{i-1,i-2}) + g_{i,i} \cdot \eta_{i,i-1} = 1 \Rightarrow g_{i,i} = \frac{1}{\eta_{i,i-1}}.$$

The lower triangular case: For $i > j > 2$, i.e. $i - 1 \geq j$,

$$a_{ij} = 0 \cdot (1 - \eta_{j-1,j-2}) + 0 \cdot \eta_{j,j-1} = 0.$$

It is easy to see that

$$a_{i2} = 0 \quad \text{for} \quad i > 2, \quad \text{and} \quad a_{i1} = 0 \quad \text{for} \quad i > 1.$$

The upper triangular case: For $j - i = 1$,

$$a_{ij} = (1 - \eta_{j-1,j-2}) \cdot g_{i,i} + \eta_{j,j-1} \cdot g_{i,i+1} = \frac{1 - \eta_{i,i-1}}{\eta_{i,i-1}} + \eta_{i+1,i} \cdot g_{i,i+1} = 0.$$

Thus,

$$g_{i,i+1} = -\frac{1 - \eta_{i,i-1}}{\eta_{i,i-1}\,\eta_{i+1,i}} = -\frac{1 - \dfrac{t_i - x_{i-1}}{t_i - t_{i-1}}}{\dfrac{t_i - x_{i-1}}{t_i - t_{i-1}}\,\dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}}$$

$$= -\frac{\dfrac{x_{i-1} - t_{i-1}}{t_i - t_{i-1}}}{\dfrac{t_i - x_{i-1}}{t_i - t_{i-1}}\,\dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}} = -\frac{t_{i+1} - t_i}{t_{i+1} - x_i}\,\frac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}.$$

For $j - i = 2$,

$$a_{ij} = (1 - \eta_{j-1,j-2}) \cdot g_{i,i+1} + \eta_{j,j-1} \cdot g_{i,i+2} = (1 - \eta_{i+1,i}) \cdot g_{i,i+1} + \eta_{i+2,i+1} \cdot g_{i,i+2} = 0.$$

Thus,

$$g_{i,i+2} = -\frac{(1 - \eta_{i+1,i}) \cdot g_{i,i+1}}{\eta_{i+2,i+1}} = \frac{\left(1 - \dfrac{t_{i+1} - x_i}{t_{i+1} - t_i}\right) \dfrac{t_{i+1} - t_i}{t_{i+1} - x_i}\,\dfrac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}}{\dfrac{t_{i+2} - x_{i+1}}{t_{i+2} - t_{i+1}}}$$

$$= \frac{\left(\dfrac{x_i - t_i}{t_{i+1} - t_i}\right) \dfrac{t_{i+1} - t_i}{t_{i+1} - x_i}\,\dfrac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}}{\dfrac{t_{i+2} - x_{i+1}}{t_{i+2} - t_{i+1}}} = \frac{t_{i+2} - t_{i+1}}{t_{i+2} - x_{i+1}}\,\frac{x_i - t_i}{t_{i+1} - x_i}\,\frac{x_{i-1} - t_{i-1}}{t_i - x_{i-1}}.$$

For $j-i=3$,

$$a_{ij}=(1-\eta_{j-1,j-2})\cdot g_{i,i+2}+\eta_{j,j-1}\cdot g_{i,i+3}=(1-\eta_{i+2,i+1})\cdot g_{i,i+2}+\eta_{i+3,i+2}\cdot g_{i,i+3}=0.$$

Thus,

$$g_{i,i+3}=-\frac{(1-\eta_{i+2,i+1})\cdot g_{i,i+2}}{\eta_{i+3,i+2}}=-\frac{\left(1-\dfrac{t_{i+2}-x_{i+1}}{t_{i+2}-t_{i+1}}\right)\dfrac{t_{i+2}-t_{i+1}}{t_{i+2}-x_{i+1}}\dfrac{x_i-t_i}{t_{i+1}-x_i}\dfrac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}}{\dfrac{t_{i+3}-x_{i+2}}{t_{i+3}-t_{i+2}}}$$

$$=-\frac{\left(\dfrac{x_{i+1}-t_{i+1}}{t_{i+2}-t_{i+1}}\right)\dfrac{t_{i+2}-t_{i+1}}{t_{i+2}-x_{i+1}}\dfrac{x_i-t_i}{t_{i+1}-x_i}\dfrac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}}{\dfrac{t_{i+3}-x_{i+2}}{t_{i+3}-t_{i+2}}}$$

$$=-\frac{t_{i+3}-t_{i+2}}{t_{i+3}-x_{i+2}}\frac{x_{i+1}-t_{i+1}}{t_{i+2}-x_{i+1}}\frac{x_i-t_i}{t_{i+1}-x_i}\frac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}.$$

We can write the general formula for $g_{i,i+k}$ with $k\geq 1$ as follows,

$$g_{i,i+k}=(-1)^k\frac{t_{i+k}-t_{i+k-1}}{t_{i+k}-x_{i+k-1}}\frac{x_{i+k-2}-t_{i+k-2}}{t_{i+k-1}-x_{i+k-2}}\cdots\frac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}. \tag{5.1.11}$$

We have verified that it is true for $k=1,2,3$. Based on mathematical induction, assume that it is true for $1\leq k\leq s$ with $s\geq 3$. Now we consider the case that $j-i=s+1$. We can write

$$a_{ij}=(1-\eta_{j-1,j-2})\cdot g_{i,i+s}+\eta_{j,j-1}\cdot g_{i,i+s+1}=(1-\eta_{i+s,i+s-1})\cdot g_{i,i+s}+\eta_{i+s+1,i+s}\cdot g_{i,i+s+1}=0.$$

Thus,

$$g_{i,i+s+1}=-\frac{(1-\eta_{j-1,j-2})\cdot g_{i,i+s}}{\eta_{i+s+1,i+s}}$$

$$=-\frac{\left(1-\dfrac{t_{i+s}-x_{i+s-1}}{t_{i+s}-t_{i+s-1}}\right)}{\dfrac{t_{i+s+1}-x_{i+s}}{t_{i+s+1}-t_{i+s}}}(-1)^s\frac{t_{i+s}-t_{i+s-1}}{t_{i+s}-x_{i+s-1}}\frac{x_{i+s-2}-t_{i+s-2}}{t_{i+s-1}-x_{i+s-2}}\cdots\frac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}$$

$$=(-1)^{s+1}\frac{t_{i+s+1}-t_{i+s}}{t_{i+s+1}-x_{i+s}}\frac{x_{i+s-1}-t_{i+s-1}}{t_{i+s}-x_{i+s-1}}\frac{x_{i+s-2}-t_{i+s-2}}{t_{i+s-1}-x_{i+s-2}}\cdots\frac{x_{i-1}-t_{i-1}}{t_i-x_{i-1}}.$$

We complete the verification.

5.2 *Inverse Shoenberg-Whitney Matrix Case II:*

- *Case II*: We consider two subcases: A) $\sigma_1 = \sigma_2 \cdots = \sigma_n = \cdots = 1$; B) $\sigma_1 = \sigma_2 = \cdots = \sigma_k = 0$ and $\sigma_{k+1} = \cdots = \sigma_n = \cdots = 1$.

Subcase A: We have that $x_j \in [t_{j-1}, t_j)$ for $j = 1, 2, \ldots, n, \ldots$, and get

$$\begin{cases} B_{j-1,2}(x_j) = \dfrac{t_j - x_j}{t_j - t_{j-1}}, \\ B_{j,2}(x_j) = \dfrac{x_j - t_{j-1}}{t_j - t_{j-1}}, \\ B_{j+1,2}(x_j) = 0. \end{cases} \tag{5.2.1}$$

$\mathcal{B}_2$ is a lower bidiagonal matrix in the form of

$$B_2^{II} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{t_1 - x_1}{t_1 - t_0} & \frac{x_1 - t_0}{t_1 - t_0} & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \frac{t_2 - x_2}{t_2 - t_1} & \frac{x_2 - t_1}{t_2 - t_1} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \frac{t_3 - x_3}{t_3 - t_2} & \frac{x_3 - t_2}{t_3 - t_2} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{t_n - x_n}{t_n - t_{n-1}} & \frac{x_n - t_{n-1}}{t_n - t_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.2}$$

We can write B_2^{II} in a simpler version:

$$B_2^{II} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \eta_{1,1} & 1 - \eta_{1,1} & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \eta_{2,2} & 1 - \eta_{2,2} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \eta_{3,3} & 1 - \eta_{3,3} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \eta_{n,n} & 1 - \eta_{n,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.3}$$

To find the left-inverse and the right-inverse of B_2^{II}, we define C_2^{II} and G_2^{II} as the following two lower tridiagonal matrices:

$$C_2^{II} = \begin{bmatrix} c_{11} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{21} & c_{22} & 0 & \ddots & \ddots & \ddots & \ddots \\ c_{31} & c_{32} & c_{33} & 0 & \ddots & \ddots & \ddots \\ \vdots & c_{42} & c_{43} & c_{44} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ c_{n,1} & c_{n,2} & \ddots & \ddots & \ddots & c_{n,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.2.4}$$

and

$$G_2^{II} = \begin{bmatrix} g_{11} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{21} & g_{22} & 0 & \ddots & \ddots & \ddots & \ddots \\ g_{31} & g_{32} & g_{33} & 0 & \ddots & \ddots & \ddots \\ \vdots & g_{42} & g_{43} & g_{44} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ g_{n,1} & g_{n,2} & \ddots & \ddots & \ddots & g_{n,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.2.5}$$

such that

$$B_2^{II} C_2^{II} = G_2^{II} B_2^{II} = I_\infty. \tag{5.2.6}$$

To ensure that $B_2^{II} C_2^{II} = I_\infty$, we denote $A_2^{II} := B_2^{II} C_2^{II}$ with its entries a_{ij} at (i, j) for $1 \leq i, j < \infty$. Notice that $a_{ij} = 0$ for $j > i$ with the given structures of B_2^{II} and C_2^{II}. We only consider the cases for a_{ij} with $i \geq j$.

Main diagonal case: For $i = j$,

$$a_{ii} = (1 - \eta_{i-1,i-1}) \cdot c_{ii} = 1$$

with $\eta_{00} = 0$ for convenience. Based on our assumption $t_{i-1} \leq x_i < t_i$, we have that $0 \leq \eta_{i,i} < 1$. Thus,

$$c_{ii} = \frac{1}{1 - \eta_{i-1,i-1}}, \qquad \text{for} \quad 1 \leq i < \infty.$$

The lower triangular case: For $i = j + 1$,

$$a_{i+1,i} = \eta_{ii} \cdot c_{ii} + (1 - \eta_{i,i}) \cdot c_{i+1,i} = 0,$$

which results in

$$c_{i+1,i} = -\frac{\eta_{ii}}{1-\eta_{ii}}\frac{1}{1-\eta_{i-1,i-1}}.$$

For $i = j+2$,

$$a_{i+2,i} = \eta_{i+1,i+1} \cdot c_{i+1,i} + (1-\eta_{i+1,i+1}) \cdot c_{i+2,i} = 0,$$

which results in

$$c_{i+2,i} = (-1)^2 \frac{\eta_{i+1,i+1}}{1-\eta_{i+1,i+1}}\frac{\eta_{ii}}{1-\eta_{ii}}\frac{1}{1-\eta_{i-1,i-1}}.$$

For $i = j+k$, we can get

$$c_{i+k,i} = (-1)^k \frac{\eta_{i+k-1,i+k-1}}{1-\eta_{i+k-1,i+k-1}} \cdots \frac{\eta_{ii}}{1-\eta_{ii}}\frac{1}{1-\eta_{i-1,i-1}}.$$

Similarly, to ensure that $G_2^{II} B_2^{II} = I_\infty$, we still denote $A_2^{II} := G_2^{II} B_2^{II}$ with its entries a_{ij} at (i,j) for $1 \le i,j < \infty$ without causing much confusion. Notice that $a_{ij} = 0$ for $j > i$ with the given structures of B_2^{II} and G_2^{II}. We only consider the cases for a_{ij} with $i \ge j$.

Main diagonal case: For $i = j$,

$$a_{ii} = g_{ii} \cdot (1-\eta_{i-1,i-1}) = 1$$

with $\eta_{00} = 0$ for convenience. Based on our assumption $t_{i-1} \le x_i < t_i$, we have that $0 \le \eta_{i,i} < 1$. Thus,

$$g_{ii} = \frac{1}{1-\eta_{i-1,i-1}}, \qquad \text{for} \quad 1 \le i < \infty.$$

The lower triangular case: For $i = j+1$,

$$a_{i+1,i} = g_{i+1,i} \cdot (1-\eta_{i-1,i-1}) + g_{i+1,i+1} \cdot \eta_{ii} = 0,$$

which results in

$$g_{i+1,i} = -\frac{\eta_{ii}}{1-\eta_{i-1,i-1}}\frac{1}{1-\eta_{i,i}}.$$

For $i = j+2$,

$$a_{i+2,i} = g_{i+2,i} \cdot (1-\eta_{i-1,i-1}) + g_{i+2,i+1} \cdot \eta_{i,i} = 0,$$

which results in

$$g_{i+2,i} = (-1)^2 \frac{\eta_{i,i}}{1-\eta_{i-1,i-1}}\frac{\eta_{i+1,i+1}}{1-\eta_{i+1,i+1}}\frac{1}{1-\eta_{i,i}}.$$

For $i = j + k$, we can get

$$g_{i+k,i} = (-1)^k \frac{\eta_{i+k-1,i+k-1}}{1 - \eta_{i+k-1,i+k-1}} \cdots \frac{\eta_{ii}}{1 - \eta_{ii}} \frac{1}{1 - \eta_{i-1,i-1}}.$$

Here we notice that $g_{i+k,i} = c_{i+k,i}$ (We need to double-check this case later).

Subcase B: In this case, we have that $x_j \in [t_j, t_{j+1})$ for $j = 1, 2, \ldots, k$, and get

$$\begin{cases} B_{j-1,2}(x_j) = 0, \\ B_{j,2}(x_j) = \dfrac{t_{j+1} - x_j}{t_{j+1} - t_j}, \\ B_{j+1,2}(x_j) = \dfrac{x_j - t_j}{t_{j+1} - t_j}. \end{cases} \tag{5.2.7}$$

We also have that $x_j \in (t_{j-1}, t_j)$ for $j = k+1, \ldots, n, \ldots$, and get

$$\begin{cases} B_{j-1,2}(x_j) = \dfrac{t_j - x_j}{t_j - t_{j-1}}, \\ B_{j,2}(x_j) = \dfrac{x_j - t_{j-1}}{t_j - t_{j-1}}, \\ B_{j+1,2}(x_j) = 0. \end{cases} \tag{5.2.8}$$

$\mathcal{B}_2$ becomes a special tridiagonal matrix in the form of

$$B_2^{II} =$$

$$\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{t_2 - x_1}{t_2 - t_1} & \frac{x_1 - t_1}{t_2 - t_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & \frac{t_3 - x_2}{t_3 - t_2} & \frac{x_2 - t_2}{t_3 - t_2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & \frac{t_{k+1} - x_k}{t_{k+1} - t_k} & \frac{x_k - t_k}{t_{k+1} - t_k} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \frac{t_{k+1} - x_{k+1}}{t_{k+1} - t_k} & \frac{x_{k+1} - t_k}{t_{k+1} - t_k} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \frac{t_{k+2} - x_{k+2}}{t_{k+2} - t_{k+1}} & \frac{x_{k+2} - t_{k+1}}{t_{k+2} - t_{k+1}} & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_n - x_n}{t_n - t_{n-1}} & \frac{x_n - t_{n-1}}{t_n - t_{n-1}} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}$$

In order to make the inverse calculation easier to handle, we would like to work on a special example with concrete numbers first. We will calculate the left-inverse and the right-inverse for the following infinite matrix:

$$B_2^{II} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/3 & 1/3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1/3 & 2/3 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.2.9}$$

where we take the knots $\{t_j\}_{j=0}^{\infty}$ and the sample locations $\{x_j\}_{j=0}^{\infty}$ as follows:

$$t_j = j, \qquad \text{for} \quad j = 0, 1, 2, \ldots,$$

$$x_0 = 0, \quad x_1 = \frac{3}{2}, \quad x_2 = \frac{5}{2}, \quad x_3 = \frac{7}{2}, \quad x_4 = \frac{13}{3}, \quad x_5 = \frac{14}{3},$$

and

$$x_j = j - \frac{1}{2}, \qquad j = 6, 7, 8, \ldots.$$

Now we apply a rotation transform R_θ on B_2^{II} from the left with the following rotation matrix

$$R_\theta = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \cos\theta & \sin\theta & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\sin\theta & \cos\theta & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.2.10}$$

and get

$$R_\theta \, B_2^{II} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & (2\cos\theta+\sin\theta)/3 & (\cos\theta+2\sin\theta)/3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & (\cos\theta-2\sin\theta)/3 & (2\cos\theta-\sin\theta)/3 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.11}$$

We would like to choose $\theta = \tan^{-1}(1/2)$ to make $\cos\theta - 2\sin\theta = 0$. Let $\theta_0 = \tan^{-1}(1/2)$. We update the above matrix as follows,

$$R_{\theta_0} B_2^{II} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \sqrt{5}/3 & 4\sqrt{5}/15 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \sqrt{5}/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.12}$$

To make the non-zero entry in the 6th row to 1, we multiply the above matrix with a scale matrix from the left:

$$S_{\sqrt{5}} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \sqrt{5} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.13}$$

Now we have the following matrix:

$$S_{\sqrt{5}}\, R_{\theta_0}\, B_2^{II} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \sqrt{5}/3 & 4\sqrt{5}/15 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.2.14}$$

which can be written as

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix},$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & \sqrt{5}/3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots \\ 4\sqrt{5}/15 & 0 & \cdots & \cdots & \cdots \end{bmatrix},$$

and

$$B = \begin{bmatrix} 1 & 0 & \ddots & \ddots & \ddots \\ 1/2 & 1/2 & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 & \ddots \\ \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Notice that

$$\begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = I.$$

Let us find A^{-1} and B^{-1} first. We get

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -2 & 2 & -3/\sqrt{5} \\ 0 & 0 & 2 & -2 & 3/\sqrt{5} \\ 0 & 0 & 0 & 2 & -3/\sqrt{5} \\ 0 & 0 & 0 & 0 & 3/\sqrt{5} \end{bmatrix},$$

and

$$B^{-1} = \begin{bmatrix} 1 & 0 & \ddots & \ddots & \ddots \\ -1 & 2 & 0 & \ddots & \ddots \\ 1 & -2 & 2 & 0 & \ddots \\ -1 & 2 & -2 & 2 & \ddots \\ \ddots & -2 & 2 & \ddots & \ddots \end{bmatrix}.$$

Then we calculate

$$A^{-1}CB^{-1} = A^{-1}C = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots \\ -4/5 & 0 & \ddots & \ddots & \ddots \\ 4/5 & 0 & \ddots & \ddots & \ddots \\ -4/5 & 0 & \ddots & \ddots & \ddots \\ 4/5 & 0 & \cdots & \cdots & \cdots \end{bmatrix}.$$

We put the above three matrices together, and get

$$\begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -3/\sqrt{5} & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 3/\sqrt{5} & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -3/\sqrt{5} & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 3/\sqrt{5} & 4/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Now we can calculate C_2^{II}, which is

$$G_2^{II} = \begin{bmatrix} A^{-1} & -A^{-1}CB^{-1} \\ 0 & B^{-1} \end{bmatrix} S_{\sqrt{5}}\, R_{\theta_0}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -3/\sqrt{5} & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 3/\sqrt{5} & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -3/\sqrt{5} & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 3/\sqrt{5} & 4/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \sqrt{5} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \times$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/\sqrt{5} & 1/\sqrt{5} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1/\sqrt{5} & 2/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 2 & -2 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 2 & -2 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 2/5 & 1/5 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 2 & -2 & 2 & -3/\sqrt{5} & -4/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 2 & -2 & 3/\sqrt{5} & 4/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 2 & -3/\sqrt{5} & -4/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 3/\sqrt{5} & 4/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & \sqrt{5} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & -\sqrt{5} & 2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & \sqrt{5} & -2 & 2 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & -\sqrt{5} & 2 & -2 & 2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 2/\sqrt{5} & 1/\sqrt{5} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1/\sqrt{5} & 2/\sqrt{5} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
$$

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 2 & -2 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 2 & -2 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

Finally, we need to verify that $G_2^{II}B_2^{II} = I_\infty$, i.e.

$$
= \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\
0 & 2 & -2 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 2 & -2 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 2/3 & 1/3 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1/3 & 2/3 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

(*When we do verification using the above matrix, it does not give us the inverse matrix. We can use this example as an counter example against the associative law for the infinite matrices.*)

Now we use the elementary transformations to calculate the inverse of B_2^{II}, and get

$$
C_2^{II} = \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 2 & -2 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 2 & -2 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & 0 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}. \tag{5.2.15}
$$

Next, we verify if $B_2^{II}C_2^{II} = I_\infty$.

$$
B_2^{II}C_2^{II} =
$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/3 & 1/3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1/3 & 2/3 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 2 & -2 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$= I_\infty.$$

Consider

$$C_2^{II} B_2^{II} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 2 & -2 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 2 & -2 & 1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2 & -1 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/3 & 1/3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1/3 & 2/3 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$= I_\infty.$$

That is, C_2^{II} is the inverse of B_2^{II} in this example.

Next, we consider the following general case:

$$B_2^{II} =$$

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \eta_{21} & 1-\eta_{21} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & \eta_{32} & 1-\eta_{32} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \eta_{k+1,k} & 1-\eta_{k+1,k} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \eta_{k+1,k+1} & 1-\eta_{k+1,k+1} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \eta_{k+2,k+2} & 1-\eta_{k+2,k+2} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \eta_{n,n} & 1-\eta_{n,n} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.2.16}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2-x_1}{t_2-t_1} & \frac{x_1-t_1}{t_2-t_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & \frac{t_3-x_2}{t_3-t_2} & \frac{x_2-t_2}{t_3-t_2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \frac{t_{k+1}-x_k}{t_{k+1}-t_k} & \frac{x_k-t_k}{t_{k+1}-t_k} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \frac{t_{k+1}-x_{k+1}}{t_{k+1}-t_k} & \frac{x_{k+1}-t_k}{t_{k+1}-t_k} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{t_{k+2}-x_{k+2}}{t_{k+2}-t_{k+1}} & \frac{x_{k+2}-t_{k+1}}{t_{k+2}-t_{k+1}} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_n-x_n}{t_n-t_{n-1}} & \frac{x_n-t_{n-1}}{t_n-t_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.17}$$

Next, we do the elementary transformations to get the inverse of B_2^{II}.

$$\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \frac{x_1-t_1}{t_2-x_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & \frac{x_2-t_2}{t_3-x_2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & \frac{(t_{k+1}-t_k)(x_{k+1}-x_k)}{(t_{k+1}-x_k)(x_{k+1}-t_k)} & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \frac{t_{k+1}-x_{k+1}}{x_{k+1}-t_k} & 1 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \frac{t_{k+2}-x_{k+2}}{x_{k+2}-t_{k+1}} & 1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_n-x_n}{x_n-t_{n-1}} & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{5.2.18}$$

$$\sim \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{t_2-t_1}{t_2-x_1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & \frac{t_3-t_2}{t_3-x_2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & \frac{t_{k+1}-t_k}{t_{k+1}-x_k} & -\frac{(t_{k+1}-t_k)(x_k-t_k)}{(t_{k+1}-x_k)(x_{k+1}-t_k)} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & \frac{t_{k+1}-t_k}{x_{k+1}-t_k} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_{k+2}-t_{k+1}}{x_{k+2}-t_{k+1}} & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_n-t_{n-1}}{x_n-t_{n-1}} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}. \tag{5.2.19}$$

$$
\rightarrow \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \frac{x_1-t_1}{t_2-x_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & \frac{x_2-t_2}{t_3-x_2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \frac{t_{k+1}-x_{k+1}}{x_{k+1}-t_k} & 1 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \frac{t_{k+2}-x_{k+2}}{x_{k+2}-t_{k+1}} & 1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_n-x_n}{x_n-t_{n-1}} & 1 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{5.2.20}
$$

$$
\sim \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{t_2-t_1}{t_2-x_1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & \frac{t_3-t_2}{t_3-x_2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & \frac{x_{k+1}-t_k}{x_{k+1}-x_k} & -\frac{x_k-t_k}{x_{k+1}-x_k} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & \frac{t_{k+1}-t_k}{x_{k+1}-t_k} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_{k+2}-t_{k+1}}{x_{k+2}-t_{k+1}} & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_n-t_{n-1}}{x_n-t_{n-1}} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix} \tag{5.2.21}
$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \frac{x_1-t_1}{t_2-x_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & \frac{x_2-t_2}{t_3-x_2} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \frac{t_{k+2}-x_{k+2}}{x_{k+2}-t_{k+1}} & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_n-x_n}{x_n-t_{n-1}} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.2.22}$$

$$\sim \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2-t_1}{t_2-x_1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & \frac{t_3-t_2}{t_3-x_2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & \frac{x_{k+1}-t_k}{x_{k+1}-x_k} & -\frac{x_k-t_k}{x_{k+1}-x_k} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -\frac{t_{k+1}-x_{k+1}}{x_{k+1}-x_k} & \frac{t_{k+1}-x_k}{x_{k+1}-x_k} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_{k+2}-t_{k+1}}{x_{k+2}-t_{k+1}} & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \frac{t_n-t_{n-1}}{x_n-t_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.2.23}$$

To see the transformation chain better, we split the above two matrices into upper and lower blocks, and do the transformations respectively.

In order to handle the complexity of the computation gracefully, we introduce the following notations:

$$\gamma_j = \frac{x_{j-1} - t_{j-1}}{t_j - x_{j-1}}, \qquad \phi_j = \frac{t_{j-1} - x_{j-1}}{x_j - x_{j-1}}.$$

Upper Part:

$$
\begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \frac{x_1-t_1}{t_2-x_1} & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & \frac{x_2-t_2}{t_3-x_2} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1 & \frac{x_{k-2}-t_{k-2}}{t_{k-1}-x_{k-2}} & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1 & 0
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{t_2-t_1}{t_2-x_1} & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & \frac{t_3-t_2}{t_3-x_2} & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & \frac{t_{k-1}-t_{k-2}}{t_{k-1}-x_{k-2}} & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & \frac{x_{k+1}-t_k}{x_{k+1}-x_k} & -\frac{x_k-t_k}{x_{k+1}-x_k}
\end{bmatrix}
$$

$$
\to \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \gamma_2 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & \gamma_3 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1 & \gamma_k & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1 & 0
\end{bmatrix}
\sim \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1+\gamma_2 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1+\gamma_3 & 0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1+\gamma_k & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1-\phi_{k+1} & \phi_{k+1}
\end{bmatrix}
$$

$$
\to \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \gamma_2 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1 & \gamma_3 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0
\end{bmatrix}
$$

$$
\sim \begin{bmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1+\gamma_2 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & 0 & 1+\gamma_3 & 0 & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & 0 & 1+\gamma_{k-1} & -\gamma_{k-1}(1+\gamma_k) & \gamma_{k-1}\gamma_k(1-\phi_{k+1}) & \gamma_{k-1}\gamma_k\phi_{k+1} \\
\vdots & \ddots & \ddots & \ddots & 0 & 1+\gamma_k & -\gamma_k(1-\phi_{k+1}) & -\gamma_k\phi_{k+1} \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1-\phi_{k+1} & \phi_{k+1}
\end{bmatrix}
$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \gamma_2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & \gamma_3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1+\gamma_2 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1+\gamma_3 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1+\gamma_{k-1} & -\gamma_{k-1}(1+\gamma_k) & \gamma_{k-1}\gamma_k(1-\phi_{k+1}) & \gamma_{k-1}\gamma_k\phi_{k+1} \\ \vdots & \ddots & \ddots & \ddots & 0 & 1+\gamma_k & -\gamma_k(1-\phi_{k+1}) & -\gamma_k\phi_{k+1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1-\phi_{k+1} & \phi_{k+1} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1+\gamma_2 & -\gamma_2(1+\gamma_3) & \gamma_2\gamma_3(1+\gamma_4) & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1+\gamma_3 & -\gamma_3(1+\gamma_4) & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1+\gamma_j & -\gamma_j(1+\gamma_{j+1}) & \ddots & \gamma_j\cdots\gamma_k(1-\phi_{k+1}) & \gamma_j\cdots\gamma_k\phi_{k+1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1+\gamma_k & -\gamma_k(1-\phi_{k+1}) & -\gamma_k\phi_{k+1} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1-\phi_{k+1} & \phi_{k+1} \end{bmatrix}.$$

Its general row can be written as:

$$[\underbrace{0,\ \cdots\ 0,}_{j-1}\ 1+\gamma_j,\ -\gamma_j(1+\gamma_{j+1}),\ \cdots\ ,\ (-1)^{k+1-j}\gamma_j\cdots\gamma_k(1-\phi_{k+1}),\ (-1)^{k+1-j}\gamma_j\cdots\gamma_k\phi_{k+1}].$$

Lower Part:

We still introduce some special notations to make the computation easier to handle:

$$\omega_j = \frac{t_j - x_j}{x_j - t_{j-1}}, \qquad \psi_j = \frac{t_j - x_{j-1}}{x_j - x_{j-1}}.$$

$$\begin{bmatrix} 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \ddots & \frac{t_{k+2}-x_{k+2}}{x_{k+2}-t_{k+1}} & 1 & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \frac{t_n-x_n}{x_n-t_{n-1}} & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\sim \begin{bmatrix} -\frac{t_{k+1}-x_{k+1}}{x_{k+1}-x_k} & \frac{t_{k+1}-x_k}{x_{k+1}-x_k} & 0 & \ddots & \ddots & \ddots \\ \ddots & 0 & \frac{t_{k+2}-t_{k+1}}{x_{k+2}-t_{k+1}} & 0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 & \frac{t_n-t_{n-1}}{x_n-t_{n-1}} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \omega_{k+2} & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \omega_{k+3} & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \omega_j & 1 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \omega_n & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\sim \begin{bmatrix} 1-\psi_{k+1} & \psi_{k+1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & 1+\omega_{k+2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & 0 & 1+\omega_{k+3} & 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 & 1+\omega_j & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1+\omega_n & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \omega_{k+3} & 1 & 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \omega_j & 1 & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \omega_n & 1 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.2.24}$$

$$\sim \begin{bmatrix} 1-\psi_{k+1} & \psi_{k+1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ -\omega_{k+2}(1-\psi_{k+1}) & -\omega_{k+2}\psi_{k+1} & 1+\omega_{k+2} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & 0 & 1+\omega_{k+3} & 0 & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & 0 & 1+\omega_j & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1+\omega_n & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.2.25}$$

5.3 *Inverse Shoenberg-Whitney Matrix Case III:*

- *Case III*: $\sigma_1 = \sigma_3 = \cdots = \sigma_{2k-1} = \cdots = 0$ and $\sigma_2 = \sigma_4 = \cdots = \sigma_{2k} = \cdots = 1$.

In this case, we have that $x_{2k-1} \in [t_{2k-1}, t_{2k})$ for all $k = 1, 2, \ldots$, and get

$$\begin{cases} B_{2k-2,2}(x_{2k-1}) = 0, \\ B_{2k-1,2}(x_{2k-1}) = \dfrac{t_{2k} - x_{2k-1}}{t_{2k} - t_{2k-1}}, \\ B_{2k+1,2}(x_{2k-1}) = \dfrac{x_{2k-1} - t_{2k-1}}{t_{2k} - t_{2k-1}}. \end{cases} \tag{5.3.1}$$

We also have that $x_{2k} \in (t_{2k-1}, t_{2k})$ for $k = 1, 2, \ldots$, and get

$$\begin{cases} B_{2k-1,2}(x_{2k}) = \dfrac{t_{2k} - x_{2k}}{t_{2k} - t_{2k-1}}, \\ B_{2k,2}(x_{2k}) = \dfrac{x_{2k} - t_{2k-1}}{t_{2k} - t_{2k-1}}, \\ B_{2k+1,2}(x_{2k}) = 0. \end{cases} \tag{5.3.2}$$

$\mathcal{B}_2$ becomes a block diagonal matrix in the form of

$$B_2^{III} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{t_2 - x_1}{t_2 - t_1} & \frac{x_1 - t_1}{t_2 - t_1} & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \frac{t_2 - x_2}{t_2 - t_1} & \frac{x_2 - t_1}{t_2 - t_1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & \frac{t_4 - x_3}{t_4 - t_3} & \frac{x_3 - t_3}{t_4 - t_3} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \frac{t_4 - x_4}{t_4 - t_3} & \frac{x_4 - t_3}{t_4 - t_3} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_{n-2} - x_{n-3}}{t_{n-2} - t_{n-3}} & \frac{x_{n-3} - t_{n-3}}{t_{n-2} - t_{n-3}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{t_{n-2} - x_{n-2}}{t_{n-2} - t_{n-3}} & \frac{x_{n-2} - t_{n-3}}{t_{n-2} - t_{n-3}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.3.3}$$

Denote

$$D_j = \begin{bmatrix} \dfrac{t_j - x_{j-1}}{t_j - t_{j-1}} & \dfrac{x_{j-1} - t_{j-1}}{t_j - t_{j-1}} \\ \\ \dfrac{t_j - x_j}{t_j - t_{j-1}} & \dfrac{x_j - t_{j-1}}{t_j - t_{j-1}} \end{bmatrix}. \tag{5.3.4}$$

Then we can write B_2^{III} as

$$B_2^{III} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & D_2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & D_4 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & D_n & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.3.5}$$

Thus the inverse of B_2^{III} can be written as

$$\left(B_2^{III}\right)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & D_2^{-1} & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & D_4^{-1} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & D_n^{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.3.6}$$

It is easy to fnd that

$$D_j^{-1} = \begin{bmatrix} \dfrac{x_j - t_{j-1}}{x_j - x_{j-1}} & \dfrac{t_{j-1} - x_{j-1}}{x_j - x_{j-1}} \\ \\ \dfrac{x_j - t_j}{x_j - x_{j-1}} & \dfrac{t_j - x_{j-1}}{x_j - x_{j-1}} \end{bmatrix}.$$

Now we can write $\left(B_2^{III}\right)^{-1}$ explicitly as follows,

$$\left(B_2^{III}\right)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{x_2 - t_1}{x_2 - x_1} & \frac{t_1 - x_1}{x_2 - x_1} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \frac{x_2 - t_2}{x_2 - x_1} & \frac{t_2 - x_1}{x_2 - x_1} & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & \frac{x_4 - t_3}{x_4 - x_3} & \frac{t_3 - x_3}{x_4 - x_3} & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \frac{x_4 - t_4}{x_4 - x_3} & \frac{t_4 - x_3}{x_4 - x_3} & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{x_n - t_{n-1}}{x_n - x_{n-1}} & \frac{t_{n-1} - x_{n-1}}{x_n - x_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{x_n - t_n}{x_n - x_{n-1}} & \frac{t_n - x_{n-1}}{x_n - x_{n-1}} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{5.3.7}$$

5.4 ***Inverse Shoenberg-Whitney Matrix Case IV:***

Case IV: Assume that $\sigma_1 = \cdots = \sigma_{k_1} = 0, \sigma_{k_1+1} = \cdots = \sigma_{k_2} = 1, \cdots, \sigma_{k_{2s-2}+1} = \cdots = \sigma_{k_{2s-1}} = 0, \sigma_{k_{2s-1}+1} = \cdots = \sigma_{k_{2s}} = 1, \cdots$.

Note: In order to make the discussion a little easier, we allow $k_1 = 0$. When it happens, we have $\sigma_1 = \cdots = \sigma_{k_2} = 1$ and we treat the part $\sigma_1 = \cdots = \sigma_{k_1} = 0$ empty.

Under this assumption, $\mathcal{B}_2$ can be written as a block-diagonal matrix as follows,

$$B_2^{IV} := \begin{bmatrix} \mathcal{B}_2^1 & 0 & \cdots & \cdots & \cdots \\ 0 & \mathcal{B}_2^2 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathcal{B}_2^s & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{5.4.1}$$

where $\mathcal{B}_2^1$ has different structures based on the value of k_1.

More specifically, when $k_1 > 0$, we have a tridiagonal matrix

$$\mathcal{B}_2^1 := \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & b_{11} & b_{21} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & b_{k_1,k_1} & b_{k_1+1,k_1} & 0 & 0 \\ \vdots & \ddots & \ddots & b_{k_1,k_1+1} & b_{k_1+1,k_1+1} & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & b_{k_2-1,k_2} & b_{k_2,k_2} \end{bmatrix}. \tag{5.4.2}$$

When $k_1 = 0$, its structure becomes a lower bidiagonal matrix

$$\mathcal{B}_2^1 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ b_{01} & b_{11} & 0 & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & b_{k_2-1,k_2} & b_{k_2,k_2} \end{bmatrix}. \tag{5.4.3}$$

For $\mathcal{B}_2^s$ with $s > 1$, we have a tridiagonal matrix. In order to see the structure clearly, we write the case for $s = 2$ as follows

$$\mathcal{B}_2^2 := \begin{bmatrix} b_{k_2+1,k_2+1} & b_{k_2+2,k_2+1} & 0 & 0 & 0 & 0 \\ 0 & \ddots & \ddots & 0 & 0 & 0 \\ 0 & \ddots & b_{k_3,k_3} & b_{k_3+1,k_3} & 0 & 0 \\ 0 & \ddots & b_{k_3,k_3+1} & b_{k_3+1,k_3+1} & 0 & \ddots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \cdots & b_{k_4-1,k_4} & b_{k_4,k_4} \end{bmatrix}. \tag{5.4.4}$$

We can use $\mathcal{B}_2^2$ as a representative for the general case. In other words, if we can write out the inverse of $\mathcal{B}_2^2$, then we can easily write out the inverse of $\mathcal{B}_2^s$ for any $s > 1$.

To find the inverse matrix formula for the above matrix, we define a special tridiagonal matrix from the given independent variables $u_1, \ldots, u_p, v_1, \ldots, v_q$ with $p \geq 1$ and $q \geq 1$ as follows,

$$\Omega(\vec{u},\vec{v}) = \begin{bmatrix} u_1 & 1-u_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & u_2 & 1-u_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & u_p & 1-u_p & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & v_1 & 1-v_1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & v_2 & 1-v_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & v_q & 1-v_q \end{bmatrix}_{m\times m}, \tag{5.4.5}$$

where we denote $\vec{u} := (u_1, \ldots, u_p)$ and $\vec{v} := (v_1, \ldots, v_q)$ and $m = p+q$. In the following lemma, we give the formula for $\Omega^{-1}(\vec{u},\vec{v})$.

Lemma. Given a p-vector $\vec{u} := (u_1, \ldots, u_p)$ and a q-vector $\vec{v} := (v_1, \ldots, v_q)$ that satisfy the conditions: $p \geq 1, q \geq 1$ and $p+q = m$, define a tridiagonal matrix $\Omega(\vec{u},\vec{v})$ as above. In order to make $\Omega(\vec{u},\vec{v})$ invertible, we require that

$$u_i \neq 0 \quad \text{for } 1 \leq i \leq p, \quad u_p \neq v_1, \text{ and } \quad v_j \neq 1 \quad \text{for } 2 \leq j \leq q. \tag{5.4.6}$$

Then we can write $\Omega^{-1}(\vec{u},\vec{v})$ as $[\alpha_{ij}]$, where the non-zero entries in $\{\alpha_{ij}\}$ are given by the following formulas:

- For the diagonal entries, we have

  $$\alpha_{ii} = \frac{1}{u_i}, \quad \text{for } 1 \leq i \leq p-1, \quad \alpha_{pp} = \frac{1-v_1}{u_p - v_1},$$

  and

  $$\alpha_{p+1,p+1} = \frac{u_p}{u_p - v_1}, \quad \alpha_{jj} = \frac{1}{1 - v_{j-p}} \quad \text{for } p+2 \leq j \leq m.$$

- For the lower triangular entries $\alpha_{i,j}$ with $p \leq j < i \leq m$, we represent them in the following general formulas, for $r = 1, \ldots, q-1$:

  $$\alpha_{p+r,p} = -\frac{v_1}{u_p}\,\alpha_{p+r,p+1}, \quad \alpha_{p+r+1,p+1} = -\frac{u_p\,v_2}{u_p - v_1}\,\alpha_{p+r+1,p+2},$$

  and for $p+2 \leq j \leq m-r$,

  $$\alpha_{j+r,j} = -\frac{v_{j-p+1}}{1 - v_{j-p}}\,\alpha_{j+r,j+1}.$$

- For the upper triangular entries, for $s = 1, \ldots, q$,

$$\forall\, 1 \le i \le p, \qquad \alpha_{i,i+s} = \Big(1 - \frac{1}{u_i}\Big)\alpha_{i+1,i+s},$$

  and for $t = q+1, \ldots, m-1$,

$$\forall\, 1 \le i \le m-t, \qquad \alpha_{i,i+t} = \Big(1 - \frac{1}{u_i}\Big)\alpha_{i+1,i+t}.$$

Since each diagonal block matrix $\mathcal{B}_2^j$ in B_2^{IV} is invertible for $j \ge 1$, we can easily write the inverse of B_2^{IV} as follows,

$$\big(B_2^{IV}\big)^{-1} := \begin{bmatrix} (\mathcal{B}_2^1)^{-1} & 0 & \cdots & \cdots & \cdots \\ 0 & (\mathcal{B}_2^2)^{-1} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & (\mathcal{B}_2^s)^{-1} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.4.7}$$

Explicit formulas for $\big(\mathcal{B}_2^j\big)^{-1}$ can also be given by the result of the lemma above.

The explicit formulas for $\big(\mathcal{B}_2^j\big)^{-1}$ with $j \ge 1$ in B_2^{IV} can be given in the following two cases:

- For $j = 1$, when $k_1 = 0$, we choose $\vec{u_1} := (1)$ and $\vec{v_1} := (b_{01}, \ldots, b_{k_2-1,k_2})$. Then $\big(\mathcal{B}_2^1\big)^{-1} = \Omega^{-1}(\vec{u_1}, \vec{v_1})$.

  When $k_1 > 0$, we take $\vec{u_1} := (b_{11}, \ldots, b_{k_1,k_1})$ and $\vec{v_1} := (b_{k_1,k_1+1}, \ldots, b_{k_2-1,k_2})$. Then
$$\big(\mathcal{B}_2^1\big)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \Omega^{-1}(\vec{u_1}, \vec{v_1}) \end{bmatrix}.$$

- For $j \ge 2$, we note by $\vec{u_j} := (b_{k_{2(j-1)}+1,k_{2(j-1)}+1}, \ldots, b_{k_{2j-1},k_{2j-1}})$ and $\vec{v_j} := (b_{k_{2j-1},k_{2j-1}+1}, \ldots, b_{k_{2j}-1,k_{2j}})$. Then $\big(\mathcal{B}_2^j\big)^{-1} = \Omega^{-1}(\vec{u_j}, \vec{v_j})$.

5.5 *Inverse Shoenberg-Whitney Matrix Case V:*

Case V: There are two subcases to be considered: A) $\sigma_1 = \cdots = \sigma_{k_1} = 0, \sigma_{k_1+1} = \cdots = \sigma_{k_2} = 1, \cdots, \sigma_{k_{2s}+1} = \cdots = 0$; and B) $\sigma_1 = \cdots = \sigma_{k_1} = 0, \sigma_{k_1+1} = \cdots = \sigma_{k_2} = 1, \cdots, \sigma_{k_{2s-2}+1} = \cdots = \sigma_{k_{2s-1}} = 0, \sigma_{k_{2s-1}+1} = \cdots = 1$.

For subcase A, we can write B_2^V as the following block diagonal matrix

$$B_2^V := \begin{bmatrix} \mathcal{B}_2^1 & 0 & \cdots & \cdots \\ 0 & \mathcal{B}_2^2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathcal{B}_2^s \end{bmatrix}, \tag{5.5.1}$$

where $\mathcal{B}_2^1, \ldots, \mathcal{B}_2^{s-1}$ are the same as those in *Case IV*, and $\mathcal{B}_2^s$ has the following structure:

$$\mathcal{B}_2^s := \begin{bmatrix} b_{k_s+1,k_s+1} & b_{k_s+2,k_s+1} & 0 & \cdots & \cdots & \cdots \\ 0 & b_{k_s+2,k_s+2} & b_{k_s+3,k_s+2} & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & b_{k_s+w,k_s+w} & b_{k_s+w+1,k_s+w} & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5.5.2}$$

Then we have

$$\left(B_2^V\right)^{-1} = \begin{bmatrix} (\mathcal{B}_2^1)^{-1} & 0 & \cdots & \cdots \\ 0 & (\mathcal{B}_2^2)^{-1} & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & (\mathcal{B}_2^s)^{-1} \end{bmatrix}, \tag{5.5.3}$$

where $(\mathcal{B}_2^1)^{-1}, \ldots, \left(\mathcal{B}_2^{s-1}\right)^{-1}$ can be written using the formulas in *Case IV*, and $(\mathcal{B}_2^s)^{-1}$ can be written using the formulas in *Case I* with both the left inverse and the right inverse.

For subcase B, we can write B_2^V as the following block diagonal matrix

$$B_2^V := \begin{bmatrix} \mathcal{B}_2^1 & 0 & \cdots & \cdots \\ 0 & \mathcal{B}_2^2 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \mathcal{B}_2^s \end{bmatrix}, \tag{5.5.4}$$

where $\mathcal{B}_2^1, \ldots, \mathcal{B}_2^{s-1}$ are the same as those in *Case IV*, and $\mathcal{B}_2^s$ has the following structure:

$$\mathcal{B}_2^s :=$$

$$\begin{bmatrix}
b_{k_{2s-2}+1,k_{2s-2}+1} & b_{k_{2s-2}+2,k_{2s-2}+1} & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & b_{k_{2s-1},k_{2s-1}} & b_{k_{2s-1}+1,k_{2s-1}} & \ddots & \ddots & \ddots \\
\vdots & \ddots & b_{k_{2s-1},k_{2s-1}+1} & b_{k_{2s-1}+1,k_{2s-1}+1} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & b_{k_{2s-1}+w-1,k_{2s-1}+w} & b_{k_{2s-1}+w,k_{2s-1}+w} & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix},$$

and its inverse can be found using the method in *Case II*.

Chapter 6

Conclusion and Future Research

Our technique to construct a local quasi-interpolant for linear B-splines on the infinite interval $[0, \infty)$ heavily relies on the operations of the infinite matrices that have quite different properties comparing with their finite-dimensional counterpart. To our knowledge, there is no existing theory developed for the operations of the general infinite-dimensional matrices. People tend to use the properties for the finite-dimensional matrices on the infinite-dimensional matrices directly, which could result in incorrect results. In our theory development, we encounter several examples for the infinite-dimensional matrices that do not follow certain obvious properties for the finite-dimensional matrices. In order to ensure that the results developed on the infinite-dimensional matrices correct, we verify all our results based on the original definitions.

More specifically, we found that in the following three situations, the behaviors for the infinite-dimensional matrices are quite different.

- *On invertible matrices and singular matrices*

  In the following example,

$$\begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & -1 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 1 & -1 & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & -1 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \ddots & \ddots & \ddots & 0 & 1 & -1 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix}, \qquad (6.0.1)$$

  we observe that an infinite-dimensional upper triangular matrix with all the diagonal entries 1's, which is supposed to be invertible, multiplies a non-zero vector and results in the zero vector. This behavior cannot happen in the finite-dimensional matrix case, in which if an invertible matrix multiplies a non-zero vector, the result vector can never be zero. Thus, we have to re-examine the meaning of the invertible infinite-dimensional matrices. Should we use the view of the linear operators or linear transformations to study the invertibility of the infinite-dimensional matrices?

- *On associative law for matrix products*

The *associative law* for the matrix products is an ordinary property for the finite-dimensional case. Unfortunately, we lose this fundamental property for the infinite-dimensional matrices. Look at the following example: Given three infinite matrices A, B, and C,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{6.0.2}$$

$$B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/5 & 1/5 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{6.0.3}$$

and

$$C = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 2/3 & 1/3 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1/3 & 2/3 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{6.0.4}$$

we compare two matrix products using different orders: $(AB)C$ and $A(BC)$, and get

$$A\,(BC) = A \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1/3 & 4/15 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} = I_\infty, \tag{6.0.5}$$

and

$$(AB)\,C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -2/5 & -11/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 2/5 & 11/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & 1 & -2 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} C \tag{6.0.6}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 & 0 & -8/5 & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1 & 0 & 0 & 8/5 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1 & 0 & -8/5 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1 & 8/5 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \neq I_\infty. \tag{6.0.7}$$

Since we cannot use the associative law for the products of infinite-dimensional matrices, the impact is huge. Many ordinary matrix operations will not be true anymore. Another consequence of this property is that the left inverse and the right inverse for certain matrix could be different.

- *On left inverse and right inverse*

We consider the following matrix,

$$A = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & 0 & 1/2 & 1/2 & \ddots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & 0 & 1/3 & 4/15 & \ddots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 1/2 & 1/2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6.0.8}$$

We found its left inverse as follows,

$$A_L^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6.0.9}$$

We also found its right inverse as follows,

$$A_R^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots \\ 0 & 2 & -2 & 2 & -3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 2 & -2 & 3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 2 & -3 & 4/5 & 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & 3 & -4/5 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & 0 & 1 & 0 & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & 0 & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 & -2 & 2 & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -1 & 2 & -2 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{6.0.10}$$

We would like to extend our results to more general B-splines. But the main difficulty is at the manipulations of the infinite-dimensional matrices. If we do not understand the infinite-dimensional matrices well, it is very hard to get the correct results. To this end, we should find good explanations for the above three properties. We hope that we can develop a theory that provides reliable operations for the infinite-dimensional matrices.

Bibliography

[1] Marsden, M.: *An identity for spline functions with applications to variation-diminishing spline approximation*. J. Approx. Th. 3 (1970), 7-49.

[2] Chen, D. R., and Xiang, D. H.: *A construction of multires- olution analysis on interval*. Acta Mathematica Sinica, English Series 23.4 (2007): 705-710.

[3] Chui, C. K.: *Wavelets: A Mathematical Tool for Signal Process- ing*. Philadelphia: Society for Industrial and Applied Mathematics; (1997).

[4] Chui, C. K.: *An introduction to wavelets*. Vol. 1. Aca- demic press, (2014).

[5] Chui, C. K. and Diamond, H.: *A general framework for local interpolation*. Numerische Mathematik, 58(1): 569-581, 1990.

[6] Chui, C. K., and Jiang, Q.: *Applied Mathematics: Data Compression, Spectral Methods, Fourier Analysis, Wavelets, and Ap- plications*. vol. 2, Atlantis Press, Paris, France, (2013).

[7] Daubechies, I.: *Ten lectures on wavelets*. Vol. 61. Philadel- phia: Society for industrial and applied mathematics (1992).

[8] De Boor, C., and Fix, G.: *Spline approximation by quasi-interpolants*. Journal of Approximation Theory, 8:96-110, 1973.

[9] Jiang, Q.: *On the regularity of matrix refinable functions*. SIAM journal on mathematical analysis 29.5 (1998): 1157-1176.

[10] Joy, K. I.: *Bernstein polynomials*. On-Line Geometric Model- ing Notes13 (2000).

[11] Gustafson, P., Savir, N., and Spears, E.: *A characterization of refinable rational functions*. American Journal of Undergraduate Research 5.3 (2006): 11-20.

[12] Lyche, T., and Morken, K.: *Spline methods draft*. Depart- ment of Informatics, University of Oslo, URL= http://heim. ifi. uio. no/knutm/komp04. pdf (2004).

[13] Malone, D.: *A sinister view of dilation equations*. International Journal of Wavelets, Multiresolution and Information Processing 3.01 (2005): 67-77.

[14] Marsden, M.: *An identity for spline functions with applications to variation-diminishing spline approximation*. J. Approx. Th. 3 (1970), 7-49.

[15] Gori, L., and Pitolli, F.: *A class of totally positive refinable functions.* Rendiconti di Matematica, Serie VII 20 (2000): 305-322.

[16] Piegl, L., and Tiller, W.: *The NURBS book.* Springer Science , Business Media, (2012).

[17] Van der Walt, M. D.: *Wavelet analysis of non-stationary signals with applications*, UMSL, 2015.

[18] Strang, Gilbert, and George J. Fix.: *An analysis of the finite element method.* (1973).